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"Best" Estimations on the Distribution of the Length of Sums of Two Random Vectors

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1. Introduction

Let ξ and η be independent and identically distributed random variables taking values in a Hilbert-space X. The aim of the paper is to give lower estimations on the probability $P(\|\xi+\eta\| \geqq x)$ in terms of the quantity $P(\|\xi\| \geqq cx)$ where $0<c$ is a fixed constant. More precisely, we are looking for a function f satisfying

$$P(\|\xi+\eta\| \geqq x) \geqq f(P(\|\xi\| \geqq cx)). \tag{1}$$

We say that f (or the estimate) is the *best possible* if for any $x>0$ and any value of $P(\|\xi\| \geqq cx)$ there is a pair ξ, η with equality in (1).

Inequalities of this type can be proved for one dimensional x with the classical methods (convolution).

Higher dimensions need, however, another technique. The elements of this technique can be found in [3], where

$$P(\|\xi+\eta\| \geqq x) \geqq \tfrac{1}{2} P^2(\|\xi\| \geqq x) \tag{2}$$

is proved for any Hilbert-space (with a technical side-condition which turned out to be unnecessary). (2) is best possible if the dimension is at least 2.

The idea of the method of [3] is the following. We associate an oriented graph $G=(X,E)$ to the random variables, where x is the space and the set E of edges consists of the pairs (a,b) satisfying $\|a+b\| \geqq x$, where X is a fixed positive real. By some geometry, certain conditions can be imposed on this graph. As $P(\|\xi+\eta\| \geqq x)$ is the measure of E, we obtain a lower estimation minimizing the measure of edges under the conditions obtained for the graph. That is, we have to solve some extremal graph-problems for a "continuous" vertex set. The analogous minimization problems for a finite number of vertices are either solved or easier treatable. Thus, we start with the finite case and then we take the "continuous" version of it. This method is more or less developed in [4], but it did not work smoothly enough in the form

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presented there. The reason is that the question "when does the continuous version follow the finite extremal graph theorem?" was not completely answered in [4]. Since then [5] has settled this question by more appropriate definitions and theorems. In the present paper these improved results will be used. On the other hand, the aim of [4] was to show the method with kaleidoscopic results, only. The aim of the present paper, in contrary to [4], is to go deeper and to determine the best lower estimate of $P(|\xi+\eta|\geqq x)$ in terms of $P(|\xi|\geqq cx)$ for any $c>0$.

Section 2 lists the theorems, Sects. 3 and 4 contain the geometrical lemmas and finite graph-results, respectively, Sect. 5 gives the "continuous" versions and Sect. 6 proves the theorems. Finally, the last section mentiones some further questions.

2. Results

Theorem 1. *If X is an infinite dimensional Hilbert-space, ξ and η are X-valued i.i.d. random variables, then the best possible functions f in the inequality $P(\|\xi+\eta\|\geqq x)\geqq f(P(\|\xi\|\geqq cx))$ are the following ones:*

$$f(p)=\begin{cases}1/2 & \text{if } p\geqq 1/3,\\ 2p-\frac{3}{2}p^2 & \text{otherwise,}\end{cases}\quad \text{when}\quad 5/2\leqq c<\infty; \tag{3}$$

$$f(p)=\begin{cases}1/2 & \text{if } p\geqq 1/2,\\ 2p(1-p) & \text{otherwise,}\end{cases}\quad \text{when}\quad 3/2\leqq c<5/2; \tag{4}$$

$$f(p)=\begin{cases}-1/2+2p-p^2 & \text{if } p\geqq 1/2,\\ p^2 & \text{otherwise,}\end{cases}\quad \text{when}\quad \sqrt{5}/2\leqq c<3/2; \tag{5}$$

$$f(p)=\tfrac{1}{2}p^2 \quad \text{when} \quad 1\leqq c<\frac{\sqrt{5}}{2}; \tag{6}$$

$$f(p)=\frac{1}{k-1}p^2 \quad \text{when} \quad \sqrt{\frac{k-1}{2(k-2)}}\leqq c<\sqrt{\frac{k-2}{2(k-3)}} \qquad (4\leqq k<\infty); \tag{7}$$

$$f(p)=0 \quad \text{when} \quad 0<c\leqq 1/\sqrt{2} \qquad \text{(see Fig. 1)} \tag{8}$$

