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CONTINUOUS VERSIONS OF SOME EXTREMAL HYPERGRAPH PROBLEMS. II

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Introduction

In the previous paper [1] the simplest kind of extremal hypergraph problems was considered: given a condition on the hypergraph, minimize the number of edges (the number of vertices is fixed). In the present paper we treat another class of problems. A good representative of this class is the following theorem of KRUSKAL [2]. (By some kind authors it is called Kruskal—Katona theorem, based on [3]; for simple proofs see [10], [11] and [12].): Given the number of vertices and edges (g-tuples with a fixed g) of the hypergraph, the theorem determines the minimum number of $(g-1)$-tuples contained by at least one edge. To have a "continuous" version of this we take first all the "oriented" copies of the edges and of the $(g-1)$-tuples. In this way and edge of the hypergraph becomes simply $g!$ elements of the direct product X^g of the vertex set X. We have to minimize the size of the set of projections of these elements on X^{g-1}. It is easy to find a continuous analogue of this problem; Take a measure space $M=(X, \sigma, \mu)$ and choose a measurable and symmetric set $E \subset M^g$ with a prescribed measure $\mu_g(E)$, so that the (outer) measure of the projection of E on M^{g-1} is minimal.

It is not hard to prove the continuous version of this problem by approximating E with finitely many cubes (if it has a good shape), as it was made by DAYKIN [4] who proved the continuous version of this problem, independently. However, if we take a more complicated mapping in place of projection or we take some assumptions on E, then we need a more complex proof. BOLLOBÁS [6] suggested a way, simpler than ours, which works for a wider class of problems. But their methods fail to work in the generality of the present paper.

We try to make the continuous version of the following extremal combinatorial problems. The number of vertices and edges of a hypergraph is fixed. The hypergraph satisfies certain conditions. Given a transformation which makes a new hypergraph on the same vertex-set, but the sizes of the edges may be new. The number of edges of this transformed hypergraph is to be minimized.

The condition is of the following form: all finite spanned (induced) subhypergraphs belong to a prescribed family. The transformation maps any family of g-tuples into a family of h-tuples ($h \leqq g$) in a hereditary manner.

Although the paper is a continuation of [1], we shall repeat the necessary definitions to make the paper selfcontained.

The methods of the proof are very similar to that of [1].

Definitions, results

Let X be a finite or infinite set. $G=(X, E)$ is called a *directed g-graph (hypergraph)*, where $E \subset X^g$, that is, E consists of some ordered sequences of form $e=(x_{i_1}, \dots, x_{i_g})$. The elements of X and E are called *vertices* and *edges*, respectively. Multiple edges are excluded. The edges having less than g different vertices are called *loops*. If $Y \subset X$ then the *spanned subgraph* $G_Y=(X, E)_Y=(Y, E_Y)$ consists of all those edges of G which satisfy $x_{i_j} \in Y$ for all j. If $X' \subset X$, $E' \subset E$, then (X', E') is a *subgraph* of (X, E).

Let $\mathcal{G}$ be a set of finite directed g-graphs. We say that $\mathcal{G}$ is *hereditary* if for any spanned subgraph G_1 of $G \in \mathcal{G}$, $G_1 \in \mathcal{G}$ holds. If $\mathcal{G}$ is not *hereditary* the *hereditary kernel* $\hat{\mathcal{G}}$ of $\mathcal{G}$ can be produced in the following way: $G \in \hat{\mathcal{G}}$ if and only if all the spanned subgraphs (including G) are in $\mathcal{G}$. It is easy to see that $\hat{\mathcal{G}}$ is always hereditary.

Let $M=(X, \sigma, \mu)$ be a measure space with a finite measure. (In this paper we shall consider only finite measures.) Furthermore, let $E \subset X^g$ be a measurable set in the product space $(X, \sigma, \mu)^g=(X^g, \sigma_g, \mu_g)$. We define the measure of a graph $G=(X, E)$ in the following way: $\mu(G)=\mu_g(E)$.

