COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
18. COMBINATORICS, KESZTHELY (HUNGARY), 1976.

CONTINUOUS VERSIONS OF SOME EXTREMAL HYPERGRAPH PROBLEMS

G.O.H. KATONA

INTRODUCTION

In [6] there is an application of the well-known graph-theorem of Turán [1] in the probability theory. As [6] is written in Hungarian, it is wortwhile to repeat here briefly the statement and the idea of the proof. Let us first repeat Turán's theorem in a special case:

If a graph (no loops or multiple edges) with vertices $v_1, \dots, v_n$ *has no empty triangle (for any 3 different vertices there is at least one edge among them), then the number of edges is at least* $\frac{n}{2}\left(\frac{n}{2} - 1\right)$.

We need it in a somewhat modified form, namely if the graph has no empty triangle then the number of pairs (v_i, v_j) such that v_i and v_j are connected or equal is at least $\frac{n^2}{2}$. This follows easily from the theorem, as this number is equal to the double of the number of edges $+ n$.

Now the inequality from probability theory, what we want to prove:

Let ξ and η be independent and identically distributed random variables in a d-dimensional Euclidean space.

Then

$$(0) \qquad \mathsf{P}(|\xi + \eta| \geqslant x) \geqslant \frac{1}{2} \mathsf{P}(|\xi| \geqslant x)^2 .$$

Observe that for any 3 vectors a_1, a_2, a_3 with $|a_i| \geqslant x$ $(i = 1, 2, 3)$ there is a pair $i \neq j$ such that $|a_i + a_j| \geqslant x$. Define a graph G with the vertex-set $\{\omega: |\xi(\omega)| \geqslant x\}$, where ω_1 and ω_2 are connected iff $|\xi(\omega_1) + \xi(\omega_2)| \geqslant x$. This graph, by the above remark, does not contain an empty triangle. $\mathsf{P}(|\xi + \eta| \geqslant x) \geqslant \mathsf{P}(|\xi + \eta| \geqslant x, \ |\xi| \geqslant x, \ |\eta| \geqslant x)$ and this is the measure of the pairs (ω_1, ω_2) such that ω_1 and ω_2 are connected. $\omega_1 = \omega_2$ is trivially counted. For the discrete case we know that this number is at least the half of the total number of pairs. Thus we may expect the same for the measure of the pairs (ω_1, ω_2) in the direct product.

The aim of this paper is to investigate, under what conditions can we transmit the Turán-type discrete theorems to continuous cases. It is a very easy task if we suppose that the set of edges in the product space is "nice" (e.g. its boundary is a Jordan-curve if the measure space is the $[0, 1]$ interval with the Lebesgue measure). However, our above example shows that usually we cannot suppose anything else but measurability.

It should be remarked that [11] already contains theorems of this type, but it is written in Russian and the results of the present paper go much further.

The idea, that there is a need of continuous versions of combinatorial results, is not new. E.g. Nash-Williams [9] suggested to explore this field. Moshe Katz [3] worked out the continuous version of a certain combinatorial result of himself. Vera T. Sós [2] and N. Sauer [7] also have unpublished results of this kind. Very likely there are other similar papers and results producing the continuous versions of some given discrete result, but – I think – up to now there is no systematic treatment of this subject.

RESULTS

Let X be a finite or infinite set. $G = (X, E)$ is called a *directed g-graph (hypergraph),* where $E \subset X^g$, that is, E consists of some ordered sequences of form $e = (x_{i_1}, \dots, x_{i_g})$, $x_{i_j} \in X$ $(1 \leqslant j \leqslant g)$. The elements of X and E are called *vertices* and *edges,* respectively. Multiple edges are excluded. The edges having less than g different vertices are called *loops.* If $Y \subset X$, then the *spanned subgraph* $G_Y = (X, E)_Y = (Y, E_Y)$ consists of all those edges of G which satisfy $x_{i_j} \in Y$ *for all* j. If $X' \subset X$, $E' \subset E$ then (X', E') is a *subgraph* of (X, E).

Let $\mathbf{G}$ be a set of finite graphs. $\mathbf{G}$ is *symmetric* if $\mathbf{G}$ is closed for the permutations of the vertices. As the vertices are not numbered this property automatically holds. It will be always supposed without saying it. We say that we *double* a vertex x of $G = (\{x, x_1, \dots\}, E)$ if the new graph $G_d = (\{x', x'', x_1, \dots\}, E_d)$ contains the edges $(\dots, x', \dots)$, $(\dots, x'', \dots)$ if and only if E contains the corresponding edge $(\dots, x, \dots)$. $\mathbf{G}$ is called *doublable* if for any member of $\mathbf{G}$ $G_d \in \mathbf{G}$ holds.

$\mathbf{G}$ is called *hereditary* if for any spanned subgraph G_1 of $G \in \mathbf{G}$, $G_1 \in \mathbf{G}$ holds. If $\mathbf{G}$ is not hereditary, the *hereditary kernel* $\hat{\mathbf{G}}$ of $\mathbf{G}$ can be produced in the following way: $G \in \hat{\mathbf{G}}$ if and only if all the spanned subgraphs (including G) are in $\mathbf{G}$. It is easy to see that $\hat{\mathbf{G}}$ is always hereditary. On the other hand, if $\mathbf{G}$ is doublable, $\hat{\mathbf{G}}$ is also doublable.

Example 1. Put $g = 2$. Let $\mathbf{G}_1$ consist of the graphs G having the property that $G_{\{x_1, x_2, x_3\}}$ contains at least one non-loop edge for any 3 different vertices x_1, x_2, x_3. (Thus (or and?) $\mathbf{G}$ contains all the graphs with 1 and 2 vertices.) It is easy to see that $\mathbf{G}_1$ is hereditary. However it is not doublable as the graph $G = (\{x_1, x_2, x_3\}, (x_1, x_2))$ shows: $G \in \mathbf{G}_1$ but if we double the vertex x_3, the graph $G_d = (\{x_1, x_2, x_3', x_3''\}, (x_1, x_2))$ is not in $\mathbf{G}$, the subgraph spanned by x_1, x_3' and x_3'' contains no edge.

Example 2. Put $g = 2$ and let $\mathbf{G}_2$ consist of the graphs G containing all the possible loops and having the property that $G_{\{x_1, x_2, x_3\}}$

contains at least one non-loop edge for any 3 different vertices x_1, x_2, x_3. It is easy to see that $\mathbf{G}_2$ is hereditary and doublable.

Let $M = (X, \sigma, \mu)$ be a measure space with a finite measure. (In this paper we shall consider only finite measures.) Furthermore, let $E \subset X^g$ be a measurable set in the product space $(X, \sigma, \mu)^g = (X^g, \sigma_g, \mu_g)$. We define the measure of a graph $G = (X, E)$ in the following way: $\mu(G) = \mu_g(E)$. If $\mathbf{G}$ is a family of finite (directed) g-graphs then introduce the notation

$$(1) \qquad H(M, \mathbf{G}) = \inf_E \frac{\mu_g(E)}{\mu_g(X^g)}$$

where the infimum is taken over all measurable $E \subset X^g$ satisfying the condition

(2) all the finite spanned subgraphs G_Y $(Y \subset X)$ of $G = (X, E)$ are isomorphic with some graphs in $\mathbf{G}$.