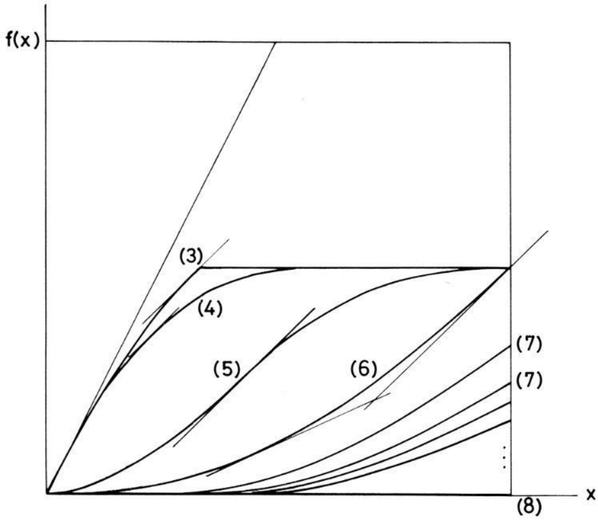
Remarks 1. Only (7) is known from these estimates (see [4]).

2. In our theorems we implicitely suppose that $\{\|\xi+\eta\|\geqq x\}$ is measurable in the product space. This automatically holds when X is a separable Hilbert space.

The fact that these estimates are sharp show that our method, using extremal graph theory, cannot be too artificial.

For finite dimensional (Euclidean) space we cannot expect such a "perfect" theorem. The reason is that the following question of geometry is unsolved in general: Determine

$$e_k(d)=\sup \min_{1\leqq i<j\leqq k} (\text{angle between } a_i \text{ and } a_j), \tag{9}$$



where a_i $(1 \leqq i \leqq k)$ are vectors of length 1 in the d-dimensional Euclidean space. In terms of these numbers, best estimates can be obtained, again:

Theorem 2. *If X is a d-dimensional $(d \geqq 2)$, Euclidean space, ξ and η are X-valued i.i.d. random variables, then the following estimates are best possible:*

(3) *when* $5/2 \leqq c < \infty$,

(4) *when* $3/2 \leqq c < 5/2$,

(5) *when* $\sqrt{5}/2 \leqq c < 3/2$,

(6) *when* $1 \leqq c < \dfrac{\sqrt{5}}{2}$,

(7) *when* $\dfrac{1}{2\cos\dfrac{e_k(d)}{2}} \leqq c < \dfrac{1}{2\cos\dfrac{e_{k-1}(d)}{2}}$ $(4 \leqq k < \infty)$;

(8) *when* $0 < c \leqq 1/2$.

Some known values of $e_k(d)$ can be found in [4]. In paper [4] only the case (7) is formulated in some special cases.

Let us formulate the case $d = 1$ separately:

Theorem 3. *If ξ and η are i.i.d. real-valued random variables, then the following lower estimates of $P(|\xi + \eta| \geqq x)$ are best possible:*

(3) *when* $5/2 \leqq c < \infty$;

(4) *when* $3/2 \leqq c < 5/2$;

(5) *when* $1 \leqq c < 3/2$;

(6) *when* $1/2 \leqq c < 1$;

(8) *when* $0 < c < 1/2$.

3. Some Trivial Geometry

Lemma 3.1. *If $a_1, \dots, a_k$ are vectors of length at least 1 in a Hilbert-space then there is a pair $i \neq j$ satisfying.*

$$\|a_i + a_j\| \geqq \sqrt{\frac{2(k-2)}{k-1}}.$$

This is a special case of Lemma 5 of [4].

Lemma 3.2. *If a_1, a_2, a_3 are vectors in a Hilbert-space and $\|a_1\| \geqq 1$, then there is a pair $i \neq j$, satisfying*

$$\|a_i + a_j\| \geqq 2/3. \tag{10}$$

Proof. The statement follows from the triangle inequality.

