Europ. J. Combinatorics (1981) **2**, 145–154

Sums of Vectors and Turán's Problem for 3-graphs

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Let X be a Hilbert space, $0 \leq m \leq n$ integers. If $a_1, \ldots, a_n \in X, \|a_i\| \geq 1 (1 \leq i \leq n)$ then $N_3(a_1, \ldots, a_n)$ denotes the number of sums $\|a_{i_1} + a_{i_2} + a_{i_3}\| \geq 1$. The asymptotic behaviour of $N_3(X, n) = \min N_3(a_1, \ldots, a_n)$ is studied.

1. INTRODUCTION

Let X be a Hilbert space, $0 \leq m \leq n$ integers. If $a_1, \ldots, a_n \in X$, then $N_g(a_1, \ldots, a_n)$ denotes the number of sets $A = \{i_1, \ldots, i_g\}$, $|A| = g$ such that

$$\left\| \sum_{j=1}^{g} a_{i_j} \right\| \geq 1. \tag{1}$$

The main aim of the paper is to give lower estimates on the minimum

$$N_3(X, n, m) = \min N_3(a_1, \ldots, a_n), \tag{2}$$

where the minimum is taken over all the sequences $a_1, \ldots, a_n$ where $\|a_i\| \geq 1$ $(1 \leq i \leq m)$, $\|a_i\| < 1$ $(m < i \leq n)$, n and m are fixed.

More precisely, we will only consider the asymptotic behaviour of $N_3(X, n, m)$, when $n, m \to \infty$ and m/n tends to a constant p. Most of the work deals with the special case $m = n$.

The corresponding questions for $g = 2$ are solved in [3] (see also [1, 2]).

The problem suggested might be interesting in itself, but its real significance is given by the following obvious connection with probability theory. If ξ_1, ξ_2 and ξ_3 are independent and uniformly distributed on the values $a_1, \ldots, a_n$ $(P(\xi_i = a_j) = 1/n$ $(1 \leq i \leq 3, 1 \leq j \leq n))$, then $6N_3(a_1, \ldots, a_n)/n^3$ is "almost" equal to the probability $P(\|\xi_1 + \xi_2 + \xi_3\| \geq 1)$. Thus, the solution of the minimization problem (2) gives the "best" lower estimate on this probability. These consequences will be briefly described in Section 4. Results of this type for $g = 2$ are obtained in [1, 2, 3].

Let us illustrate the method of the paper by the simplest result. Though it is a special case of Lemma 5 in [2], we present its trivial proof.

LEMMA 1.1. *If $a_1, \ldots, a_4 \in X$ and $\|a_i\| \geq 1$ $(1 \leq i \leq 4)$, then, for some $i \neq j \neq k \neq i$,*

$$\|a_i + a_j + a_k\| \geq 1.$$

PROOF

$$\begin{aligned}(a_1 + a_2 + a_3)^2 + (a_1 + a_2 + a_4)^2 + (a_1 + a_3 + a_4)^2 + (a_2 + a_3 + a_4)^2 \\ = 2(a_1 + a_2 + a_3 + a_4)^2 + (a_1^2 + a_2^2 + a_3^2 + a_4^2) \geq 0 + 4\end{aligned}$$

and hence one of the terms in the first row is ≥ 1. The proof is completed.

Let $T(m, 4, 3)$ denote the minimum number of edges of a 3-graph (no loops, no multiple edges) on m vertices satisfying the condition that

$$\text{any four-set of vertices contain at least one edge.} \tag{3}$$

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Suppose $a_1, \ldots, a_m \in X$ and define a 3-graph $G = (\{1, \ldots, m\}, E)$, where $\{i, j, k\} \in E$ iff $i \neq j \neq k \neq i$ and $\|a_i + a_j + a_k\| \geq 1$. Lemma 1.1 expresses that G satisfies condition (3), that is,

$$N_3(a_1, \ldots, a_n) \geq T(m, 4, 3),$$

consequently

$$N_3(X, n, m) \geq T(m, 4, 3). \tag{4}$$

There are two problems with this inequality. The first one is that it is not sharp, since the right-hand side does not even depend on n. In addition to this, the value of $T(m, 4, 3)$ is not known. There are only conjectures and estimates concerning it.