Let φ_X be a function which maps the finite directed g-graphs to directed h-graphs ($h \leqq g$, fixed integer) with the same vertex-set. In other words φ_X maps the subsets of X^g ($|X|<\infty$) for subsets of X^h in such a way that

(1) $\varphi_X(E)$ is invariant under the permutations of X

and

(2) $$\varphi_X(E) = \varphi_{X_1}(E) \quad \text{when} \quad X \subset X_1 \quad \text{and} \quad E \subset X^g.$$

(1) and (2) imply that φ_X does not depend on X, just on the "configuration" E. It means that if φ is determined on a set X, then it is determined in any set of a smaller cardinality. Thus, we write simply φ rather than φ_X. We call φ *hereditary* if it satisfies (1), (2) and the condition

(3) $$\varphi(E) \subset \varphi(E_1) \quad \text{if} \quad E \subset E_1.$$

For infinite X's φ is defined in the following way:

(4) $$\varphi(E) = \varphi_X(E) = \bigcup_{\substack{Y \subset X \\ Y \text{ finite}}} \varphi(E_Y).$$

Let us introduce the next notation:

(5) $$\beta(\alpha, \mathcal{G}, \varphi, M) = \inf \frac{\bar{\mu}_h(\varphi(E))}{\mu(X)^h},$$

where $\bar{\mu}_h$ is the outer measure generated by μ_h and inf is taken subject to the following conditions:

(6) E is measurable in M^g,

(7) $$\frac{\mu_g(E)}{\mu(X)^g} \geqq \alpha,$$

(8) all the finite spanned subgraphs of $G=(X, E)$ are isomorphic to some graph in $\mathcal{G}$.

It is easy to see that in condition (8) and in definition (5) $\hat{\mathcal{G}}$ can be used equivalently in place of $\mathcal{G}$. Thus, in the future we can always assume that $\mathcal{G}$ is hereditary without any loss of generality.

If there is no graph with $|X|$ vertices satisfying (6)—(8), then $\beta(\alpha, \mathcal{G}, \varphi, M)$ is undefined. We assume throughout this paper, that this is not the case; when $\beta(\alpha, \mathcal{G}, \varphi, M)$ is used, it is always understood that there exists such a graph with vertex-set X.

If X is finite, σ will always be the family of all subsets of X. M_n denotes the measure space $(\{x_1, \ldots, x_n\}, \sigma, \mu)$, where $\mu(x_i)=1/n$ $(1 \leqq i \leqq n)$. M is called *atomless* if for any $A \in \sigma$, $\mu(A)>0$ there is a set $B \subset A$, $B \in \sigma$ such that $0<\mu(B)<\mu(A)$.

THEOREM. *If $\mathcal{G}$ is a class of directed g-graphs and φ is a hereditary function then the limit*

$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \beta_1(\alpha, \mathcal{G}, \varphi)$$

exists for all but countable many values of α $(0 \leqq \alpha \leqq 1)$; it exists for $\alpha=0$. $\beta_2(\alpha, \mathcal{G}, \varphi)$ is defined to be equal to $\beta_1(\alpha, \mathcal{G}, \varphi)$, where the latter is continuous and $\beta_2(\alpha, \mathcal{G}, \varphi)=\lim_{\varepsilon\to 0} \beta_1(\alpha-\varepsilon, \mathcal{G}, \varphi)$ otherwise $(\alpha>0)$; $\beta_2(0, \mathcal{G}, \varphi)=\lim_{n\to\infty} \beta(0, \mathcal{G}, \varphi, M_n)$. Then for any atomless measure space M

$$\beta_2(\alpha, \mathcal{G}, \varphi) \leqq \beta(\alpha, \mathcal{G}, \varphi, M)$$

holds.

Examples, remarks

EXAMPLE 1. We call a g-graph *symmetric* if it contains all the permutations of its edges $(x_{i_1}, \ldots, x_{i_g})$. Let $\mathcal{G}$ be the class of all symmetric g-graphs without loops. Choose $h=g-1$ and define φ by $\varphi(E)=\{(x_1, \ldots, x_{g-1}):(x_1, \ldots, x_g) \in E\}$. Then for a graph $G=(X, E)$ $\varphi(E)$ denotes the set of ("oriented") non-loop $(g-1)$-tuples being subsets of some edge in E. It is known (see [2], [3], simple proofs: [10], [11] and [12]) that $\binom{N}{g}$ non-oriented g-edges contain at least $\binom{N}{g-1}$ non-oriented $(g-1)$-tuples (N is an integer). This means, that