It is easy to see that in condition (2) and in definition (1) $\hat{\mathbf{G}}$ can be used equivalently in place of $\mathbf{G}$. Thus, in the future we can always assume that $\mathbf{G}$ is hereditary without any loss of generality.

If there is no graph with $|X|$ vertices containing all of its spanned subgraphs from $\hat{\mathbf{G}}$, then $H(M, \hat{\mathbf{G}})$ is undefined. We assume in this paper, that this is not the case; when $H(M, \hat{\mathbf{G}})$ is used, it is always understood that there exists such a graph with vertex-set X.

If X is finite, σ will always be the family of all subsets of X. M_n denotes the measure space $(\{x_1, \dots, x_n\}, \sigma, \mu)$, where $\mu(x_i) = \frac{1}{n}$ $(1 \leqslant i \leqslant n)$. M is called *atomless* if for any $A \in \sigma$, $\mu(A) > 0$ there is a set $B \subset A$, $B \in \sigma$ such that $0 < \mu(B) < \mu(A)$.

Example 3. Let $\mathbf{G}_1$ be as in Example 1. The theorem of Turán [1] says that if a non-directed 2-graph (without loops) with n vertices contains no empty triangle then the minimal number of edges is $\frac{n}{2}\left(\frac{n}{2} - 1\right)$ for even n and $\left(\frac{n-1}{2}\right)^2$ for odd n. Thus, if $M = M_n$ and $G = (X, E)$ is a graph satisfying (2), then

$$\inf_E \frac{\mu_2(E)}{\mu_2(X^2)} = \begin{cases} \dfrac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{n^2} = \dfrac{1}{4} - \dfrac{1}{2n} & (n \text{ is even}) \\ \dfrac{\left(\frac{n-1}{2}\right)^2}{n^2} = \dfrac{1}{4} - \dfrac{1}{2n} + \dfrac{1}{4n^2} & (n \text{ is odd}). \end{cases}$$

Using our notation (1) we can write

$$H(M_n, G_1) = \begin{cases} \dfrac{1}{4} - \dfrac{1}{2n} & \text{if } n \text{ is even} \\ \dfrac{1}{4} - \dfrac{1}{2n} + \dfrac{1}{4n^2} & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

Observe, that for the class $\boldsymbol{G}_2$ of Example 2 we can write

$$H(M_n, \boldsymbol{G}_2) \geqslant \frac{1}{4} + \frac{1}{2n} \geqslant \frac{1}{4}. \tag{4}$$

The next lemma will express that if we identify elements in X, $H(M, \boldsymbol{G})$ is not decreased for doublable $\boldsymbol{G}$'s. Let $M = (X, \sigma, \mu)$ be an arbitrary measure space and $A \in \sigma$ be a measurable set. Define $M^A = (X^A, \sigma^A, \mu^A)$ in the following way: $X^A = (X - A) \cup \{a\}$, where $a \notin X$ (a stands for symbolizing A), $\sigma_A = \{B\colon (B \subset X - A$ and $B \in \sigma)$ or $(B = B' \cup \{a\}$ and $B' \cup A \in \sigma$ for some $B' \subset X - A)\}$ and $\mu(B) = \mu^A(B)$ if $a \notin B$, $\mu^A(B) = \mu(B - A) + \mu(A)$ otherwise.

Lemma 1. *Let* $\boldsymbol{G}$ *be a doublable class of directed g-graphs,* $M = (X, \sigma, \mu)$ *be an arbitrary measure space and* $A \in \sigma$ *be a measurable set. Then*

$$H(M^A, \boldsymbol{G}) \geqslant H(M, \boldsymbol{G}). \tag{5}$$

Proof. Let E^A be a measurable set in $(M^A)^g$ such that all the spanned subgraphs of $G^A = ((X^A)^g, E^A)$ are in $\boldsymbol{G}$. Define the graph $G = (X^g, E)$ in the following way: $(x_1, \ldots, x_g) \in E$ $(x_i \in X)$ iff $(y_1, \ldots, y_g) \in E^A$, where $y_i = x_i$ when $x_i \notin A$ and $y_i = a$ otherwise. We prove that G also has the property that all of its finite spanned subgraphs are in $\boldsymbol{G}$.

Suppose, in the contrary, that G contains a (finite) spanned subgraph $G_T \notin \boldsymbol{G}$. There are two possibilities:

(a) $|T \cap A| \leq 1$,

(b) $|T \cap A| > 1$.

In case (a) put $T' = (T - A) \cup \{a\}$. Then $(G^A)_{T'}$ is isomorphic to G_T, consequently G^A has a spanned subgraph not in $\mathbf{G}$, what is a contradiction. In case (b) we can obtain $(G^A)_{T'}$ from G_T by identifying the vertices in $T \cap A$. It means that G_T can be obtained from $(G^A)_{T'}$ by repeated application of doubling the vertex a. As $\mathbf{G}$ is doublable $(G^A)_{T'} \notin \mathbf{G}$ follows from $G_T \notin \mathbf{G}$, and this is a contradiction again.

Choose any E^A satisfying (2) (in M^A). We just proved that the corresponding E satisfies (2) (in M), too. Consequently for such E's

$$\frac{\mu_g(E)}{\mu_g(X^g)} \geq H(M, \mathbf{G}) \tag{6}$$

holds by (1). It is easy to see, that $\mu_g^A(E^A) = \mu_g(E)$ and $\mu_g^A((X^A)^g) = \mu_g(X^g)$, thus we obtain by (6) that

$$\frac{\mu_g^A(E^A)}{\mu_g^A((X^A)^g)} \geq H(M, \mathbf{G})$$

holds for any E^A satisfying (2). Hence (5) follows using (1), again. The proof is completed.

Lemma 2. *Let* $\mathbf{G}$ *be a doublable class of directed g-graphs. Then the limit*

$$\lim_{n \to \infty} H(M_n, \mathbf{G}) \tag{7}$$

exists. If furthermore M *is an arbitrary measure space then*

$$H(M, \mathbf{G}) \geq \lim_{n \to \infty} H(M_n, \mathbf{G}). \tag{8}$$

Proofs.

1. From Lemma 1 it follows by induction that

$$H(M', \mathbf{G}) \geq H(M_n, \mathbf{G}), \tag{9}$$

where

$$M' = (X', \sigma', \mu'), \quad X' = \{x_1, \ldots, x_d\}, \quad \mu(x_i) = k_i$$

$$\text{(integers, } 1 \leqslant i \leqslant d), \quad \sum_{i=1}^{d} k_i = n.$$

2. Let $M = (X, \sigma, \mu)$ be an arbitrary measure space and $G = (X, E)$ be a graph satisfying (2) ($E \subset X^g$ measurable). The relation

$$\begin{aligned} &n^g H(M_n, G)\mu(X)^n \leqslant \\ &\leqslant \frac{n!}{(n-g)!}\mu_g(E)\mu(X)^{n-g} + \left(n^g - \frac{n!}{(n-g)!}\right)\mu(X)^n \end{aligned} \tag{10}$$

will be proved soon. Define the function f on X^n in the following way:

$$\begin{aligned} &f(y_1, \ldots, y_n) = \\ &\quad = \text{(the number of ordered sequences } (y_{i_1}, \ldots, y_{i_g}) \in E, \\ &\quad 1 \leqslant i_l \leqslant n, \quad 1 \leqslant l \leqslant g) \quad ((y_1, \ldots, y_n) \in X^n). \end{aligned} \tag{11}$$

Assume that $y_1, \ldots, y_n$ are all different. In this case $f(y_1, \ldots, y_n)$ is simply the number of edges of $G_{\{y_1, \ldots, y_n\}}$. The spanned subgraphs of $G_{\{y_1, \ldots, y_n\}}$ are spanned subgraphs of G, consequently $G_{\{y_1, \ldots, y_n\}}$ satisfies condition (2).

$$H(M_n, G) \leqslant \frac{f(y_1, \ldots, y_n)}{n^g} \tag{12}$$

follows from (1).