The famous *conjecture of Turán* says that the following graph has $T(m, 4, 3)$ edges: Let A_1, A_2, A_3 be disjoint sets with equal or almost equal sizes, where $|A_1| + |A_2| + |A_3| = m$. Take all the 3-edges being in one A_i, or containing two vertices from A_i and one from A_{i+1} $(1 \leq i \leq 3, A_4 = A_1)$. This conjecture would imply

$$\lim T(m, 4, 3) \Big/ \binom{m}{3} = \frac{4}{9}.$$

Thus the inequality

$$\lim_{m/n \to p} \frac{N_3(X, n, m)}{\binom{n}{3}} \geq \tfrac{4}{9} p^3 \tag{5}$$

follows from (4) and the Turán conjecture. A (probably the best published) estimation ([4]) is $T(m, 4, 3) \geq \frac{5}{14}$. Hence follows

$$\lim_{m/n \to p} \frac{N_3(X, n, m)}{\binom{n}{3}} \geq \tfrac{5}{14} p^3. \tag{6}$$

In what follows we will improve inequality (6). The tools will be geometrical lemmas (like Lemma 1.1) and extremal problems for 3-graphs (like the problem of $T(m, 4, 3)$). They are treated in Sections 2 and 3.

2. Geometrical Lemmas

The first of these lemmas shows that the presence of small vectors increases the number of large sums.

Lemma 2.1. *Suppose $a_1, a_2, a_3, b_1, b_2 \in X$ $\|a_i\| \geq 1$ $(1 \leq i \leq 3)$. Then either there exist indices $1 \leq i < j \leq 3$ and $1 \leq k \leq 2$ such that*

$$\|a_i + a_j + b_k\| \geq 1 \tag{7}$$

or

$$\|a_i + b_1 + b_2\| \geq 1$$

holds for some i $(1 \leq i \leq 3)$.

Proof. Suppose that, contrary to the assertion $a_1, a_2, a_3, b_1, b_2 \in X$ are such that $\|a_i\| \geq 1$ $(1 \leq i \leq 3)$,

$$\|a_i + a_j + b_k\| < 1 \qquad (1 \leq i < j \leq 3, 1 \leq k \leq 2) \tag{8}$$

and

$$\|a_l + b_1 + b_2\| < 1 \qquad (1 \leq l \leq 3). \tag{9}$$

Let us sum the two inequalities (8) at fixed i and j. The triangle inequality yields

$$\|2a_i+2a_j+b_1+b_2\|<2 \qquad (1\leq i<j\leq 3). \tag{10}$$

From (9) and (10), again by the triangle inequality, we have

$$\|2a_i+2a_j-a_l\|<3 \qquad (1\leq i<j<l\leq 3). \tag{11}$$

Hence, by squaring it,

$$4a_i^2+4a_j^2+a_l^2+8a_ia_j-4a_ia_l-4a_ja_l<9.$$

Let us sum these inequalities for the triples

$$(i, j, l)=(1, 2, 3), (1, 3, 2), (2, 3, 1):$$
$$9(a_1^2+a_2^2+a_3^2)<27.$$

As this inequality contradicts the assumptions $a_1^2, a_2^2, a_3^2\geq 1$, the proof is complete.

The following two lemmas show that there is no system of vectors for which the triples with length ≥ 1 are exactly the ones in the Turán conjecture. If X is one-dimensional, then it is true in a stronger sense.

LEMMA 2.2. *Suppose $a_1, \ldots, a_5$ are real numbers with $|a_i|\geq 1$ $(1\leq i\leq 5)$. Then*

$$|a_i+a_j+a_k|\geq 1 \tag{12}$$

for at least one of the triples $1\leq i<j<k\leq 5$ different from $(1, 2, 3)$, $(1, 2, 4)$ *and* $(3, 4, 5)$.