Lemma 3.3. *If a_1, a_2, a_3 are vectors in a Hilbert-space and $\|a_1\|$, $\|a_2\| \geqq \sqrt{5}/2$, then there is a pair $i \neq j$ satisfying*

$$\|a_i + a_j\| \geqq 1.$$

Proof. If $\|a_1 + a_2\| < 1$, then $\cos\alpha < -3/5$ holds for their angle α $(0 \leqq \alpha \leqq \pi)$. The smaller angle β $(0 \leqq \beta \leqq \pi)$ from the other two ones (say the angle of a_1 and a_3) satisfies $\beta \leqq \frac{2\pi - \alpha}{2}$. Hence

$$\cos\beta \geqq \cos(\pi - \alpha/2) = -\sqrt{\frac{1+\cos\alpha}{2}} > -\sqrt{\frac{1}{5}},$$

that is,

$$\|a_1 + a_3\| = \sqrt{\|a_1\|^2 + \|a_3\|^2 + 2\|a_1\|\,\|a_3\|\cos\beta}$$
$$> \sqrt{\|a_1\|^2 + \|a_3\|^2 - 2\|a_1\|\,\|a_3\|\sqrt{1/5}}.$$

The minimum of the two-variable function $x^2 + y^2 - 2xy\sqrt{1/5}$ in the domain $x \geqq \frac{\sqrt{5}}{2}$, $y \geqq 0$ is 1. $\|a_1 + a_3\| \geqq 1$ follows.

Lemma 3.4. *If a_1, a_2, a_3, a_4 are vectors in a Hilbert-space and $\|a_1\| \geqq 5/2$ $\|a_1 + a_2\| < 1$, $\|a_2 + a_3\| < 1$, $\|a_2 + a_4\| < 1$, then $\|a_3 + a_4\| \geqq 1$.*

Proof. $\|a_2\| \geqq \big|\|a_1\| - \|a_1 + a_2\|\big| > 5/2 - 1 = 3/2$ follows from the triangle inequality. Now Lemma 3.2 implies that one of the vectors $a_2 + a_3$, $a_2 + a_4$, $a_3 + a_4$ is of length at least 1. By the conditions, this vector must be $a_3 + a_4$.

4. Two-Part Turán Type Theorems for Graphs

Let us start with the well-known

Turán Theorem [9]. *Suppose that an undirected (no loops, no multiple edges) graph $G=(X,E)$ on $n=|X|$ vertices contains no empty k-set, that is, for any k vertices there is at least one edge in E among them. Then the number of edges is at least*

$$r\binom{q+1}{2}+(k-1-r)\binom{q}{2}, \tag{11}$$

where $n=(k-1)q+r$ $(0\leqq r\leqq k-1)$ and the only optimal graph is the following one: take $k-1$ disjoint complete graphs with $q+1,\dots,q+1,\ q,\dots,q$ vertices (the number of $q+1$'s is r).

If $Y\subset X$ then the subgraph $G_Y=(Y,E')$ induced by Y from $G=(X,E)$ is defined by $E'=\{\{x,y\}: x,y\in Y, \{x,y\}\in E\}$. The condition of the above theorem can be formulated that the subgraphs of G induced by 3 different vertices are not empty.

We need some extensions of the case $k=3$. Some of them are formulated and proved in [4]. To save place for better papers we repeat only one of them. Observe that in the case $k=3$ (11) is $\left\lfloor\left(\frac{n-1}{2}\right)^2\right\rfloor$ ($\lfloor x\rfloor$ denotes the largest integer $\geqq x$).

Lemma 4.1. (Lemma 1 of [4]). *If the undirected graph $G=(X,E)$ $(X=X_1\cup X_2$, $X_1\cap X_2=\emptyset$, $|X_1|=n_1$, $|X_2|=n_2$, $n_1+n_2=n)$ contains no empty triangle with exactly one vertex from X_1, then*

$$|E|\geqq\begin{cases} n_1 n_2-\binom{n_1+1}{2} & \text{if } n_1\leqq n_2 \\ \binom{n_2}{2} & \text{if } n_1\geqq n_2. \end{cases} \tag{12}$$

Remark. We need the lemma in a slightly stronger way. Namely, (12) holds if E denotes the set of edges within X_2 and between X_1 and X_2. This follows easily. If G satisfies the condition of the lemma, then omitting all the edges within X_1, the new graph G' still satisfies this condition. Thus, the lemma can be applied for G'.