Assume now that there are only d different values among the y's. Let the multiplicity of the i-th value u_i be k_i $\left(1 \leqslant i \leqslant d, \ \sum_{i=1}^{d} k_i = n\right)$ Let M' be defined in the following way: $M' = (U, \sigma, \mu')$, $U = \{u_1, \ldots \ldots, u_d\}$, $\mu'(u_i) = k_i$ $(1 \leqslant i \leqslant d)$. Consider the graph G_U. Its spanned subgraphs are all from G, therefore (1) implies that

$$H(M', G) \leqslant \frac{\mu'(E_U)}{n^g}. \tag{13}$$

It is easy to see that

$$\text{(14)} \qquad \mu'(E_U) = f(y_1, \dots, y_n).$$

It means that (12) holds in this case, too; it follows from (9), (13) and (14).

(12) yields

$$\text{(15)} \qquad n^g H(M_n, G)\mu(X)^n \leqslant \int_{X^n} f(y_1, \dots, y_n)\, d\mu.$$

We need some more functions;

$$I(i_1, \dots, i_g, y_1, \dots, y_n) = \begin{cases} 1 & \text{if} \quad (y_{i_1}, \dots, y_{i_g}) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$(1 \leqslant i_j \leqslant n, \quad 1 \leqslant j \leqslant g).$$

The equality

$$f(y_1, \dots, y_n) = \sum_{\substack{1 \leqslant i_j \leqslant n \\ 1 \leqslant j \leqslant g}} I(i_1, \dots, i_g, y_1, \dots, y_n$$

is obvious and leads to

$$\text{(16)} \qquad \int_{X^n} f(y_1, \dots, y_n)\, d\mu = \sum_{\substack{1 \leqslant i_j \leqslant n \\ 1 \leqslant j \leqslant g}} \int_{X^n} I(i_1, \dots, i_g, y_1, \dots, y_n)\, d\mu.$$

Assume that $i_1, \dots, i_g$ are different. Then

$$\int_{X^n} I(i_1, \dots, i_g, y_1, \dots, y_n)\, d\mu =$$

$$\text{(17)} \qquad = \int_{X^n} I(1, \dots, g, y_1, \dots, y_n)\, d\mu =$$

$$= \int_{E \times X^{n-g}} 1\, d\mu = \mu_g(E)\mu(X)^{n-g}.$$

The inequality

$$\text{(18)} \qquad \int_{X^n} f(y_1, \dots, y_n)\, d\mu \leqslant$$

$$\leqslant \frac{n!}{(n-g)!} \mu_g(E)\, \mu(X)^{n-g} + \left(n^g - \frac{n!}{(n-g)!}\right) \mu(X)^n$$

follows from (16), (17) and the trivial estimation $\int_{X^n} I\, d\mu \leqslant \mu(X)^n$. (10) is a consequence of (15) and (18).

3. As (10) holds for any E satisfying (2), the inequality

$$n^g H(M_n, G) \leqslant \frac{n!}{(n-g)!} H(M, G) + \left(n^g - \frac{n!}{(n-g)!}\right)$$

follows from (1). $\dfrac{n!}{(n-g)!n^g}$ tends to 1, if $n \to \infty$, consequently

(19) $$\overline{\lim_{n \to \infty}} H(M_n, G) \leqslant H(M, G).$$

If we prove (7), (19) is equivalent to (8). However (19) is valid for $M = M_n$, too:

$$\overline{\lim}\, H(M_n, G) \leqslant H(M_n, G).$$

Hence

$$\overline{\lim}\, H(M_n, G) \leqslant \underline{\lim}\, H(M_n, G)$$

and (7) holds. The lemma is proved.

Example 4. Let G_2 be chosen as in Examples 2 and 3. Choose $M = (X, \sigma, \mu)$ to be the Lebesgue-measure on the $[0, 1]$ interval. Condition (2) on $E \subset X^2$ can be interpreted as follows: any rectangle having one corner on the diagonal of the unite square or its symmetric picture on the diagonal must have another corner in E (see Fig. 1a), and E contains the diagonal.

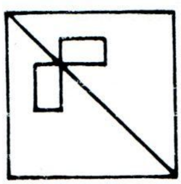

Fig. 1a

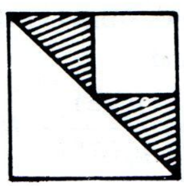

Fig. 1b

From (4) and Lemma 2 we obtain

$$H(M, G) \geqslant \frac{1}{4}.$$

Fig. 1b shows that $H(M, G) = \frac{1}{4}$.

Example 5. If we use G_1 in the previous lemma, it gives the same result. In general, we shall see that the condition of doublability is unnecessary when M is atomless. However, this is not the case when M contains atoms. Let, for instance, $M = (X, \sigma, \mu)$ be defined in the following way: $X = \{x_1, x_2, x_3\}$, $\mu(x_1) = \mu(x_2) = \epsilon$, $\mu_3 = 1 - 2\epsilon$, where ϵ is a small positive number. Choose the set $E = (\{x_1, x_2\})$. Obviously E satisfies (2) with G_1, but $\mu_2(E) = \epsilon^2$ what is $< \frac{1}{4}$, when $\epsilon < \frac{1}{2}$.

Consequently, if G is not doublable, we are not able to give such a good-looking lower estimation for $H(M, G)$ as (8). However, it is possible, even in this case, to give a lower estimation by discrete M's, where the μ's are not uniform. First we present two easy lemmas, what we need later.

Lemma 3. *Let* $M = (X, \sigma, \mu)$ *and* $M' = (X, \sigma, \mu')$ *be two measure spaces, assume* $2\mu'(X) \geqslant \mu(X) \geqslant \mu'(X)$ *and suppose that*

$$(20) \qquad \sum_{i=1}^{n} |\mu(A_i) - \mu'(A_i)| < \epsilon$$

for any collection of disjoint sets $A_i \in \sigma$. *Then*

$$(21) \qquad |H(M, G) - H(M', G)| \leqslant 2^{g+1} g\epsilon \frac{1}{\mu(X)}.$$

Proof. (Warning, it is easier to prove, than to read.)