PROOF. If four of the a_i are positive (negative), then there is obviously a triple satisfying (12). Thus, by symmetry, we may suppose that three of the a_i are positive and two of them are negative. If the set of the three positive ones is different from $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_4\}$ and $\{a_3, a_4, a_5\}$, then we are done. Consequently, it is sufficient to consider three cases.

(i) $a_1, a_2, a_3\geq 1$, $a_4, a_5\leq -1$. Here $|a_2|\geq|a_5|$ implies $|a_1+a_2+a_5|\geq 1$ and $|a_2|\leq|a_5|$ implies $|a_2+a_4+a_5|\geq 1$ in accordance with the statement of the lemma.
(ii) $a_1, a_2, a_4\geq 1$, $a_3, a_5\leq -1$. By symmetry, this case is like the previous one.
(iii) $a_3, a_4, a_5\geq 1$, $a_1, a_2\leq -1$. Here $|a_1|\geq|a_5|$ implies $|a_1+a_2+a_5|\geq 1$ and $|a_1|\leq|a_5|$ implies $|a_1+a_4+a_5|\geq 1$ in accordance with the statement of the lemma. The proof is completed.

LEMMA 2.3. *Suppose $a_1, a_2, b_1, b_2, c_1, c_2$ are vectors in a Hilbert space X and for $i=1$ and 2 we have*

$$\|a_1+a_2+b_i\|\geq 1, \tag{13}$$

$$\|b_1+b_2+c_i\|\geq 1, \tag{14}$$

$$\|c_1+c_2+a_i\|\geq 1. \tag{15}$$

Then at least one of the following six inequalities fails:

$$\|a_1+a_2+c_i\|<1, \quad i=1, 2, \tag{16}$$

$$\|b_1+b_2+a_i\|<1, \quad i=1, 2, \tag{17}$$

$$\|c_1+c_2+b_i\|<1, \quad i=1, 2. \tag{18}$$

PROOF. Suppose that all the inequalities (13)–(18) hold. The squares of (13) and (17) are

$$a_1^2+a_2^2+b_i^2+2a_1a_2+2a_1b_i+2a_2b_i\geq 1,$$
$$a_2^2+b_1^2+b_2^2+2a_ib_1+2a_ib_2+2b_1b_2<1,$$

respectively. Take the sum of the first inequality for $i=1, 2$ and subtract from it the second inequality, again, for $i=1, 2$:

$$a_1^2+a_2^2-b_1^2-b_2^2+4a_1a_2-4b_1b_2>0,$$

that is,

$$a_1^2+a_2^2+4a_1a_2>b_1^2+b_2^2+4b_1b_2. \tag{19}$$

The inequalities

$$b_1^2+b_2^2+4b_1b_2>c_1^2+c_2^2+4c_1c_2 \tag{20}$$

and

$$c_1^2+c_2^2+4c_1c_2>a_1^2+a_2^2+4a_1a_2 \tag{21}$$

can be proved in the same way. However (19), (20) and (21) obviously form a contradiction. The proof is complete.

As we shall see, the condition guaranteed by Lemma 2.2 for 3-graphs is not strong enough for our purposes. The next lemma gives an additional condition.

LEMMA 2.4. *If a_1, a_2, a_3, b_1, b_2 are real numbers $|a_i|, |b_j|\geq 1$ $(1\leq i\leq 3, 1\leq j\leq 2)$ then all the inequalities*

$$|a_1+a_2+a_3|<1, \tag{22}$$
$$|a_i+a_j+b_k|\geq 1 \quad (1\leq i<j\leq 3, 1\leq k\leq 2), \tag{23}$$
$$|a_i+b_1+b_2|<1 \quad (1\leq i\leq 3) \tag{24}$$

cannot hold simultaneously.