Lemma 4.2. *Suppose that the undirected graph $G=(X,E)$ $(X_1\cup X_2=X, X_1\cap X_2=\emptyset$, $|X_1|=n_1$, $|X_2|=n_2$, $n_1+n_2=n)$ satisfies the following properties*

it contains no empty triangle with at least one vertex from X_2 (13)

all vertices of an empty triangle in X_1 are connected with all vertices in X_2. (14)

Then

$$|E| \geqq g(n_1, n_2) = \begin{cases} \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor & \text{if } n_1 \leqq 2n_2 + 2 \\ n_1 n_2 + \left\lfloor \left(\frac{n_2-1}{2}\right)^2 \right\rfloor & \text{if } n_1 \geqq 2n_2 + 2. \end{cases} \tag{15}$$

Proof. We use induction on n_1 with fixed n_2. If $n_1 \leqq n_2$ then (13) and Lemma 3 of [4] imply the desired result. Therefore suppose $n_1 > n_2$ and (15) is proved for smaller values of n_1.

Let $G(X, E)$ be a graph satisfying the conditions (13) and (14). The union of the empty triangles (3-element sets) is denoted by Y. Put $j = |X_1 - Y|$. Here, of course, $0 \leqq j \leqq n_1$. The following two cases will be distinguished: 1) $j \geqq \frac{n_1}{2}$, 2) $j < \frac{n_1}{2}$.

1) Take the partition Y, $(X_1 - Y) \cup X_2$ of X. There is no empty triangle with at least one vertex from $(X_1 - Y) \cup X_2$. This follows from (13) if one vertex from the triangle is in X_2, and from the definition of Y, otherwise. Consequently, Lemma 3 of [4] can be applied for this partition. Since $j \geqq \frac{n_1}{2}$ implies $n_1 - j \leqq n_2 + j$, the first case of the lemma applies:

$$|E| \geqq \left\lfloor \left(\frac{n_1 + n_2 - 1}{2}\right)^2 \right\rfloor. \tag{16}$$

2) By (14) and the definition of Y there are $n_2(n_1 - j)$ edges between Y and X_2.

The triangles having two vertices from Y and one vertex from $X_1 - Y$ can not be empty. The stronger version of Lemma 4.1. (see remark) can be applied for the partition Y, $X_1 - Y$ of X_1. Namely, the first row of (14), as $j < \frac{n_1}{2}$, implies $j < n_1 - j$. So, the number of edges of G within Y and between Y and $X_1 - Y$ is at least $j(n_1 - j) - \binom{j+1}{2}$.

Finally, the number of edges in the graph induced from G by $(X_1 - Y) \cup X_2$ is at least $g(j, n_2)$ by the inductional hypothesis. Altogether:

$$|E| \geqq n_2(n_1 - j) + j(n_1 - j) - \binom{j+1}{2} + g(j, n_2). \tag{17}$$

It means that either (16) is valid for E or (17) with some $0 \leqq j < \frac{n_1}{2}$. All we have to show that the right hand sides of (16) and (17) are $\geqq g(n_1, n_2)$.

Start with (16). The case $n_1 \leqq 2n_2 + 2$ is trivial. Suppose that $n_1 \geqq 2n_2 + 2$.

$$\left\lfloor \left(\frac{n_1 + n_2 - 1}{2}\right)^2 \right\rfloor \geqq n_1 n_2 + \left\lfloor \left(\frac{n_2 - 1}{2}\right)^2 \right\rfloor$$

should be proved. It is sufficient to show this without the sings $\lfloor\ \rfloor$. This latter inequality is equivalent to the condition $n_1 \geqq 2n_2+2$.

In case of (17) we have to prove

$$n_2(n_1-j)+j(n_1-j)-\binom{j+1}{2}+g(j,n_2)\geqq g(n_1,n_2) \qquad \left(0\leqq j<\frac{n_1}{2}\right). \tag{18}$$

Cases will be distinguished:

a) $n_1 \leqq 2n_2+2$. This inequality and $j<\frac{n_1}{2}$ imply $j<2n_2+2$, that is, the first row of (15) should be considered in both cases $g(n_1,n_2)$ and $g(j,n_2)$. (18) obtains the form

$$n_2(n_1-j)+j(n_1-j)-\binom{j+1}{2}+\left\lfloor\left(\frac{j+n_2-1}{2}\right)^2\right\rfloor\geqq\left\lfloor\left(\frac{n_1+n_2-1}{2}\right)^2\right\rfloor. \tag{19}$$