$$\beta\left(\frac{\binom{N}{g} g!}{n^g}, \mathcal{G}, \varphi, M_n\right) = \frac{(g-1)!\binom{N}{g-1}}{n^{g-1}}$$

holds. As $\beta(\alpha, \mathcal{G}, \varphi, M_n)$ is a monotonic function of α, the inequality

$$\frac{(g-1)!\binom{N_1}{g-1}}{n^{g-1}} \geqq \beta(\alpha, \mathcal{G}, \varphi, M_n) \geqq \frac{(g-1)!\binom{N_2}{g-1}}{n^{g-1}} \tag{9}$$

follows from

$$\frac{\binom{N_1}{g} g!}{n^g} \geqq \alpha \geqq \frac{\binom{N_2}{g} g!}{n^g}.$$

It is easy to see that the latter inequalities can be satisfied with $N_1 = n\alpha^{1/g} + o_1(n)$ and $N_2 = n\alpha^{1/g} + o_2(n)$. Consequently,

$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \alpha^{(g-1)/g}$$

follows from (9). Using the theorem, we obtain that

$$\beta(\alpha, \mathcal{G}, \varphi, M) \geqq \alpha^{(g-1)/g}$$

for any atomless M. The equality can be shown by the "cube", that is the direct product (g times with itself) of a set with measure $\alpha\mu(X)$.

This result was independently deduced from the discrete case by DAYKIN [4]. On the other hand, it can be proved directly using the Hölder-inequality, as it was observed by LOOMIS and WHITNEY [5], and A. MEIR [17].

EXAMPLE 2. Let $\mathcal{G}$ consist of the symmetric graphs without loops, containing no empty $(g+1)$-tuple ($g+1$ vertices containing no g-edge). Choose an h $(1 \leqq h \leqq g)$ and φ as in the above example: $\varphi(E) = \{(x_1, \dots, x_h) : (x_1, \dots, x_g) \in E\}$. Even the discrete problem $(\beta(\alpha, \mathcal{G}, \varphi, M_n))$ is unsolved if $g \geqq 3$ and $h \geqq 2$. The case $g=h=2$ is the well known TURÁN theorem [8]. The case $g=2$, $h=1$ is a consequence of it.

EXAMPLE 3. The discrete question is the following open problem due to P. FRANKL [15]. Given the number of vertices and g-edges of a symmetric g-graph. Determine the minimal number of $(g-1)$-tuples contained by any union of two edges. It is easy to see that this also fits to our conditions. $h=g-1$, and

$$\varphi(E) = \big\{(x_1, \dots, x_{g-1}) \colon x_i \neq x_j \ (i \neq j);\ \exists (y_1, \dots, y_g), (z_1, \dots, z_g) \in E, \\ \{x_1, \dots, x_{g-1}\} \subset \{y_1, \dots, y_g, z_1, \dots, z_g\}\big\}.$$

EXAMPLE 4. Put $g=3$, $h=2$. $\mathcal{G}$ consists of the graphs without loops, containing (x_3, x_2, x_1) and (x_1, x_2, x_3) simultaneously, φ as in Example 1. It is known (see [9]) that

$$\beta\left(\frac{(n-1)(n-2)(n-m)+m(n-m)(n-m-1)}{n^3}, \mathcal{G}, \varphi, M_n\right) = \frac{n(n-1)-m(m-1)}{n^2}$$

and

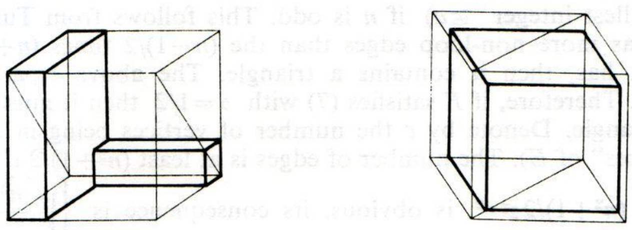
$$\beta\left(\frac{m(m-1)(m-2)}{n^3}, \mathcal{G}, \varphi, M_n\right) = \frac{m(m-1)}{n^2}$$

if $m(m-1) > \dfrac{n(n-1)}{2} + n$. It is easy to deduce

$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \begin{cases} f^{-1}(\alpha) & \text{if } \alpha \leqq \frac{1}{\sqrt{8}} \leqq 1 \\ \alpha^{2/3} & \text{if } \alpha \geqq \frac{1}{\sqrt{8}} \geqq 0, \end{cases} \tag{10}$$

where $f^{-1}(\alpha)$ is the inverse function of $f(x) = (1-x)^{3/2} + 2x - 1$.