PROOF. If four of the vectors are positive (negative) then one of (22) and (24) is violated. Hence we may suppose, by symmetry, that three of them are positive and two are negative. (22) implies that $a_1, a_2, a_3\geq 1$ is impossible. Similarly, (24) implies that $b_1, b_2\geq 1$ is impossible too. Thus, again by symmetry, $a_1, a_2, b_1\geq 1$, $a_3, b_2\leq -1$ can be supposed. If $|b_1|\geq|b_2|$ then $|b_1+b_2+a_2|\geq 1$, if $|b_1|\leq|b_2|$ then $|b_1+b_2+a_3|\geq 1$ gives a contradiction. The proof is completed.

3. LEMMAS AND CONJECTURES ON 3-GRAPHS

LEMMA 3.1. *Let $G=(V, E)$ be a 3-graph (no loops no multiple edges), where $V=V_1\cup V_2$, $V_1\cap V_2=\emptyset$, $|V_1|=n_1$, $|V_2|=n_2\leq 2n_1-1$ and E contains an edge either of the form $\{x_i, x_j, y_k\}$ $(1\leq i<j\leq 3, 1\leq k\leq 2)$ or of the form $\{x_i, y_1, y_2\}(1\leq i\leq 3)$ for any choice of $x_1, x_2, x_3\in V_2$ and $y_1, y_2\in V_1$. Then*

$$|E|\geq \tfrac{1}{6}n_1\binom{n_2}{2}. \tag{25}$$

PROOF. $E_i\subset E$ will denote the set of the edges whose intersection with V_1 is i and e_i is the size of E_i. Clearly

$$|E|\geq e_1+e_2. \tag{26}$$

Let us count the number of pairs (A, B), where the sets A and B satisfy $A \supset B$, $|A| = 5$, $B \in E$, $|A \cap V_2| = 3$, $|A \cap V_1| = 2$ and $1 \leqslant |B \cap V_1| \leqslant 2$.

For $B \in E_1$ there are $(n_1 - 1)(n_2 - 2)$ sets A satisfying the conditions. On the other hand, if $B \in E_2$ then this number is $\binom{n_2-1}{2}$. The exact number of the above pairs is

$$e_1(n_1-1)(n_2-2) + e_2\binom{n_2-1}{2}.$$

On the other hand, for any fixed one of the possible $\binom{n_1}{2}\binom{n_2}{3}$ sets A, there is at least one $A \supset B \in E_1 \cup E_2$. Hence

$$\binom{n_1}{2}\binom{n_2}{3} \leqslant e_1(n_1-1)(n_2-2) + e_2\binom{n_2-1}{2}. \tag{27}$$

Since the condition $n_2 \leqslant 2n_1 - 1$ implies

$$\binom{n_2-1}{2} \leqslant (n_1-1)(n_2-2),$$

the inequality

$$\binom{n_1}{2}\binom{n_2}{3} \leqslant (e_1+e_2)(n_1-1)(n_2-2) \tag{28}$$

follows from (27). Finally, (25) is an easy consequence of (28) and (26).

THEOREM 3.1. *If $p \leqslant \frac{1}{3}$ then*

$$\lim_{m/n \to p} \frac{N_3(X, n, m)}{\binom{n}{3}} \geqslant \tfrac{1}{2}p^2(1-p). \tag{29}$$

PROOF. Choose arbitrary vectors $a_1, \ldots, a_n \in X$ with $\|a_i\| \geqslant 1$ $(1 \leqslant i \leqslant m)$. Put $V_2 = \{1, \ldots, m\}$, $V_1 = \{m+1, \ldots, n\}$ and let $G = (V_2 \cup V_1, E)$ be the 3-graph in which E consists of the triples $\{i, j, k\}$ $(1 \leqslant i < j < k \leqslant n)$ satisfying $\|a_i + a_j + a_k\| \geqslant 1$. By Lemma 2.1, the conditions of Lemma 3.1 are satisfied. Hence

$$|E| \geqslant \tfrac{1}{6}(n-m)\binom{m}{2} \tag{30}$$

follows from (25). As E can be chosen to satisfy $|E| = N_3(X, n, m)$, inequality (29) is a consequence of (30). The proof is completed.