Omitting the signs $\lfloor\ \rfloor$, again, we obtain the inequality

$$-n_1^2+2n_1n_2+4n_1j-2n_2j+2n_1-5j^2-4j\geqq 0.$$

The coefficient of n_2 is $2n_1-2j$, consequently the quantity is not increased by taking the minimal value of n_2: $n_2=\frac{n_1-2}{2}$. The reduced inequality is

$$3n_1j-5j^2-2j\geqq 0.$$

This follows from $0\leqq j<\frac{n_1}{2}$.

b) $n_1\geqq 2n_2+2$.

ba) $j\geqq 2n_2+2$.

(18) is now of the form

$$n_2(n_1-j)+j(n_1-j)-\binom{j+1}{2}+jn_2+\left\lfloor\left(\frac{n_2-1}{2}\right)^2\right\rfloor\geqq n_1n_2+\left\lfloor\left(\frac{n_2-1}{2}\right)^2\right\rfloor.$$

This can be reduced by the above method for

$$n_1j-\frac{3j^2}{2}-\frac{j}{2}\geqq 0.$$

This follows from the supposition $0\leqq j<\frac{n_1}{2}$, again.

bb) $j\leqq 2n_2+2$.

In this case we have to prove

$$n_2(n_1-j)+j(n_1-j)-\binom{j+1}{2}+\left\lfloor\left(\frac{j+n_2-1}{2}\right)^2\right\rfloor\geqq n_1n_2+\left\lfloor\left(\frac{n_2-1}{2}\right)^2\right\rfloor.$$

This can be reduced for

$$4n_1 \geqq 2n_2 + 5j + 4.$$

This inequality easily follows from the conditions $n_1 \geqq 2n_2 + 2$, $0 \leqq j < \frac{n_1}{2}$.

5. Continuous Versions

In [5], we have proved some theorems giving a tool for making the continuous versions of discrete statements. E.g. from Lemma 2 of [5] and the Turán theorem one can deduce the

Continuous Turán Theorem. *Let (X, σ, P) be a probability space $G=(X, D)$ a symmetric measurable graph (i.e. $D \subset X^2$ is measurable and $(x, y) \in D$ implies $(y, x) \in D$). Suppose that it contains all the loops (x, x) and G_Y is non-empty for any k-element Y then*

$$P(D) \geqq \frac{1}{k-1}. \tag{20}$$

Hint for Proof. The right hand side of (20) is the limit of the ratio "(11) divided by $n^2/2$."

Similarly, Theorem 3 of [5], Lemmas 3, 2 of [4] and Lemma 4.2 of the present paper imply the next lemmas.

Lemma 5.1. *Let (X, σ, P) be a probability space, $X = X_1 \cup X_2$ $(X_1 \cap X_2 = \emptyset)$ a partition of X, X_1 measurable, $G=(X, D)$ a symmetric, measurable graph with the following properties*

1) *If $|Y|=3$, $|Y \cap X_2| \geqq 1$ then G_Y is not empty;*
2) *contains all the loops in X_2;*
3) *For $x_1 \in X_1$, $x_2 \in X_2$ either $(x_1, x_2) \in D$ or $(x_1, x_1) \in D$. Then*

$$P(D) \geqq \begin{cases} 1/2 & \text{if } P(X_1) \leqq 1/2 \\ 2P(X_1)(1-P(X_1)) & \text{if } P(X_1) \geqq 1/2. \end{cases} \tag{21}$$

Lemma 5.2. *Let (X, σ, P) be a probability space, $X = X_1 \cup X_2$ $(X_1 \cap X_2 = \emptyset)$ a partition of X, X_1 measurable, $G=(X, D)$ a symmetric, measurable graph with the following properties:*

1) *G_Y is not empty whenever $|Y|=3$, $|Y \cap X_2| \geqq 2$;*
2) *G contains all the loops in X_2. Then*

$$P(D) \geqq \begin{cases} 1/2 - P(X_1)^2 & \text{if } P(X_1) \leqq 1/2 \\ P(X_2)^2 & \text{if } P(X_1) \geqq 1/2. \end{cases} \tag{22}$$