It follows from (1) that for any $\delta > 0$ there exists an $E \subset X^g$ satisfying (2) and

$$(22) \qquad 0 \leqslant \frac{\mu_g(E)}{\mu_g(X^g)} - H(M, G) < \delta.$$

On the other hand there is a set E_1 which is a union of "rectangles" $A_1 \times \ldots \times A_g$ $(A_i \in \sigma;\ 1 \leqslant i \leqslant g)$ and satisfies

$$(\mu + \mu')g((E - E_1) \cup (E_1 - E)) < \delta$$

(see e.g. [8]), since $\mu + \mu'$ is also a measure on (X, σ). This inequality obviously implies

$$(23) \qquad |\mu_g(E) - \mu_g(E_1)| < \delta \quad \text{and} \quad |\mu'_g(E) - \mu'_g(E_1)| < \delta.$$

Without any loss of generality we can suppose that there is a partition $X = B_1 \cup \ldots \cup B_n$ $(B_i \cap B_j = 0,\ i \neq j,\ B_i \in \sigma)$ such that E_1 is a union of rectangles of the form $B_{i_1} \times \ldots \times B_{i_g}$. Then we can write

$$|\mu_g(E_1) - \mu'_g(E_1)| =$$

$$= \Big| \sum_{B_{i_1} \times \ldots \times B_{i_g} \subset E} (\mu_g(B_{i_1} \times \ldots \times B_{i_g}) -$$

$$- \mu'_g(B_{i_1} \times \ldots \times B_{i_g}))\Big| \leqslant$$

$$\leqslant \sum_{B_{i_1} \times \ldots \times B_{i_g} \subset E} |\mu_g(B_{i_1} \times \ldots \times B_{i_g}) -$$

$$- \mu'_g(B_{i_1} \times \ldots \times B_{i_g})| \leqslant$$

$$\leqslant \sum_{B_{i_1} \times \ldots \times B_{i_g} \subset X^g} |\mu_g(B_{i_1} \times \ldots \times B_{i_g}) -$$

$$- \mu'_g(B_{i_1} \times \ldots \times B_{i_g})| =$$

$$= \sum_{1 \leqslant i_1 \leqslant n} \ldots \sum_{1 \leqslant i_g \leqslant n} |\mu(B_{i_1}) \ldots \mu(B_{i_g}) -$$

$$- \mu'(B_{i_1}) \ldots \mu'(B_{i_g})| =$$

$$= \sum_{1 \leqslant i_1 \leqslant n} \ldots \sum_{1 \leqslant i_g \leqslant n} |\mu(B_{i_1}) \ldots \mu(B_i B_{i_g}) -$$

$$- \mu(B_{i_1})\mu(B_{i_2}) \ldots \mu'(B_{i_g}) + \mu(B_{i_1})\mu(B_{i_2}) \ldots \mu'(B_{i_g}) -$$

$$- \mu'(B_{i_1}) \ldots \mu'(B_{i_g})| \leqslant$$

$$\leqslant \sum_{1 \leqslant i_1 \leqslant n} \ldots \sum_{1 \leqslant i_g \leqslant n} |\mu(B_{i_1}) \ldots \mu(B_{i_{g-1}})| \cdot$$

$$\cdot \ |\mu(B_{i_g}) - \mu'(B_{i_g})| +$$

$$+ \sum_{1 \leq i_1 \leq n} \ldots \sum_{1 \leq i_g \leq n} |\mu'(B_{i_g})| \cdot |\mu(B_{i_1}) \ldots \mu(B_{i_{g-1}}) -$$

$$- \mu'(B_{i_1}) \ldots \mu'(B_{i_{g-1}})| \leq \epsilon\mu(X)^{g-1} +$$

$$+ \mu(X) \sum_{1 \leq i_1 \leq n} \ldots \sum_{1 \leq i_{g-1} \leq n} |\mu(B_{i_1}) \ldots \mu(B_{i_{g-1}}) -$$

$$- \mu'(B_{i_1}) \ldots \mu'(B_{i_{g-1}})| \leq \ldots \leq g\epsilon\mu(X)^{g-1},$$

using (20). Observe that this inequality holds for X^g in place of E_1. It follows that

$$|\mu_g(E_1)\mu_g'(X^g) - \mu_g'(E_1)\mu_g(X^g)| =$$

$$= |\mu_g(E_1)\mu_g'(X^g) - \mu_g'(E_1)\mu_g'(X^g) +$$

$$+ \mu_g'(E_1)\mu_g'(X^g) - \mu_g'(E_1)\mu_g(X^g)| \leq$$

$$\leq |\mu_g'(X^g)| \cdot |\mu_g(E_1) - \mu_g'(E_1)| +$$

$$+ |\mu_g'(E_1)| \cdot |\mu_g(X^g) - \mu_g'(X^g)| \leq$$

$$\leq \mu(X)^g g\epsilon\mu(X)^{g-1} + \mu(X^g)g\epsilon\mu(X)^{g-1} =$$

$$= 2g\epsilon\mu(X)^{2g-1} \leq 2^{g+1}g\epsilon \frac{1}{\mu(X)} \mu g(X^g)\mu_g'(X^g),$$

that is

$$\left| \frac{\mu_g(E_1)}{\mu_g(X)^g} - \frac{\mu_g'(E_1)}{\mu_g'(X^g)} \right| \leq 2^{g+1}g\epsilon \frac{1}{\mu(X)}.$$

From this inequality, (22) and (23) we obtain

$$\left| \frac{\mu_g'(E)}{\mu_g'(X^g)} - H(M,G) \right| \leq \left| \frac{\mu_g'(E)}{\mu_g'(X^g)} - \frac{\mu_g'(E_1)}{\mu_g'(X^g)} \right| +$$

$$+ \left| \frac{\mu_g'(E_1)}{\mu_g'(X^g)} - \frac{\mu_g(E_1)}{\mu_g(X)^g} \right| + \left| \frac{\mu_g(E_1)}{\mu_g(X)^g} - \frac{\mu_g(E)}{\mu_g(X)^g} \right| +$$

$$+ \left| \frac{\mu_g(E)}{\mu_g(X)^g} - H(M, \mathbf{G}) \right| \leqslant$$

$$\leqslant \delta \left(1 + \frac{1}{\mu(X)^g} - \frac{1}{\mu'(X)^g} \right) + 2^{g+1} g \epsilon \frac{1}{\mu(X)},$$

consequently, tending to 0 with δ,

$$\left| \frac{\mu'_g(E)}{\mu'_g(X^g)} - H(M, \mathbf{G}) \right| \leqslant 2^{g+1} g \epsilon \frac{1}{\mu(X)} \tag{24}$$

holds. We know from (1) that

$$\frac{\mu'_g(E)}{\mu'_g(X^g)} \geqslant H(M', \mathbf{G})$$

as E satisfies (2). This implies by (24) that

$$H(M', \mathbf{G}) - H(M, \mathbf{G}) \leqslant 2^{g+1} g \epsilon \frac{1}{\mu(X)}. \tag{25a}$$

Similarly,

$$H(M, \mathbf{G}) - H(M', \mathbf{G}) \leqslant 2^{g+1} g \epsilon \frac{1}{\mu(X)}, \tag{25b}$$

and (25a) with (25b) are equivalent to (21). The proof is completed.