Theorem 3.1 improves even the order of magnitude of (5) or (6). In fact, this order of magnitude is already correct. If $a_1, \ldots, a_m = 1$ and $a_{m+1} = \cdots = a_n = -\frac{1}{4}$ then

$$N_3(a_1, \ldots, a_m) = \binom{m}{3} + \binom{m}{2}(n-m)$$

that is,

$$\lim_{m/n \to p} \frac{N_3(X, n, m)}{\binom{n}{3}} \leqslant 3p^2(1-p) + p^3;$$

however, this is not the best upper bound.

Let us remark that in low dimensions the constant in (29) can be improved. E.g. in two dimensions, Lemma 2.1 holds with one b, consequently $\frac{1}{2}$ can be changed to $\frac{3}{4}$.

To determine the exact value of the limit seems to be very hard. Therefore, in the rest of this section we will consider only the number

$$N_3(X, n) = N_3(X, n, n)$$

and the limit

$$\lim_{n\to\infty} \frac{N_3(X, n)}{\binom{n}{3}}.$$

We will try to improve (4), (5) (which is a consequence of a conjecture, only) and (6) (here $p = 1$).

Let us repeat the proof of (4). There we had only one condition on the 3-graph: any 4-set contains at least one edge. However from Lemma 2.2 we have an additional condition:

no five vertices can induce the graph having just the edges $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$ and $\{x_3, x_4, x_5\}$ (see Figure 1). (31)

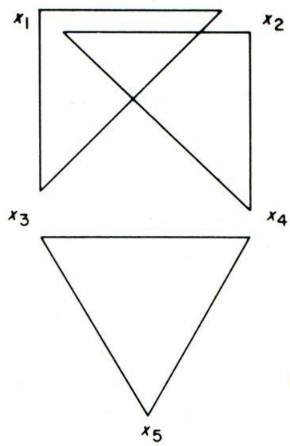

FIGURE 1.

We will now consider the problem of minimizing the number of edges of a 3-graph with n vertices and satisfying the conditions (3) and (31). This minimum will be denoted by $h(n)$. It is obvious from the proof of (4) that if X is the real line, then

$$N_3(X, n) \geq h(n). \tag{32}$$

If, as we hope, Lemma 2.2 can be proved for any Hilbert space X then (32) also holds for any Hilbert space. Before stating our first conjecture, we construct a graph $G(v_1, \ldots, v_r)$ with vertex set $V = V_1 \cup \cdots \cup V_r$, where the sets V_i are disjoint and $|V_i| = v_i$. For $v \in V$ let $f(v)$ denote the index i satisfying $v \in V_i$. A triple (a, b, c) $(a \neq b \neq c \neq a)$ is an edge of our graph $G(v_1, \ldots, v_r)$ iff either $f(a) = f(b) = f(c)$ or there is a unique smallest one among $f(a)$, $f(b)$ and $f(c)$. It is easy to see that $G(v_1, \ldots, v_r)$ satisfies (3) and (31) for any $1 \leq v_1, \ldots, v_r$. On the other hand, if $v_r \leq 2$ then no edge can be omitted without violating the conditions. Denote by $h_1(n)$ the minimal number of edges in a graph $G(v_1, \ldots, v_r)$, $v_i \geq 0$, $\sum_1^r v_i = n$.

CONJECTURE 3.1. *$h_1(n) = h(n)$. Moreover,*

$$h = \lim \frac{h(n)}{\binom{n}{3}} = 4 - 2\sqrt{3} = 0{\cdot}5358984 \ldots. \tag{33}$$