What is now the continuous variant of this problem for the case when X is (e.g.) the [0, 1] interval and μ is the Lebesgue measure? Given a measurable set E with volume α in the unit cube. E is symmetric on the plane connecting two opposite edges of the cube, minimize the area of the projection on the side-plane which is not cut by the above plane. It follows from our theorem, that the right hand side of (10) is a lower estimation on $\beta(\alpha, \mathcal{G}, \varphi, M)$. The contructions of Figure 1 show that this estimation is the best possible. For this example see also [13].



$\alpha \leqq 1/\sqrt{8}$ $\alpha \geqq 1/\sqrt{8}$

Fig. 1

EXAMPLE 5. The following problem of G. HALÁSZ [14] is not solved even in the discrete case: In an undirected graph, the number of circuits of length g is given. What is the minimal number of edges? It does fit to our model:

Let $\mathcal{G}$ consist of graphs (with non-loop edges) G in which $(x_1, \ldots, x_g) \in G$ is followed by $(x_i, x_{i+1}, \ldots, x_g, x_1, \ldots, x_{i-1}) \in G$ and $(x_i, x_{i-1}, \ldots, x_1, x_g, \ldots, x_{i+1}) \in G$ for all $1 \leqq i \leqq g$. $h=2$ and φ is as in Example 2.

REMARK 1. Sometimes it is easier to prove the continuous version than the discrete one. The aim of our theorem is not necessarily to show a way of proof for the continuous cases. Its aim is only to show the connection. However, it can happen that there is only an inductional proof and in this case our theorem gives a good way to the continuous through the discrete. On the other hand, the continuous version can be better visualized and this geometric picture can give a hint for the proof of both cases.

REMARK 2. It is very concievable that we have equality in the theorem. However, we were not able to prove it.

REMARK 3. We did not work out here the case when M has "atoms". However [1] shows how it could be made.

The last example shows that the limit $\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n)$ does not exist in general, and $\overline{\lim}_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n)$ is not necessarily a continuous function of α.

EXAMPLE 6. Let $\mathcal{G}$ consist of the symmetric 2-graphs. Let further E be a set of (2-) edges, then

$$\varphi(E) = \begin{cases} \text{the set of vertices contained by the edges in } E \\ \text{if it contains a triangle (3 vertices with all the} \\ \text{6 non-loop edges) or a loop} \\ \emptyset \text{ otherwise.} \end{cases}$$

It is easy to see that φ satisfies (1), (2) and (3) that is, φ is hereditary. Choose $\alpha=1/2$. If n is even, then $\beta(1/2, \mathcal{G}, \varphi, M_n)=0$, since the complete bipartitie graph with $n/2$, $n/2$ vertices has exactly $n^2/2$ edges, contains no loop or triangle, consequently its $\varphi=\emptyset$. On the other hand,

$$\beta(1/2, \mathcal{G}, \varphi, M_n) = \left\{\sqrt{\frac{n^2+1}{2}}\right\}\Big/ n$$

($\{a\}$ is the smallest integer $\geqq a$) if n is odd. This follows from Turán's theorem: If the graph has more non-loop edges than the $(n-1)/2$ times $(n+1)/2$ complete bipartite graph has, then it contains a triangle. The above bipartite graph has $(n^2-1)/2$ edges. Therefore, if E satisfies (7) with $\alpha=1/2$ then it must contain either a loop or a triangle. Denote by v the number of vertices being in the edges of E (the "real vertices" of E). The number of edges is at least $(n^2+1)/2$ if n is odd. Thus the inequality $(n^2+1)/2\leqq v^2$ is obvious, its consequence is $\left\{\sqrt{\frac{n^2+1}{2}}\right\}\leqq v$. On the other hand, $\left\{\sqrt{\frac{n^2+1}{2}}\right\}$ can be easily constructed by a complete graph.