Lemma 4. *Let* $M = (X, \sigma, \mu)$ *be a measure space, where* $X = \{x_1, \ldots\} \cup Z$ $(x_i \neq x_j,\ i \neq j;\ x_i \notin Z)$ *the restriction of* M *on* Z *is atomless. Let* A *denote the set of sequences* $(y_1, \ldots, y_n) \in X^n$ *containing a pair* (i, j) $(i \neq j)$ *such that* $y_i = y_j \in Z$. *Then* $\mu_n(A) = 0$.

If $Z = \{z_1, \ldots, z_s\}$ *and* $\mu(z_i) = \frac{1}{m}$ $(1 \leqslant i \leqslant s)$ *then* $\mu_n(A) \leqslant \leqslant \binom{n}{2} \mu(X)^{n-2} s \frac{1}{m^2}$.

Proof. Let B denote the set of sequences $(y_1, y_1, y_3, \ldots, y_n) \in \in X^n$, where $y_1 \in Z$. Obviously

$$\mu_n(A) \leqslant \binom{n}{2} \mu_n(B)$$

holds; it is sufficient to prove $\mu_n(B) = 0$. If C denotes the set of pairs

(y_1, y_1) with $y_1 \in Z$, then $B = C \times X^{n-2}$ and

$$\mu_n(B) = \mu_2(C)\mu(X)^{n-2}$$

holds. It is sufficient to prove $\mu_n(C) = 0$. The measure μ on Z is an atomless measure. It is known (e.g. [4]) that in an atomless measure space there is a set with any prescribed measure between 0 and $\mu(Z)$. Choose $U_1 \subset Z$ in the following way: $\mu(U_1) = \frac{\mu(Z)}{N}$. Continuing this method we obtain a partition $Z = U_1 \cup \ldots \cup U_N$ $(U_i \cap U_j = \phi,\ i \neq j)$ with $\mu(U_i) = \frac{\mu(Z)}{N}$ $(1 \leqslant i \leqslant N)$. It is clear, that

$$C \subset \bigcup_{i=1}^{N} (U_i \times U_i)$$

holds. Here

$$\mu_2\Big(\bigcup_{i=1}^{N} (U_i \times U_i)\Big) = \sum_{i=1}^{N} \mu_2(U_i \times U_i) =$$

$$= \sum_{i=1}^{N} \mu_2^2(U_i) = N\frac{\mu(Z)^2}{N^2} = \frac{\mu(Z)^2}{N}$$

tends to 0 if $N \to \infty$, consequently $\mu_2(C)\ \Big(\leqslant \mu_2\Big(\bigcup_{i=1}^{N} (U_i \times U_i)\Big)\Big)$ is 0. The first statement of the lemma is proved.

The second statement follows by the same argument.

Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a sequence of real numbers, β a real number. Then $M_n(\alpha, \beta)$ denotes the finite measure space with measures $\alpha_1, \alpha_2, \ldots, \frac{1}{n}, \ldots, \frac{1}{n}$, where the number of $\frac{1}{n}$'s is $[\beta n]$, and α^n denotes the sequence $\alpha_1, \ldots, \alpha_n, \sum_{i=n+1}^{\infty} \alpha_i, 0, \ldots$. $\sum_{i=n+1}^{\infty} \alpha_i$ is finite, because the measure must be finite.

Lemma 5. *Let* $M = (X, \sigma, \mu)$ *be a measure space, where* $X = \{x_1, x_2, \ldots\} \cup Z$ $(x_i \neq x_j,\ i \neq j;\ x_i \notin Z)$, $\mu(\{x_i\}) = \alpha_i$, $\mu(Z) = \beta > 0$ *and the restriction of* M *on* Z *is atomless. If* **G** *is a class of directed g-graphs, then the limit*

(26) $$\lim_{n\to\infty} H(M_n(\alpha^n, \beta), G)$$

exists and

(27) $$H(M, G) \geqslant \lim_{n\to\infty} H(M_n(\alpha^n, \beta), G).$$

Proofs.

1. For sake of simplicity assume first that $\alpha_i = 0$ when $i > r$ for some integer $r > 0$. The first section of the proof of Lemma 2 does not work here, because Lemma 2 holds only for doublable G's. However, the second section works with a modification. (15) does not hold, as (12) is not true, when the values y_i are not necessarily different.

In this case we have to use the law of the large numbers. For any $\epsilon > 0$ and $\delta > 0$ there is an $n_0(\epsilon, \delta)$ such that the numbers k_i (l) of the x_i's (of the elements of Z) in the sequence $(y_1, \ldots, y_n) \in X^n$ satisfy the conditions

(28) $$\left|\frac{k_i}{n} - \alpha_i\right| < \epsilon \qquad (1 \leqslant i \leqslant r)$$
$$\left|\frac{l}{n} - \beta\right| < \epsilon$$

with a measure

$$\mu(X)^n - \delta.$$

On the other hand the measure of the sequences $(y_1, \ldots, y_n) \in X^n$ having $y_i = y_j \in Z$ $(i \neq j)$ is 0 by Lemma 4. Thus, it holds with a measure $\mu(X)^n - \delta$ that $(y_1, \ldots, y_n) \in X^n$ satisfies (28) and the l members being in Z are different. It means that $f(y_1, \ldots, y_n)$ (see (11)) can be bounded from below (apart from a set of measure δ in X^n) by $\min H(M, G)$, where M is a discrete measure space with measures $k_1, \ldots, k_r, 1, \ldots, 1$ (the number of 1's is l; of course $\sum_{i=1}^{r} k_i + l = n$) satisfying (28). This is what we have instead of (12):

(29) $$\min_M H(M, G) \leqslant \frac{f(y_1, \ldots, y_n)}{n^g}$$

which holds with a measure $\mu(X)^n - \delta$. It is easy to see that

$$(30) \qquad H(M', G) = H(M, G)$$

if M' is the normalized version of M (divide the measures by n). So we can write

$$(31) \qquad \min_{M'} H(M', G) \leqslant \frac{f(y_1, \ldots, y_n)}{n^g}$$

if M' runs over the measure spaces with measures $\frac{k_1}{n}, \ldots, \frac{k_r}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$ $\left(\sum_{i=1}^{n} k_i + l = n\right)$ satisfying (28). However, M' differs from $M_n(\alpha, \beta)$ only a little. It follows from (28) that (20) is satisfied with $\epsilon(r+2)$ (if $\frac{1}{n} \leqslant \epsilon$) for the spaces M' and $M_n(\alpha, \beta)$. Using Lemma 3 we obtain

$$(32) \qquad |H(M', G) - H(M_n(\alpha, \beta), G)| \leqslant 2^{g+1} g\epsilon(r+2) \frac{1}{\mu(X)}.$$