Note that the inequality

$$h(n) \geqslant \left\lceil \frac{h(n-1)n}{n-3} \right\rceil \tag{34}$$

is trivial ($\lceil x \rceil$ denotes the smallest integer $\geqslant x$). For small n the function $h(n)$ can be determined by (34). $h(4) = 1$ is trivial, the optimal graph is $G(3, 1)$. $h(5) \geqslant 3$ follows from (34), but with a little effort $h(5) \geqslant 4$ can be shown. The graphs $G(4, 1)$ and $G(3, 2)$ show $h(5) = 4$. From (34) and $G(4, 2)$, $h(6) = 8$ can be obtained, but it should be remarked that there are two more optimal graphs non-isometric to each other and to $G(4, 2)$. (34) gives only $h(7) \geqslant 14$, but a tedious analysis of the cases implies $h(7) \geqslant 15$. Then the graph $G(5, 2)$ shows that $h(7) = 15$. In fact, for $n = 7$ there are at least seven, non-isomorphic optimal graphs. For $n = 8$, (34) gives $h(8) \geqslant 24$ and $G(5, 2, 1)$ shows that $h(8) \leqslant 25$. (It is worth mentioning, that, as shown in [4], $T(8, 4, 3) = 20$.)

Inequality (34) implies

$$\frac{h(n-1)}{\binom{n-1}{3}} \leqslant \frac{h(n)}{\binom{n}{3}} \leqslant \cdots \leqslant h. \tag{35}$$

Putting $h(7) = 15$ and $h(8) \geqslant 24$ into (35) we find the same lower estimate for h:

$$\tfrac{24}{56} = \tfrac{3}{7} = 0{\cdot}428571 \cdots \leqslant h. \tag{36}$$

Inequality (36) already improves our best result, since $\frac{5}{14} < \frac{3}{7}$. However, we shall obtain some further improvements. It would be good to prove $h(8) = 25$ since this would give $h \geqslant \frac{25}{56} (> \frac{4}{9})$, a better estimate than the one resulting from the Turán conjecture. (Let us remark that $T(8, 4, 3) = 20$ gives $\frac{5}{14}$ in the same way [4].)

However, if $\lceil\ \rceil$ is not omitted from (34), it gives a stronger estimate than (35):

$$\frac{1}{\binom{M}{3}} \left\lceil \cdots \left\lceil \left\lceil h(n) \frac{n+1}{n-2} \right\rceil \frac{n+2}{n-1} \right\rceil \cdots \frac{M}{M-3} \right\rceil \leqslant h, \tag{37}$$

whenever $M \geqslant n$. The theory of diophantine approximation might give a way to determine the limit of the left-hand side of (37). We simply calculated the left-hand side of (37) for large M with a hand-calculator. Starting with $n = 7$ we obtained $0{\cdot}44444 \leqslant h$. It seems to be very likely that the limit for $M \to \infty$ is $\frac{4}{9}$, but we were not able to prove this. However, even in this case we almost have reached the estimate that would follow from the Turán conjecture (by (32)):

THEOREM 3.2. *If X is the real line, then*

$$\lim_{n \to \infty} \frac{N_3(X, n)}{\binom{n}{3}} \geqslant 0{\cdot}44444. \tag{38}$$

Of course, (38) holds for any X satisfying Lemma 2.2. For a general Hilbert space X, we have only Lemma 2.3. Even this fact could improve the earlier results for any X. However, it does not seem to be wise to spend too much time on this, since it is likely that Lemma 2.2 holds for any Hilbert space.

LEMMA 3.2

$$h \leqslant 4 - 2\sqrt{3}.$$

PROOF. Put $a=(\sqrt{3}-1)/2$ and take the graph

$$G([(1-a)n],[(1-a)an],[(1-a)a^2n],\ldots,[(1-a)a^rn]), \tag{39}$$

where r is determined by the inequalities $(1-a)a^rn\leqslant 1$ and $(1-a)^{r-1}n>1$. $[\ ]$ denotes either $\lfloor\ \rfloor$ or $\lceil\ \rceil$, chosen in such a way that the sum of the quantities is equal to n. It is easy to see that this can be done, since the sum without the symbols $[\ ]$ differs from n by less than 1. The number of edges missing from the graph (39) is

$$\binom{[(1-a)n]}{2}\sum_{i=1}^{r}[(1-a)a^in]+\binom{[(1-a)an]}{2}$$

$$\times\sum_{i=2}^{r}[(1-a)a^in]+\cdots+\binom{[(1-a)a^{r-1}n]}{2}\times[(1-a)a^rn]$$