We have obtained

$$\varliminf_{n\to\infty} \beta(1/2, \mathcal{G}, \varphi, M_n) = 0$$

and

$$\varlimsup_{n\to\infty} \beta(1/2, \mathcal{G}, \varphi, M_n) = 1/\sqrt{2},$$

i.e. the limit does not exist.

It is easy to see, using similar ideas, that

$$\lim_{n\to\infty} \beta(\alpha, \mathcal{G}, \varphi, M_n) = \begin{cases} 0 & \text{if} \quad \alpha < 1/2 \\ \sqrt{\alpha} & \text{if} \quad \alpha > 1/2 \end{cases}$$

(see Fig. 2). The limit "function" is not continuous.

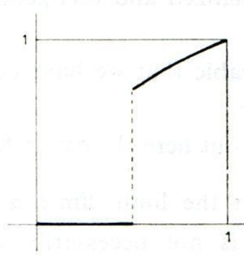


Fig. 2

On the other hand, it is easy to construct an example, when $\beta(1/2, \mathcal{G}, \varphi, M)=0$, that is, the $\varliminf$ is the exact estimation. Let M be the [0, 1] interval with the Lebesgue measure and let E be the set of pairs (x, y) where $(1/2<x$ and $y<1/2)$ or $(x<1/2$ and $1/2<y)$. Then E satisfies conditions (6)—(8), but it does not contain a complete triangle, consequently $\varphi(E)=\emptyset$ and $\beta(1/2, \mathcal{G}, \varphi, M)=0$.

REMARK 4. Condition (8) could be substituted by $\varphi(E)=X^h$ when (8) is not satisfied. In this case we would not use $\mathscr{G}$ just φ. However, in this case it is more complicated to formulate the conditions assumed on φ. This is why we choose this way of formulation. (See problem 3.)

The proof

For the proof of the theorem we need a lemma which is a special case of the law of the large numbers. We did not find it in the same form, but there are many close versions (see e.g. [16]).

LEMMA 1. *Let* $\xi_1, \xi_2, \dots$ *be identically distributed random variables with existing expectation* M_1 *and variance* D_1. *Denote by* $f(n)$ *the number of pairs* ξ_i, ξ_j $(1\leqq i, j\leqq n)$ *such that* ξ_i *and* ξ_j *are not independent. If*

$$\frac{f(n)}{n^2}\to 0 \tag{11}$$

then for any $\varepsilon>0$ *and* $\delta>0$

$$\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-M_1\right|<\varepsilon \tag{12}$$

with probability $1-\delta$ *when n is large enough.*

PROOF. Let us consider the variance of the random variable $\zeta_n=\frac{1}{n}\sum_{i=1}^{n}\xi_i$. The equalities

$$D^2(\zeta_n)=M((\zeta_n-M_1)^2)=M\left(\left(\frac{1}{n}\sum_{i=1}^{n}(\xi_i-M_1)\right)^2\right)= \tag{13}$$

$$=\frac{1}{n^2}M\left(\left(\sum_{i=1}^{n}(\xi_i-M_1)\right)^2\right)=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}M((\xi_i-M_1)(\xi_j-M_1))$$

are obvious.

Observe that $M((\xi_i-M_1)(\xi_j-M_1))=0$ if ξ_i and ξ_j are independent, and

$$|M((\xi_i-M_1)(\xi_j-M_1))|\leqq\sqrt{M((\xi_i-M_1)^2)M((\xi_j-M_1)^2)}=D_1^2$$

otherwise. We have

$$D^2(\zeta_n)\leqq\frac{1}{n^2}f(n)D_1^2$$

by (13). Hence

$$\lim_{n\to\infty}D^2(\zeta_n)=0 \tag{14}$$

follows from (11). Apply the well-known Čebyšev-inequality:

$$P(|\zeta_n-M_1|>\varepsilon)<\frac{D^2(\zeta_n)}{\varepsilon^2}. \tag{15}$$

If n is large enough, $D^2(\zeta_n)/\varepsilon^2<\delta$ holds by (14) and (15) gives the statement of the lemma.

Now we give another form of this lemma, closer to our needs.