(31) and (32) result in

$$(33) \qquad H(M_n(\alpha, \beta), G) - 2^{g+1} g(r+2)\epsilon \leqslant \frac{f(y_1, \ldots, y_n)}{n^g}$$

for a set $A \subset X^n$ with a measure at least $\mu(X)^n - \delta$ and hence

$$(34) \qquad \begin{aligned} &\int_{X^n} f(y_1, \ldots, y_n)\, d\mu \geqslant \int_A f(y_1, \ldots, y_n)\, d\mu \geqslant \\ &\geqslant (\mu(X)^n - \delta)(H(M_n(\alpha, \beta), G) - 2^{g+1} g(r+2)\epsilon) n^g \end{aligned}$$

follows. This is the substitute of (15). (18) remains unchanged. From (18) and (34) we obtain now

$$(35) \qquad \begin{aligned} &(\mu(X^n) - \delta)(H(M_n(\alpha, \beta), G) - 2^{g+1} g(r+2)\epsilon) \leqslant \\ &\leqslant \frac{n!}{n^g (n-g)!} \mu_g(E) \mu(X)^{n-g} + \left(1 - \frac{n!}{n^g (n-g)!}\right) \mu(X)^n \end{aligned}$$

instead of (10). This inequality holds with arbitrary small ϵ and δ if $n \geqslant n_0(\epsilon, \delta)$. On the other hand $\frac{n!}{n^g (n-g)!} \to 1$, thus

(36) $\qquad H(M_n(\alpha, \beta), G) - \epsilon_1 \leqslant \dfrac{\mu_g(E)}{\mu(X)^g}$

holds for any $\epsilon_1 > 0$ if n is large enough, and E satisfies (2).

(37) $\qquad H(M_n(\alpha, \beta), G) - \epsilon_1 \leqslant H(M, G)$

follws from (1), consequently

(38) $\qquad \overline{\lim_{n\to\infty}}\ H(M_n(\alpha, \beta), G) \leqslant H(M, G)$

holds.

2. Repeat the proof with $M_m(\alpha, \beta)$ in place of M (n is fixed, large, $n << m$). $M_m(\alpha, \beta) = (X, \sigma, \mu)$, where $X = \{x_1, \dots, x_r, z_1, \dots \dots, z_{[\beta m]}\}$, $Z = \{z_1, \dots, z_{[\beta m]}\}$, $\mu(x_i) = \alpha_i$ $(1 \leqslant i \leqslant r)$, $\mu(z_i) = \dfrac{1}{m}$ $(1 \leqslant \leqslant i \leqslant [\beta m])$. (28) is valid, again. Lemma 4 implies that the measure of the set of sequences $(y_1, \dots, y_n)$ with equal elements from Z is $\leqslant \binom{n}{2} \mu(X)^{n-2} \dfrac{[\beta m]}{m^2}$. That is, (29), (30), (31) and (33) hold for a set of sequences with measure $\mu(X)^n - \delta - \binom{n}{2} \mu(X)^{n-2} \dfrac{[\beta m]}{m^2}$, consequently, (34) and (35) are true with $\mu(X)^n - \delta - \binom{n}{2} \mu(X)^{n-2} \dfrac{[\beta m]}{m^2}$ on the place of $\mu(X)^n - \delta$. Thus, (36) and (37) follow if n is large enough and m is large enough relative to n:

$$H(M_n(\alpha, \beta), G) - \epsilon_1 \leqslant H(M_m(\alpha, \beta), G).$$

It is easy to see that this implies that

$$\lim H(M_n(\alpha, \beta), G)$$

exists. This case (when $\alpha_i = 0$ if $i > r$) is proved.

3. Prove that the limit (26) exists in the general case, when α has infinitely many non-zero terms. Denote the sequence $\left(\alpha_1, \alpha_2, \dots, \alpha_m, \sum_{i=m+1}^{\infty} a_i\right)$ by α^m. Prove first that $\lim_{k\to\infty} H(M_k(\alpha^m, \beta), G)$ converges if $m \to \infty$. Fix an $\epsilon > 0$ and choose $m_0 = m_0(\epsilon)$ so that $\sum_{i=m_0+1}^{\infty} \alpha_i < \epsilon$ holds. Such an m_0 exists since the measure is finite. The measures used

in $M_n(\alpha^{m_1}, \beta)$ and $M_n(\alpha^{m_2}, \beta)$ "differ" by at most 2ϵ if $m_1, m_2 > > m_0$. Thus, we can use Lemma 3:

$$(39) \qquad |H(M_n(\alpha^{m_1}, \beta), G) - H(M_n(\alpha^{m_2}, \beta), G)| \leqslant 2^{g+2} \epsilon g \frac{1}{\mu(X)},$$

(where $\mu(X) = \sum_{i=1}^{\infty} \alpha_i + [\beta n] \frac{1}{n}$). On the other hand choose an $n = = n(\epsilon, m_1, m_2)$ so that

$$(40) \qquad |\lim_{k \to \infty} H(M_k(\alpha^{m_1}, \beta), G) - H(M_n(\alpha^{m_1}, \beta), G)| < \epsilon$$

and

$$(41) \qquad |\lim_{k \to \infty} H(M_k(\alpha^{m_2}, \beta), G) - H(M_n(\alpha^{m_2}, \beta), G)| < \epsilon$$

simultaneously hold. From (39), (40) and (41) it easily follows that

$$|\lim_{k \to \infty} H(M_k(\alpha^{m_2}, \beta), G) - \lim_{k \to \infty} H(M_k(\alpha^{m_2}, \beta), G)| < $$
$$< 2\epsilon + \frac{2^{g+2} \epsilon g}{\mu(X)},$$

which is arbitrarily small if m_0 is large enough. Consequently, the limit

$$h = \lim_{m \to \infty} (\lim_{k \to \infty} H(M_k(\alpha^m \ \beta), G))$$

exists. By definition

$$(42) \qquad |h - \lim_{k \to \infty} H(M_k(\alpha^m, \beta), G)| < \epsilon$$

holds if $m > m_0(\epsilon)$. On the other hand, the inequality

$$(43) \qquad |\lim_{k \to \infty} H(M_k(\alpha^m, \beta), G) - H(M_k(\alpha^m, \beta), G)| < \epsilon$$

also holds if $k > k_0(\epsilon, m)$. Finally, the distributions of $M_k(\alpha^m, \beta)$ and $M_k(\alpha^k, \beta)$ differ by at most 2ϵ if $k, m > m_0$. Thus, we can use Lemma 3:

$$(44) \qquad |H(M_k(\alpha^k, \beta), G) - H(M_k(\alpha^m, \beta), G)| < 2^{g+2} \epsilon g \frac{1}{\mu(X)}.$$

Summarizing (42), (43) and (44) we obtain

(45) $$|H(M_k(\alpha^k, \beta), G) - h| < 2\epsilon + 2^{g+2}\epsilon g \frac{1}{\mu(X)}$$

if $m > m_0(\epsilon)$, $m > m_3(\epsilon)$, $k > m_0(\epsilon)$ and $k > k_0(\epsilon, m)$. That is, (45) is arbitrarily small if k is large,

(46) $$\lim_{k \to \infty} H(M_k(\alpha^k, \beta), G) = l.$$

4. Prove now (27) for the general case. Denote by $M(n)$ the measure which is identical to M on Z and has the measures $\alpha_1, \ldots, \alpha_n$, $\sum_{i=n+1}^{\infty} \alpha_i, 0, \ldots$ on the elements of $X - Z$. It is easy to see, using Lemma 3, that

(47) $$\lim_{n \to \infty} H(M(n), G) = H(M, G).$$

Since $M(n)$ has only finitely many atoms, we can state (27) for its case:

$$H(M(n), G) \geqslant \lim_{k \to \infty} H(M_k(\alpha^n, \beta), G).$$

Here the left hand side converges to $H(M, G)$ by (47), while limit of the right hand side is h, what is equal to $\lim_{k \to \infty} H(M_k(\alpha^k, \beta), G)$ from (46). (27) and the lemma is proved.