$$=\tfrac{1}{2}((1-a)n)^2\sum_{i=1}^{r}(1-a)a^in+((1-a)an)^2\sum_{i=2}^{r}(1-a)a^in$$

$$+\cdots+((1-a)a^{r-1}n)^2(1-a)a^rn+o(n^3)$$

$$=\tfrac{1}{2}\sum_{j=0}^{\infty}((1-a)a^jn)^2\sum_{i=j+1}^{\infty}(1-a)a^in)+o(n^3)$$

$$=\frac{n^3}{2}\,\frac{(1-a)^2a}{1-a^3}+o(n^3)=\frac{n^3}{2}\,\frac{(1-a)a}{1+a+a^2}+o(n^3).$$

This shows that the ratio of the missing edges to the total number $\binom{n}{3}$ of edges tends to

$$\frac{3(1-a)a}{1+a+a^2}=4-2\sqrt{3}.$$

The proof is complete.

LEMMA 3.3. *The first part of Conjecture* 3.1 *implies* (33).

PROOF. Let us first prove

$$h(n)=\min_{1\leqslant u\leqslant n}\left(h(n-u)+\binom{u}{3}+\binom{n-u}{2}u\right). \tag{40}$$

Choose an optimal graph $G(u,\ldots)$ with n vertices. Here, the first class V_1 has u elements. $G(u,\ldots)$ contains all the triples completely in V_1. Their number is $\binom{u}{3}$. The number of triples having exactly one vertex from V_1 is $\binom{n-u}{2}u$. As these numbers do not depend on the structure of the graph induced by $V-V_1$, the number of edges in $G(u,\ldots)$ is minimal if this induced subgraph has the minimal number of edges. By the first part of the conjecture, this can be chosen to be $h(n-u)$. This proves (40).

Let $u(n)$ be one of the optimal values u in (40). If $n-u(n)$ does not tend to infinity then $n-u(n)<K$ for infinitely many n. Consequently,

$$\binom{u(n)}{3}\Big/\binom{n}{3}\to 1$$

follows for these n. In turn this implies

$$\lim\frac{h(n)}{\binom{n}{3}}\geqslant 1$$

and this is impossible by Lemma 3.2.

By the definition of h we have

$$h(n)=\binom{n}{3}(h-\varepsilon(n)) \tag{41}$$

where $\varepsilon(n)\to 0$. Furthermore, $\varepsilon(n-u)\to 0$ by the remark above. Let us examine the quantity minimized in (40):

$$\begin{aligned}\binom{n-u}{3}&(h-\varepsilon(n-u))+\binom{u}{3}+\binom{n-u}{2}u\\ &=\tfrac{1}{6}u^3(4-h)+\tfrac{1}{6}u^2n(3h-2)+\tfrac{1}{2}un^2(1-h)+\tfrac{1}{6}n^3\\ &\quad+\tfrac{1}{6}[-u^3\varepsilon(n-u)-u^2 3n\varepsilon(n-u)+u3n^2\varepsilon(n-u)\\ &\quad-3u^2(h-\varepsilon(n-u))-3un+2u+6un(h-\varepsilon(n-u))\\ &\quad-2u(h-\varepsilon(n-u))-3n^2+2n].\end{aligned}$$

The quantity in the square brackets is $o(n^3)$. On the other hand, the minimum of the rest of the right-hand side is

$$\frac{n^3}{6}\,\frac{h^2-2h\sqrt{h}-2h+8}{(4-h)^2}, \tag{42}$$

Hence the right-hand side of (40) differs from (42) by $o(n^3)$. By (41) the left-hand side of (40) is $(n^3/6)h+o(n^3)$. Hence

$$h=\frac{h^2-2h\sqrt{h}-2h+8}{(4-h)^2}. \tag{43}$$

On the right-hand side $(2-\sqrt{h})^2$ can be cancelled, since $h<1$. That is, (43) is equivalent to

$$h^2+4h\sqrt{h}+3h-2\sqrt{h}-2=0.$$

The roots of this equation (for $\sqrt{h}$) are -1, -1, $\sqrt{4-2\sqrt{3}}$ and $-\sqrt{4-2\sqrt{3}}$. The condition $0\leqslant h$ implies $h=\sqrt{4-2\sqrt{3}}$. The proof is complete.