Lemma 5.3. *Let (X, σ, P) be a probability space, $X = X_1 \cup X_2$ $(X_1 \cap X_2 = \emptyset)$ a partition of X, X_1 measurable, $G=(X, D)$ a symmetric, measurable graph with* *the following properties:*

1) (13),

2) (14),

3) *G contains all the loops in* X_2;

4) *if* $(x_1, x_3) \notin D$ $(x_1, x_3 \in X_1)$ *then* x_1 *and* x_3 *are connected with all elements of* X_2. *Then*

$$P(D) \geqq \begin{cases} 1/2 & \text{if } P(X_1) \leqq 1/3 \\ 2P(X_1)P(X_2) + 1/2 P(X_2)^2 & \text{if } P(X_1) \geqq 1/3. \end{cases} \tag{23}$$

6. Proofs of the Main Result

Proof of Theorem 1. First we prove the estimates. The constructions showing that they are sharp will follow later.

(8) is trivial.

Fix the real number $x > 0$ and put

$$X_0 = \left\{ a : a \in X, \|a\| \geqq \sqrt{\frac{k-1}{2(k-2)}}\, x \right\}.$$

The graph $G = (X_0, D)$ is defined by $D = \{(a, b) : \|a + b\| \geqq x\}$. It is obvious that $(a, a) \in D$ and that $(a, b) \in D$ iff $(b, a) \in D$. That is, $G = (X_0, D)$ is a symmetric graph containing all the loops. The measurability of the graph is our implicite supposition. Lemma 3.1 shows that G_Y contains at least one edge for any Y with $|Y| = k$. The continuous Turán theorem can be applied for the probability space $(X_0, \sigma/X_0, P(X \cap X_0)/P(X_0))$:

$$\frac{P(\|\xi + \eta\| \geqq x)}{P^2\left(\|\xi\| \geqq \sqrt{\frac{k-1}{2(k-2)}}\, x\right)} \geqq \frac{1}{k-1}.$$

(7) is proved for $c = \sqrt{\frac{k-1}{2(k-2)}}$. Increasing c, $P(\|\xi\| \geqq cx)$ does not increase, thus (7) is proved for $k \geqq 4$. For $k = 3$, we obtain (6) for any $c \geqq 1$.

Fix the real number $x > 0$ and put $X_1 = \left\{ a : a \in X, \|a\| < \frac{\sqrt{5}}{2} x \right\}$, $X_2 = X - X_1$. The graph $G = (X, D)$ is defined by $D = \{(a, b) : \|a + b\| \geqq x\}$. It is obvious that G is symmetric and it contains all the loops in X_2. Condition 1) of Lemma 5.2 follows from Lemma 3.3. Inequality (5) follows by Lemma 5.2.

Inequality (4) follows in the same way using $X_1 = \{a : a \in X, \|a\| < 3/2\}$, $X_2 = X - X_1$, Lemmas 5.1 and 3.2.

Finally, inequality (3) can be obtained by $X_1 = \{a : a \in X, \|a\| < 5/2\}$, $X_2 = X - X_1$, and Lemmas 5.3 and 3.4.

Constructions. Put $k\geqq 4$ and suppose that

$$\sqrt{\frac{k-1}{2(k-2)}}\leqq c<\sqrt{\frac{k-2}{2(k-3)}}.$$

Let the vectors $s_1,\dots,s_{k-1}$ of length cx form a $(k-2)$-dimensional regular simplex. Define ξ with $P(\xi=s_i)=\frac{p}{k-1}(1\leqq i\leqq k-1)$, $P(\xi=0)=1-p$ for any fixed $p(0\leqq p\leqq 1)$. It is obvious that $P(\|\xi\|\geqq cx)=p$ we have to prove $P(\|\xi+\eta\|\geqq x)=\frac{p^2}{k-1}$, only. However, $\|s_i+0\|=cx<x$, $\|s_i+s_j\|=\sqrt{\frac{2(k-3)}{k-2}}cx<x$ $(i\neq j)$ and $\|2s_i\|=2cx>x$ easily hold. $\|\xi+\eta\|\geqq x$ is true iff $\xi=\eta=s_i$ for some i $(1\leqq i\leqq k-1)$. The probability of this is really $\frac{p^2}{k-1}$. (7) can not be improved.