LEMMA 2. *Let $M=(X, \sigma, \mu)$ be a finite measure space, E be a measurable subset of X^g. Then*

$$\left| \frac{|E_{\{y_1, \dots, y_n\}}|}{n^g} - \frac{\mu_g(E)}{\mu(X)^g} \right| < \varepsilon$$

with a measure $\mu(X)^n(1-\delta)$.

PROOF. We may suppose that $\mu(X)=1$. Let $y_1, \dots, y_n$ be independently chosen elements of X, and define the random variables $\xi(i_1, \dots, i_g; y_1, \dots, y_n)$ by

$$\text{(16)} \qquad \xi(i_1, \dots, i_g; y_1, \dots, y_n) = \begin{cases} 1 & \text{if} \quad (y_{i_1}, \dots, y_{i_g}) \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that their expectations and variances do not depend on $i_1, \dots, i_g$, that is, they are identical, $M_1=\mu_g(E)$. Two variables of form (16) can be dependent only when there is an equal number among $i_1, \dots, i_g$ and $i'_1, \dots, i'_g$. The number of such pairs is equal to

$$\text{(17)} \qquad \sum_{i_1, \dots, i_g} \text{(the number of sequences } i'_1, \dots, i'_g \text{ non-disjoint to } i_1, \dots, i_g).$$

One term here is equal to n^g-(the number of disjoint sequences). The latter term is at least $(n-g)^g$ and this gives an upper estimation for (17):

$$\sum_{i_1, \dots, i_g} (n^g-(n-g)^g) = n^{2g}-n^g(n-g)^g.$$

Since the total number of pairs is n^{2g} and $(n^{2g}-n^g(n-g)^g)/n^{2g} \to 0$ when $n \to \infty$, (11) is satisfied. We may apply Lemma 1. $\frac{1}{n}\sum_{i=1}^{n} \xi_i$ in (12) becomes $|E_{\{y_1, \dots, y_n\}}|/n^g$ in our case. The lemma is proved.

PROOF OF THE THEOREM 1. Suppose $M=(X, \sigma, \mu)$ is an atomless measure space and $G=(X, E)$ is a graph satisfying (6), (7) and (8). Let us introduce the following functions:

$$\text{(18)} \qquad I(i_1, \dots, i_h; y_1, \dots, y_n) = \begin{cases} 1 & \text{if} \quad (yi_1, \dots, yi_h \in \varphi(E_{\{y_1, \dots, y_n\}}) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\text{(19)} \qquad f(y_1, \dots, y_n) = \sum_{\substack{1 \le i_j \le n \\ (1 \le j \le h)}} I(i_1, \dots, i_h; y_1, \dots, y_n).$$

In other words $f(y_1, \dots, y_n)$ is the number of the ordered sequences $(y_{i_1}, \dots, y_{i_h})$ being in $\varphi(E_{\{y_1, \dots y_n\}})$. Note that the functions (18) and (19) are not necessarily measurable. For this reason we introduce $\bar{I}(i_1, \dots, i_h; y_1, \dots, y_n)$ as the indicating function of a measurable set containing the support (denoted by supp) of $I(i_1, \dots, i_h; y_1, \dots, y_n)$ while

$$\text{(20)} \qquad \mu(\operatorname{supp} \bar{I}(i_1, \dots, i_h; y_1, \dots, y_n)) = \bar{\mu}(\operatorname{supp} I(i_1, \dots, i_h; y_1, \dots, y_n)).$$

Similarly to (20) we have

$$\bar f(y_1, \dots, y_n) = \sum_{\substack{1 \le i_j \le n \\ (1 \le j \le h)}} \bar I(i_1, \dots, i_h;\ y_1, \dots, y_n). \tag{21}$$

After these, the notation $\bar\varphi(E)$ is obvious.