However, we are still not able to construct a system of vectors following the construction of the optimal graphs. For instance, $G(2, 2, 1)$ can not be copied in one dimension. This is exactly the statement of Lemma 2.4.

CONJECTURE 3.2. *If a 3-graph on n vertices satisfies conditions* (3) *and* (31) *and does not contain an induced subgraph isomorphic to* $G(2, 2, 1)$ *then the graph cannot have fewer edges than the graph*

$$G\left(\left\lceil\frac{2n}{3}\right\rceil, \left\lfloor\frac{n}{3}\right\rfloor\right).$$

This conjecture has not been studied extensively, so we should call it a "Hope" rather than a conjecture. However, if it is not true then the aim is to find further additional conditions until they ensure that the number of edges cannot be smaller than in $G(\lceil 2n/3\rceil, \lfloor n/3\rfloor)$. Why is this graph so magic? Because it can be represented by vectors: $a_i=1$ $(1\leqslant i\leqslant \lceil 2n/3\rceil)$ and $a_i=-2$ $(\lceil 2n/3\rceil<i\leqslant n)$. $|a_i+a_j+a_k|\geqslant 1$ iff $\{i, j, k\}$ is an edge in $G(\lceil 2n/3\rceil, \lfloor n/3\rfloor)$. This proves one half of the next conjecture.

CONJECTURE 3.3. *For any Hilbert space* X

$$\lim_{n\to\infty}\frac{N_3(X, n)}{\binom{n}{3}}=\frac{5}{9}.$$

The inequality $\leqslant$ follows by the above construction for any Hilbert space. The inequality $\geqslant$ follows from Conjecture 3.2 for any space for which Lemmas 2.2 and 2.4 can be proved.

4. Conclusions for Probability Theory

We give here the results only. The proofs are trivial using the method of papers [2, 3]. First a consequence of Theorem 3.1:

THEOREM 4.1. *If ξ_1, ξ_2, ξ_3 are independent, identically distributed random variables in a Hilbert space and $P(\|\xi_1\| \geqslant x) \leqslant \frac{1}{3}$ then*

$$P(\|\xi_1+\xi_2+\xi_3\| \geqslant x) \geqslant \tfrac{1}{2} P^2(\|\xi_1\| \geqslant x)(1-P(\|\xi_1\| \geqslant x)).$$

A consequence of Theorem 3.2:

THEOREM 4.2. *If ξ_1, ξ_2, ξ_3 are independent, identically distributed real random variables then*

$$P(|\xi_1+\xi_2+\xi_3| \geqslant x, |\xi_1|, |\xi_2|, |\xi_3| \geqslant x) \geqslant 0{\cdot}44444 \,.\, P^3(|\xi_1| \geqslant x).$$

The above theorem is true for any space X satisfying Lemma 2.2. From the Turán conjecture we obtain $\frac{4}{9}$ (in place of 0·44444) for any X. A proof of Conjecture 3.3 would imply $\frac{5}{9}$. (As we remarked earlier, this constant cannot be improved.)

A. Sidorenko [6] has found similar results by other combinatorial methods.

5. Open Problems

As the paper contains more open problems than solutions, it seems to be useful to make a list of the open problems posed in the paper.

(i) Prove or disprove Conjecture 3.1–3.3.
(ii) For what spaces are Lemmas 2.2 and 2.4 true? It is obvious that it is sufficient to prove the lemmas up to five dimensions. (For two dimensions they are probably easy but the proof may be time consuming.)
(iii) Determine the limit of (37) as $M \to \infty$. When $n = 7$ ($h(7) = 15$) is this limit $\frac{4}{9}$?

Note added in proof. G. Bereznai and Á. Varecza recently proved that (37) tends to $\frac{4}{9}$ in a paper submitted to *Elem. Math.*

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Received 24 November 1980

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