The above construction works for $c\leqq\frac{1}{\sqrt{2}}$ at a fixed x and p with any k. So $P(\|\xi+\eta\geqq x)$ can be chosen to be $\frac{p^2}{k-1}$. This tends to 0 when k tends to infinity. Thus (8) can not be improved, either.

Let v and e be vectors in X with the properties $\|v\|=\|e\|=1$, $ve=0$. Let ξ be defined by

$$P\left(\xi=\frac{v+e/2}{\sqrt{5/2}}cx\right)=P\left(\xi=\frac{-v+e/2}{\sqrt{5/2}}cx\right)=\frac{p}{2},\quad P\left(\xi=-\frac{e}{\sqrt{5}}cx\right)=1-p,$$

where p is a fixed real $0\leqq p\leqq 1$. If $1\leqq c<\sqrt{5}/2$ then

$$\left\|\frac{\pm v+e/2}{\sqrt{5/2}}cx\right\|=cx\quad\text{and}\quad\left\|-\frac{e}{\sqrt{5}}cx\right\|<cx,$$

thus $P(\|\xi\|\geqq cx)=p$ is trivial.

On the other hand,

$$\left\|\frac{+v+e/2}{\sqrt{5/2}}cx+\frac{-v+e/2}{\sqrt{5/2}}cx\right\|<x,$$

$$\left\|\frac{\pm v+e/2}{\sqrt{5/2}}cx-\frac{e}{\sqrt{5}}cx\right\|<x$$

and $\left\|2\frac{e}{\sqrt{5}}cx\right\|<x$ hold, consequently $\|\xi+\eta\|\geqq x$ iff $\xi=\eta=\frac{\pm v+e/2}{\sqrt{5/2}}cx$. That is, $P(\|\xi+\eta\|\geqq x)=\frac{p^2}{2}$ as desired. (6) cannot be improved.

For the case of (5) we need two constructions. Suppose that $1/2\leqq p\leqq 1$, then define ξ by $P(\xi=cvx)=1/2$, $P(\xi=-cvx)=p-1/2$, $P(\xi=-(\frac{1}{2}-\varepsilon)vx)=1-p$, where v is a fixed vector of length 1 and ε is a small positive number. It is easy to see that

$$P(\|\xi+\eta\|\geqq x)=P(\xi+\eta=2cvx)+P(\xi+\eta=-2cvx)+P(\xi+\eta=-cvx-(\tfrac{1}{2}-\varepsilon)vx)$$
$$=(1/2)^2+(p-1/2)^2+2(p-1/2)(1-p)=-1/2+2p-p^2,$$

if ε is small enough. Suppose now that $0\leqq p\leqq 1/2$. ξ is defined by $P(\xi=cvx)=p$, $P(\xi=-(1/2-\varepsilon)vx)=(1-p)$. Here $P(\|\xi+\eta\|\geqq x)=P(\xi+\eta=2cvx)=p^2$ for small enough ε. (5) can not be improved.

Let us consider now (4). For $p\geqq 1/2$ and $p\leqq 1/2$ the random vectors $P(\xi=cvx)=1/2$, $P(\xi=-cvx)=p-1/2$, $P(\xi=-(\frac{3}{2}-\varepsilon)vx)=1-p$, and $P(\xi=cvx)=p$, $P(\xi=-(\frac{3}{2}-2\varepsilon)vx)=p$, $P(\xi=(\frac{1}{2}-\varepsilon)vx)=1-2p$ give the equality in (4), resp.

In case of (3) the equality is constructed in three different ways for the cases $1/2\leqq p$, $1/3\leqq p\leqq 1/2$ and $p\leqq 1/3$. In the first case the random vector $P(\xi=cvx)=1/2$, $P(\xi=-cvx)=p-1/2$, $P(\xi=-(c-\varepsilon)vx)=1-p$, in the second case $P(\xi=cvx)=p$, $P(\xi=(c-\varepsilon)vx)=1/2-p$, $P(\xi=-(c-\varepsilon)vx)=1/2$ and finally in the third case $P(\xi=\pm cvx)=p/2$, $P(\xi=0)=1-p$ give the desired equality.