Remark 1. Lemma 2 is a special case of Lemma 5. We have proved it separately to show the basic idea of the proof, which is confused with technical details in case of Lemma 5.

Remark 2. It is easy to see that (27) is valid with $M_n(\alpha^{f(n)}, \beta)$ where $f(n) \to \infty$ $(n \to \infty)$. On the other hand $M_n(\alpha, \beta)$ could have been defined in a different way: The measures would be $\alpha_1, \alpha_2, \ldots, \frac{\beta}{n}, \ldots, \frac{\beta}{n}$ where the number of $\frac{\beta}{n}$'s is n. (27) remains unchanged if we use this definition in place of the original.

Remark 3. The existence of the limit (26) cannot be considered to be new. Although it has never been formulated in such a general form, the proof used in a very special case [5] can be essentially used here, too (see also [10], [13]).

To the applications (see [11] and a forthcoming paper) we need the inequalities of type (8) and (27). All of our information concerning this is included in Lemma 5. However, theoretically it is interesting when have we equality in (27). I conjecture that it holds under the conditions of Lemma 5, but I was not able to prove it.

Theorem 1. *Let G be a doublable class of directed g-graphs and $M = (X, \sigma, \mu)$ be a measure space, where $X = \{x_1, x_2, \dots\} \cup Z$ $(x_i \neq x_j$, $i \neq j$; $x_i \notin Z)$, $\mu(\{x_i\}) = \alpha_i$, $\mu(Z) = \beta > 0$ and the restriction of M on Z is atomless. Then*

$$H(M, G) = \lim_{n \to \infty} H(M_n(\alpha^n, \beta), G). \tag{48}$$

Proof. Lemma 5 gives one side of (48), thus we have to prove

$$H(M, G) \leqslant \lim_{n \to \infty} H(M_n(\alpha^n, \beta), G), \tag{49}$$

only. E.g. by [4] we know that there is a partition $A_1 \cup \dots \cup A_n$ $(A_i \cap \cap A_j = \phi,\ i \neq j)$ of Z such that A_i's are measurable and $\mu(A_i) = \frac{\beta}{n}$. By successive application of Lemma 1 we obtain

$$\begin{aligned} &H(M, G) \leqslant H(M^{A_1}, G) \leqslant \\ &\leqslant H((M^{A_1})^{A_2}, G) \leqslant \dots \leqslant H((M^{A_1})^{\dots A_n}, G) \leqslant \\ &\leqslant H(((M^{A_1})^{\dots A_n})^B, G), \end{aligned} \tag{50}$$

where $B = \{x_{n+1}, x_{n+2}, \dots\}$. Here $((M^{A_1})^{\dots A_n})^B$ is a distribution with probabilities $\alpha_1, \dots, \alpha_n, \sum_{i=n+1}^{\infty} \alpha_i, \frac{\beta}{n}, \dots, \frac{\beta}{n}$. Consequently, by Remark 3, the limit of (49) is

$$\lim_{n \to \infty} H(M_n(\alpha^n, \beta), G).$$

This proves (49) and the theorem.

I conjecture that the theorem holds for non-doublable graphs, too. However I was not able to prove it in the general case. A very special case follows from the following theorem of Brown, Erdős and Simonovits

[10]. A class $\mathbf{G}$ of directed G-graphs is called *strongly hereditary* if $G \in \mathbf{G}$ implies $G' \in \mathbf{G}$ for any G' (not necessarily spanned) subgraph of G.

Theorem BES. *Let* $\mathbf{G}$ *be a strongly hereditary class of 2-graphs. Then there exists an* $r \times r$ 0, 1 *matrix* $A = (a_{ij})$ *with the following properties. For any* n, *let us construct graphs by taking (disjoint) partitions* $C_1^n \cup \ldots \cup C_r^n$ *of the vertex-set, connect all the vertices of* C_i^n *with all the vertices of* C_j^n $(i \neq j)$ *iff* $a_{ij} = 1$ *and form complete acyclic graphs in* C_i^n *iff* $a_{ii} = 1$; *no other pairs are connected. Denote the edge set of the graph having the minimal number of edges from the above constructed graphs by* E_n. *Then all the spanned subgraphs of this graph belong to* $\mathbf{G}$ *and*

$$(51) \qquad \lim_{n\to\infty} \frac{|E_n|}{n^2} = \lim_{n\to\infty} H(M_n, \mathbf{G}).$$

Theorem 2. *Let* $\mathbf{G}$ *be a strongly hereditary class of directed 2-graphs and* $M = (X, \sigma, \mu)$ *be an atomless measure space. Then*

$$H(M, \mathbf{G}) = \lim_{n\to\infty} H(M_n, \mathbf{G}).$$

Proof. We have to prove

$$(52) \qquad H(M, \mathbf{G}) \leqslant \lim_{n\to\infty} H(M_n, \mathbf{G}),$$

only. We do this by constructing a graph (X, E) on X. If $\beta_1, \ldots, \beta_r$ are reals and $\sum_{i=1}^{r} \beta_i = \mu(X)$ then there is a partition $X = C_1 \cup \ldots \cup C_r$ $(C_i \cap C_j = 0,\ i \neq j)$ with $\mu(C_i) = \beta_i$. This follows from the condition that M is atomless (see [4]). Let $E_N(\beta_1, \ldots, \beta_r)$ denote the edge set of the graph in which all the vertices in C_i are connected with all the vertices in C_j $(i \neq j)$ iff $a_{ij} = 1$ (see Theorem BES) the vertices $x, y \in C_i$ are not connected if $a_{ii} = 0$; if $a_{ii} = 1$ we use a disjoint partition $C_i = \bigcup_{j=1}^{N} C_{ij}$, $\mu(C_{ij}) = \frac{\beta_i}{N}$, the vertices of C_{ij} are connected with the vertices of $C_{ij'}$ iff $j < j'$. It follows from Theorem BES that all the finite spanned subgraphs of $E(\beta_1, \ldots, \beta_r)$ belong to $\mathbf{G}$ (using that it is strongly hereditary). On the other hand it is easy to see that $E_n(\beta_1, \dots, \beta_r)$ is measurable. Using (1) we obtain

$$(53) \qquad H(M, G) \leqslant \frac{\mu_2(E_N(\beta_1, \dots, \beta_r))}{\mu_2(X^2)}$$

Let now β_i be equal to $\dfrac{|C_i^n|\,\mu(X)}{n}$. Then

$$(54) \qquad \left| \frac{\mu_2(E_N(\beta_1, \dots, \beta_r))}{\mu_2(X^2)} - \frac{|E_n|}{n^2} \right| \leqslant \frac{1}{N} \sum_{i=1}^{r} \frac{|C_i^n|^2}{n^2} \leqslant \frac{1}{N}$$

easily holds for any N. Suppose $N \to \infty$. (52) and the theorem follows from (53), (54) and (51). The proof is completed.

