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IF THE INTERSECTION OF ANY r SETS HAS A SIZE $\neq r-1$

by

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Working on problems connected with data base systems, suggested by J. DEMETROVICS, we observed the following simple but interesting

THEOREM 1. *Let $A_1, \ldots, A_m$ be a family of not necessarily distinct subsets of an n element set X. Suppose that*

$$\left|\bigcap_{j=1}^{r} A_{i_j}\right| \neq r-1 \tag{1}$$

holds for any r $(1 \leqq r \leqq m)$ and any distinct indices $i_1, \ldots, i_r$ $(1 \leqq i_j \leqq m)$. Then

$$m \leqq n. \tag{2}$$

PROOF. We use induction over n. $d(x)$ is the degree of $x \in X$: $d(x) = |\{j\colon x \in A_j\}|$.

1. We prove first that $d(x) \leqq |A_i|$ follows from $x \in A_i$.

Fix an i and $x \in A_i$. Take the sets $A_i \cap A_j - \{x\}$ for all $j \neq i$ such that $x \in A_j$. If these sets do not satisfy (1), there are indices $j_1, \ldots, j_r$ such that $x \in A_{j_l}$ $(1 \leqq l \leqq r)$ and

$$\left|\bigcap_{l=1}^{r} (A_i \cap A_{j_l} - \{x\})\right| = r-1.$$

Hence $\left|A_i \cap \bigcap_{l=1}^{r} (A_i \cap A_{j_l})\right| = r$ would follow contradicting (1). The sets $A_i \cap A_j - \{x\}$ satisfy (1) on a set of size $|A_i| - 1 \leqq n-1$: so we may use the inductional hypothesis: the number of sets $x \in A_j$, $j \neq i$ is $\leqq |A_i| - 1$. Thus the number of sets $x \in A_j$ is $\leqq |A_i|$.

2. It follows from the induction hypothesis that the union of any r of the sets A_i has a size at least r if $r \leqq n$. By Hall's theorem we obtain elements $x_i \in A_i$ $(1 \leqq i \leqq r)$, where $x_1, \ldots, x_n$ lists all the elements of X. The first section gives

$$d(x_i) \leqq |A_i|.$$

Hence

$$\sum_{i=1}^{n} |A_i| \geqq \sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{m} |A_i|$$

and

$$\sum_{i=n+1}^{m} |A_i| = 0$$

follows. $|A_i| \neq 0$ by (1), consequently the sum must be empty. We have $m \leqq n$, and the proof is complete.

The n different one-element sets give equality in (2).

COROLLARY. *If $A_1, \ldots, A_m$ are non-empty subsets of a set of n elements, no two have an intersection equal to* 1 *and no three have an intersection* >1 *then* $m \leqq n$.

PROOF. It is easy to see that the sets satisfy (1).

THEOREM 2. *Let $A_1, \ldots, A_m$ be a family of not necessarily distinct subsets of an n element set X, and let $t>0$ be a fixed integer. Suppose that*

$$\left|\bigcap_{j=1}^{r} A_{i_j}\right| \neq r-1-t \tag{3}$$

holds for any r $(1 \leqq r \leqq m)$ and any distinct indices $i_1, \ldots, i_r$ $(1 \leqq i_j \leqq m)$. Then

$$m \leqq n+t \tag{4}$$

moreover

$$m \leqq n \tag{5}$$

with the additional conditions $A_i \neq A_j$ $(i \neq j)$ and $2^{t-1} \leqq n$.

PROOF. Take the sets $\left(\bigcap_{i=1}^{t} A_i\right) \cap A_j$ $(t<j \leqq m)$. The intersection of any r different ones cannot be of size $r-1$ by (3). Apply Theorem 1 for these sets: $m-t \leqq n$. The choice $A_i = X$ $(1 \leqq i \leqq n+t)$ gives equality in (4).

The proof of (5) proceeds in a similar way. The only difference is that we have to choose some distinct sets $A_{i_1}, \ldots, A_{i_t}$ with $\left|\bigcap_{j=1}^{t} A_{i_j}\right| \leqq n-t$. It can be proved by induction over t (with fixed n) that this can be done if $m \geqq n+1$: By the inductional hypothesis we can find $t-1$ sets with an intersection Y satisfying $|Y| \leqq n-t+1$. If $|Y| < n-t+1$, we are done, thus we may suppose $|Y| = n-t+1$. If there is one among the sets A_i which does not contain Y, we are done, again. It means, as the sets are distinct, that their number m is at most 2^{t-1}. By the condition $2^{t-1} \leqq n$ this contradicts $m \geqq n+1$. The proof is complete.

It is easy to see that the family of all $(n-1)$-element subsets of X give equality in (5) if $n+t$ is even. But there are no 4 distinct subsets satisfying (3) if $n=4$, $t=1$.

While the condition of the corollary did not give stronger result than Theorem 1 gave, this is not the case here. Choose $t=1$ and take the stronger conditions $\left|\bigcap_{j=1}^{2} A_{i_j}\right| \neq 0$, $\left|\bigcap_{j=1}^{3} A_{i_j}\right| \leqq 0$. Then the $\binom{m}{2}$ intersections $A_i \cap A_j$ are all disjoint. Consequently, $\binom{m}{2} \leqq n$.

THEOREM 3. *Let $A_1, \ldots, A_m$ be a family of distinct subsets of an n element set X, and let $t>0$ be a fixed integer. Suppose that*

$$\left|\bigcap_{j=1}^{r} A_{i_j}\right| \neq r-1+t \tag{6}$$

holds for any r $(2 \leqq r \leqq m)$ and any distinct indices $i_1, \ldots, i_r$ $(1 \leqq i_j \leqq m)$. Then

$$m \leqq \sum_{\nu=1}^{t+1} \binom{n}{\nu}. \tag{7}$$

PROOF. Let us count the number of pairs (A_i, G) $(1 \leqq i \leqq m,\ G \subset A_i,\ |G| = t)$ in two different ways

$$\sum_{i=1}^{m} \binom{|A_i|}{t} = \sum_{|G|=t} |\{A_i\colon\ 1 \leqq i \leqq m,\ G \subset A_i\}|. \tag{8}$$

Here $|\{A_i\colon\ 1 \leqq i \leqq m,\ G \subset A_i,\ G \neq A_i\}| \leqq n - t$ by Theorem 1. Consequently, the right-hand side of (8) is at most $\binom{n}{t}(n-t+1)$:

$$\sum_{i=1}^{m} \binom{|A_i|}{t} \leqq \binom{n}{t}(n-t+1). \tag{9}$$

Suppose, that in contradiction to (7) $m > \sum_{\nu=1}^{t+1} \binom{n}{\nu}$. It is easy to see that $\sum_{i=1}^{m} \binom{|A_i|}{t} > \sum_{\nu=1}^{t+1} \binom{\nu}{t} \binom{n}{\nu}$ (fewer subsets with smaller sizes), and this contradicts (9). The proof is complete.

If we also assume (6) for $r=1$, then we obtain the bound $\sum_{\nu=1}^{t-1} \binom{n}{\nu} + \binom{n}{t+1}$.

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