$I(1, 2, \dots, h;\ y_1, \dots, y_n)$ is necessarily zero if $(y_1, \dots, y_h) \notin \varphi(E)$ by (3) and (4). In other words, $I(1, 2, \dots, h;\ y_1, \dots, y_n)$ as a function of $y_1, \dots, y_n$ can be one only if $(y_1, \dots, y_h) \in \varphi(E)$. Consequently, $\operatorname{supp} I(1, 2, \dots, h;\ y_1, \dots, y_n) \subset \varphi(E) \times X^{n-h}$ and $\operatorname{supp}\big(I(1, 2, \dots, h;\ y_1, \dots, y_n)\big) \subset \bar\varphi(E) \times X^{n-h}$ follow, where $\bar\varphi(E) \times X^{n-h}$ is measurable. Thus $(\bar\varphi(E) \times X^{n-h}) \cap \operatorname{supp}\big(\bar I(1, 2, \dots, h;\ y_1, \dots, y_n)\big)$ is measurable and it contains $\operatorname{supp}\big(I(1, 2, \dots, h;\ y_1, \dots, y_n)\big)$. This means, that we could choose this set in place of $\bar I(1, \dots, h;\ y_1, \dots, y_n)$. Suppose, that $\bar I(1, \dots, h;\ y_1, \dots, y_n)$ is chosen in this way. Then we have

$$\operatorname{supp}\big(\bar I(1, \dots, h;\ y_1, \dots, y_n)\big) \subset \bar\varphi(E) \times X^{n-h}. \tag{22}$$

$$\int_{X^n} \bar f(y_1, \dots, y_n)\, d\mu_n = \sum_{\substack{1 \le i_j \le n \\ (1 \le j \le h)}} \int_{X^n} \bar I(i_1, \dots, i_h;\ y_1, \dots, y_n)\, d\mu_n =$$

$$= n^h(1+o(n)) \int_{X^n} \bar I(1, \dots, h;\ y_1, \dots, y_n)\, d\mu_n \le n^h(1+o(n)) \int_{\bar\varphi(E) \times X^{n-h}} 1\, d\mu_n =$$

$$= n^h \bar\mu_h\big(\varphi(E)\big)\mu(X)^{n-h}(1+o(n))$$

follow from (21), (22) and the definition of $\bar\varphi(E)$. We shall use the inequality

$$\int_{X^n} \bar f(y_1, \dots, y_n)\, d\mu_n \le n^h \bar\mu_h\big(\varphi(E)\big)\mu(X)^{n-h}(1+o(n)). \tag{23}$$

2. Assume that $y_1, \dots, y_n$ are different. $f(y_1, \dots, y_n)$ is simply the number of elements in $\varphi(E_{\{y_1, \dots y_n\}})$. Thus

$$\beta\left(\frac{|E_{\{y_1, \dots, y_n\}}|}{n^g}, \mathcal{G}, \varphi, M_n\right) \le \frac{|\varphi(E_{\{y_1, \dots, y_n\}})|}{n^h} = \frac{f(y_1, \dots, y_n)}{n^h} \tag{24}$$

follows from (5), since (6)—(8) are satisfied. As β is a monotonic function of α, we obtain

$$\beta\left(\frac{\mu_g(E)}{\mu(X)^g} - \varepsilon, \mathcal{G}, \varphi, M_n\right) \le \frac{\bar f(y_1, \dots, y_n)}{n^h} \tag{25}$$

from (24), lemma 2 and the definition of $\bar f$. (25) holds with a measure $\mu(X)^n(1-\delta)$ on the basis of Lemma 2 and the fact that in an atomless measure $y_1, \dots, y_n$ are almost surely different (see Lemma 4 of [1]).

Take the integral of (25) over X^n and use (23):

$$\mu(X)^n(1-\delta)\beta\left(\frac{\mu_g(E)}{\mu(X)^g} - \varepsilon, \mathcal{G}, \varphi, M_n\right) \le \frac{1}{n^h}\int_{X^n} \bar f(y_1, \dots, y_n)\, d\mu_n \le \tag{26}$$

$$\le \bar\mu_h\big(\varphi(E)\big)\mu(X)^{n-h}(1+o(n)).$$

Since E is supposed to satisfy (7) we can write

$$\beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n) - \delta \le (1-\delta)\beta(\alpha-\varepsilon, \mathcal{G}, \varphi, M_n) \le \frac{\bar\mu_h(\varphi(E))}{\mu(X)^h}(1+o(n)) \tag{27}$$

instead of (26). By (5), E can be chosen in this way:

$$\frac{\bar{\mu}_h(\varphi(E))}{\mu(X)^h} \leqq \beta(\alpha, \mathscr{G}, \varphi, M)+\delta.$$

From this and (27)

$$\beta(\alpha-\varepsilon, \mathscr{G}, \varphi, M_n)-\delta \leqq (\beta(\alpha, \mathscr{G}, \varphi, M)+\delta)(1+o(n))$$

is a consequence of (5). Hence

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}}\, \beta(\alpha-\varepsilon, \mathscr{G}, \varphi, M_n) \leqq \beta(\alpha, \mathscr{G}, \varphi, M) \tag{28}$$

easily follows when $\alpha > 0$.