Remark 4. Let us define $\bar{H}(M, G)$ as $H(M, G)$ in (1) but using sup rather than inf. Let further $\bar{G}$ denote the set of complements of the elements of G. Obviously, all the spanned subgraphs of (X, E) belong to G iff all the spanned subgraphs of $(X, X^g - E)$ belong to $\bar{G}$. Consequently,

$$\bar{H}(M, G) + H(M, \bar{G}) = 1.$$

It follows that Theorems 1 and 2 hold for $\bar{H}(M, G)$ and Lemma 5 holds with the opposite direction of the inequality sign. Lemma 1 also holds if $\bar{G}$ is doublable.

Remark 5. Throughout, we considered the case of directed graphs. In case of undirected graphs we simply substitute any undirected edge by the set of all oriented variants of it. Let us see an example.

Example 5. We try to form the continuous version of a theorem of Bollobás [12], asserting that in an undirected *uniform* 3-*graph* (no loops are allowed) with $3n$ vertices and containing no 3 different edges a, b, c with $(a - b) \cup (b - a) \subset c$ the number of edges is $\leqslant n^3$.

Let now G be the set of directed 3-graphs not containing edges $a_1, \dots, a_6, b_1, \dots, b_6, c_1, \dots, c_6$, where the letters with the same indices denote edges with identical (3 different) vertices but with different directions, and these (3 different) sets satisfy $(a - b) \cup (b - a) \subset c$. What is $\bar{H}(M_n, G)$ in this case? Fixing any direction for all the 3-edges we obtain from Bollobás' theorem that we can choose at most $\left(\frac{n}{3}\right)^3$ edges from these $\binom{n}{3}$. This is true for all the 6 cases. On the other hand we can choose all the $3\binom{n}{3} + n$ loops. That is,

$$\bar{H}(M_n, G) \leqslant \frac{\frac{2n^3}{9} + \frac{3n^2}{2} + n}{n^3}, \qquad \lim \bar{H}(M, G) \leqslant \frac{2}{9}.$$

From Lemma 5 and Remark 5 $\bar{H}(M, G) \leqslant \frac{2}{9}$ for atomless measure spaces M. It is easy to construct a graph (X, E) with measure $\frac{2}{9}\mu(X^3)$. Divide X into 3 disjoint classes X_1, X_2, X_3 with equal measure and take all the edges having their coordinates in different classes. This shows that $\bar{H}(M, G) = \frac{2}{9}$ for atomless measures.

Remark 7. In [13] Erdős and Simonovits proved that

$$\lim_{n \to \infty} H(M_n, G) = \frac{1}{\rho - 1},$$

where G is a class of undirected 2-graphs and ρ is the minimum of the chromatic numbers of the graphs not contained in $\bar{G}$ and $\rho \geqslant 2$. (The latter condition simply expresses that G contains all the complete graphs). Lemma 5 and a trivial construction imply that $H(M, G) = \frac{1}{\rho - 1}$ for an atomless measure M under the above conditions.

A GENERALIZATION

To some applications (see [11] and a forthcoming paper) we need a generalization. It does not differ too much from the previous results, neither in formulation nor in proving. Thus we give here briefly the necessary definitions and completely omit the proof.

Let $G(t)$ denote a class of directed g-graphs, where the vertices of the graphs are coloured by t colours (no assumption on the colouring). We say that $G(t)$ is doublable if doubling any vertex of $G \in G(t)$ and giving the same old colour to both then for the new graph $G_d \in \mathbf{G}(t)$, again.

Let $M = (X, \sigma, \mu)$ be a measure space and $X = X_1 \cup \ldots \cup X_t$ a disjoint partition. Colour X_i by the i-th colour. Then define

$$H(M, X_1, \ldots, X_t;\ \mathbf{G}(t)) = \inf \frac{\mu_g(E)}{\mu_g(X^g)}$$

where the infimum runs over all measurable $E \subset X^g$ such that every finite spanned subgraph of $G = (X, E)$ is isomorphic and coloured identically with an element of $\mathbf{G}(t)$. $M(n_1, \ldots, n_t)$ denotes the finite measure space with $n = \sum_{i=1}^{t} n_i$ elements, with uniform distribution $\mu = \frac{1}{n}$ and with partition $|X_i| = n_i$.

Theorem 3. *Let* $\mathbf{G}(t)$ *be a set of* t*-coloured directed* g*-graphs, and* $M = (X, \sigma, \mu)$ *be a measure space with a disjoint partition* $X_1, \ldots, X_t$. *Suppose further that either* $\mathbf{G}(t)$ *is doublable or* M *is atomless. Then*

$$H(M, X_1, \ldots, X_t;\ \mathbf{G}(t)) \geqslant \lim H(M(n_1, \ldots, n_t), \mathbf{G}(t))$$

holds if $n_1, \ldots, n_t$ *tend to infinity satisfying*

$$\frac{n_i}{\sum_{i=1}^{t} n_i} \to \frac{\mu(X_i)}{\mu(X)} \qquad (1 \leqslant i \leqslant t).$$

Acknowledgement. I am indebted to Vera T. Sós and M. Simonovits for their valuable remarks.

Remark added in August 1977. Recently B. Bollobás [14] informed me that he can prove Lemma 2 much easier under the condition that G is strongly hereditary.

REFERENCES

[1] P. Turán, On a graph-theoretical extremal problem, (in Hungarian), *Mat. Fiz. Lapok,* 48 (1941), 436-452.

[2] V.T. Sós, Personal communication.

[3] M. Katz, Rearrangements of (0, 1) matrices, *Israel J. Math.,* 9 (1971), 53-72.

[4] J. Neveu, *Mathematical foundation of the calculus of probability,* Holden-Day, San Francisco, 1965.

[5] G. Gatona – T. Nemetz – M. Simonovits, On a graph-problem of Turán, (in Hungarian), *Mat. Lapok,* 15 (1964), 228-238.

[6] G. Katona, Graphs, vectors and inequalities in probability theory, (in Hungarian), *Mat. Lapok,* 20 (1969), 123-127.

[7] N. Sauer, Personal communication.

[8] P. Halmos, *Measure theory,* D. van Nostrand C., N.Y., 1950.

[9] C.St.J.A. Nash-Williams, Unexplored and semi-explored territories in graph theory, *New directions in graph theory,* (Proc. 3-rd Ann Arbor Conference on Graph Theory, Univ. Mich. Ann Arbor, Mich. 1971), 149-186, Academic Press, N.Y., 1973.

[10] W.G. Brown – P. Erdős – M. Simonovits, Extremal Problems for Directed Graphs, *Journal of Combinatorial Th.,* 15 (1973), 77-93.

[11] G.O.H. Katona, Inequalities for the distribution of length of sums of random vectors, (in Russian), *Teor. Verojatnost i Prim.,* (to appear).

[12] B. Bollobás, Three-graphs without two triples whose symmetric difference is contained in a third, *Discrete Math.,* 6 (1974), 21-24.

[13] P. Erdős – M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hungar.*, 1 (1966), 51-57.

[14] B. Bollobás, Personal communication.

G.O.H. Katona

Mathematical Institute of the Hungarian Academy of Sciences, 1053 Budapest, Reáltanoda u. 13-15, Hungary.