Proof of Theorem 2. The proof of the case $1\leqq c<\infty$ coincides with that of Theorem 1. The fact that X has infinitely many dimensions was used neither in the proof nor in the constructions. The proof of inequality (7) under the condition

$$\frac{1}{2\cos\frac{e_k(d)}{2}}\leqq c<\frac{1}{2\cos\frac{e_{k-1}(d)}{2}}\qquad(4\leqq k<\infty)\tag{24}$$

is the same as at Theorem 1. The only difference is that the definition of $e_k(d)$ is used in place of Lemma 3.1. On the other hand, suppose that (24) holds for c with $4\leqq k$. Hence follows $e_k(d)<e_{k-1}(d)$. Choose the vectors $t_1,\dots,t_{k-1}$ to satisfy $\|t_i\|=cx$ $(1\leqq i\leqq k-1)$ and

$$e_{k-1}(d)=\min_{1\leqq i<j\leqq k-1}\qquad(\text{angle between } t_i \text{ and } t_j).$$

Here $\|t_i+0\|=cx<x$ follows from

$$0<e_{k-1}(d)\leqq\frac{2\pi}{3}\qquad(4\leqq k,2\leqq d);$$

$\|t_i+t_j\|\leqq 2cx\cos\frac{e_{k-1}(d)}{2}<x$ follows from the construction of t_i and (24). Consequently, if the random vector ξ is given by $P(\xi=t_i)=\frac{p}{k-1}(1\leqq i\leqq k)$, $P(\xi=0)=1-p$ then $P(\|\xi+\eta\|\geqq x)=P(\xi=\eta=t_i$ for some $1\leqq i\leqq k-1)=\frac{p^2}{k-1}$. The equality is constructed.

If $c\leqq 1/2$ then the above construction works for any k, so the probability $P(\|\xi+\eta\|\geqq x)$ can be arbitrarily small. The best estimation is really 0.

Proof of Theorem 3. The proofs of the inequalities and the constructions are the same as in the previous cases.

7. Further Possibilities

1. Our estimates are best in the sense that we determine the "best" function f of the distribution-function of $\|\xi\|$ at a given place (cx). Another problem is to find the best operator f where $P(\|\xi\| \geqq x)$ is considered to be a function. In [4] there is an easy example where f is a function of two fixed points of the distribution-function.

2. To give good lower estimates on $P(\|\xi_1+\xi_2+\xi_3\| \geqq x)$ is much more difficult. Neither the geometrical not the combinatorial tools are developed. Some results can be found in [8] and [7].

3. If ξ is symmetric (that is, $P(\xi \in A) = P(\xi \in -A)$), then

$$P(\|\xi+\eta\| \geqq x) \geqq P(\|\xi\| \geqq x) - \frac{P^2(\|\xi\| \geqq x)}{2}$$

is the best inequality for any linear normed space. A slightly weaker estimate is proved in [1], for one dimension, only (however, more terms). This was generalized by Kanter [2] using different methods. If, however $P(\|\xi\| \geqq cx)$ is used in the estimate with $c \neq 1$, then our method becomes useful again. E.g. we are able to prove the "best" estimate

$$P(|\xi+\eta| \geqq x) \geqq \tfrac{1}{4} P^2 \left(|\xi| \geqq \frac{4}{\sqrt{3}}\right)$$

for a symmetric ξ and η is two-dimensions. We are intending to work out this problem in another paper.

4. Zolotarev [10] suggested the following problem. (2) is too weak if ξ has to have a very "smooth" distribution. For instance suppose that the concentration-function $s(x)$ $(s(x) = \sup P(\xi \in S_x)$, where S_x is an arbitrary sphere of radius x) satisfies $\left|\frac{s(x_1)-s(x_2)}{x_1-x_2}\right| \leqq K$ with a small number K. Give good estimate of $P(|\xi+\eta| \geqq x)$ with $P(|\xi| \geqq x)$ under this condition. (If K is large (say $\geqq 200$) then our construction can be modified to be good, that is, (2) is the best estimate).

5. The independence and identical distribution of ξ and η are important conditions. Our method hardly works without them. However, there are other ways of extending the use of the method. Let f_1 and f_2 be a one-variable and a two-variable function, resp. Our method can always help when an estimate is needed for $P(f_2(\xi, \eta) \geqq x)$ with $P(f_1(\xi) \geqq cx)$. Examples are $f_1(\xi) = |\xi|$, $f_2 = \xi\eta$ or $f_1(\xi)$ is the vector of the coordinates of ξ.

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