3. Apply the proof of Section 2 for M_m in place of M (m is $\gg n$). In Section 2, we used that M is atomfree only in one place, namely, that $y_1, \dots, y_n$ are different with measure $\mu(X)^n$. If we use M_m in place of M, this is no longer true, but the measure of sequences $y_1, \dots, y_n$ with two identical members is small; at most $\binom{n}{2}/m$. Consequently, (25) holds with a measure $\mu(X)^n(1-\delta)-\binom{n}{2}/m$ (where $\mu(X)=1$) in place of $\mu(X)^n(1-\delta)$. If $m>\binom{n}{2}/\delta$ then the new term is less than δ, thus we can simply write $1-2\delta$ in place of $\mu(X)^n(1-\delta)$ into the formulas (26) and (27):

$$\beta(\alpha-\varepsilon, \mathscr{G}, \varphi, M_n)-2\delta \leqq \mu_h(\varphi(E))(1+o(n)).$$

That is,

$$\beta(\alpha-\varepsilon, \mathscr{G}, \varphi, M_n)-2\delta \leqq \beta(\alpha, \mathscr{G}, \varphi, M_m)(1+o(n))$$

follows if n is large enough depending on ε, δ and $m>\binom{n}{2}/\delta$. Hence

$$\overline{\lim_{n\to\infty}}\, \beta(\alpha-\varepsilon, \mathscr{G}, \varphi, M_n) \leqq \underline{\lim}_{m\to\infty}\, \beta(\alpha, \mathscr{G}, \varphi, M_m). \tag{29}$$

Denote the interval $[\underline{\lim}\, \beta(\alpha, \mathscr{G}, \varphi, M_n), \overline{\lim}\, \beta(\alpha, \mathscr{G}, \varphi, M_n)]$ by I_α. It follows by (29) that these intervals are disjoint (and, of course, they lie in [0, 1]), therefore the length of I_α is positive only for a countable set of values α. For the other values of α, I_α is a single point, that is $\overline{\lim}=\underline{\lim}$; the limit $\lim_{n\to\infty} \beta(\alpha, \mathscr{G}, \varphi, M_n)$ exists. The function $\beta_1(\alpha, \mathscr{G}, \varphi)$ is defined in all but countably many places. Note that β_1 is monotonically increasing. An increasing function is continuous with countably many exceptions. Thus $\beta_1(\alpha, \mathscr{G}, \varphi)$ is defined and continuous on a set $[0, 1]-A$ where A is a countable set. The left hand side of (28) equals

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \beta(\alpha-\varepsilon, \mathscr{G}, \varphi, M_n) \quad (\alpha-\varepsilon\in[0, 1]-A) \quad \text{for any} \quad \alpha\in[0, 1],$$

that is, $\lim_{\varepsilon\to 0} \beta_1(\alpha-\varepsilon, \mathscr{G}, \varphi)$ (for $\alpha-\varepsilon\in[0, 1]-A$). This is, by definition, equal to $\beta_2(\alpha, \mathscr{G}, \varphi)$. The inequality of the theorem follows from (28).

4. The case $\alpha=0$ can be settled by an easy modification of sections 2 and 3. The proof is complete.

Open problems

1. Is equality in the theorem?

2. What happens if we allow the existence of "atoms"?

3. For what general class of φ's can a similar theorem be proved? (See Remark 4.)

4. Under what conditions on $\mathcal{G}$ and φ can we state that $\beta_1(\alpha, \mathcal{G}, \varphi)$ is a) continuous, b) continuous from left (right) hand side, c) defined everywhere?

5. The papers [4] and [5], where the product of the volumes of the different projections is considered, suggest a more general concept of our function φ. It would be nice to work out the right concept and prove a more general theorem.

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