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ON A CONJECTURE OF ERDŐS AND A STRONGER FORM OF SPERNER'S THEOREM

by

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INTRODUCTION. A theorem of P. ERDŐS [1] states as follows: Let $|x_i| \geqq 1$ $(1 \leqq i \leqq n)$ be real numbers, and $\varepsilon_i = \pm 1$ $(1 \leqq i \leqq n)$, then the number of sums $\sum_{i=1}^{n} \varepsilon_i x_i$ lying in the interior of an arbitrary interval of length 2 is at most $\binom{n}{\left[\frac{n}{2}\right]}$. Erdős conjectured in the same paper, that this statement is true if the x_i are vectors in a Hilbert space, satisfying $\|x_i\| \geqq 1$. In the present paper we prove the two-dimensional case. P. ERDŐS used a well known theorem of SPERNER [2], we need a stronger form of it which is of independent interest.

THEOREM 1. *Let H_1 and H_2 be disjoint sets with finite cardinal numbers n_1 and n_2 $(n_1 \geqq n_2)$. If $a_1, \ldots, a_s$ is a system of subsets of $H = H_1 \cup H_2$, such that no two sets a_i and a_j $(i \neq j)$ satisfy the relations*

(1) $$a_i \cap H_1 = a_j \cap H_1 \quad \text{and} \quad a_i \cap H_2 \supset a_j \cap H_2$$

or

(1′) $$a_i \cap H_1 \supset a_j \cap H_1 \quad \text{and} \quad a_i \cap H_2 = a_j \cap H_2,$$

then

(2) $$s \leqq \binom{n_1+n_2}{\left[\frac{n_1+n_2}{2}\right]},$$

and this is the best upper limitation.

REMARKS. **1.** This theorem gives for the case $n_2 = 0$ SPERNER's theorem. Indeed, then $a_i \cap H_2 = a_j \cap H_2$ and so because of $a_i \cap H_1 = a_i$ and $a_j \cap H_1 = a_j$ the inclusion $a_i \supset a_j$ can not hold.

2. If $n_1 > 0$, $n_2 > 0$, this theorem is a stronger form of Sperner's theorem. The conditions of this theorem are weaker, but the statement is the same.

PROOF OF THEOREM 1. Let us consider a set $b \subset H_2$. Let $a_{i_1}, a_{i_2}, \ldots, a_{i_p}$ be those sets a_i for which $a_{i_j} \cap H_2 = b$. Then $H_1 \cap a_{i_j} \supset H_1 \cap a_{i_k}$ can not hold $(1 \leqq j \leqq k \leqq p)$ because of (1′), that is, $a_{i_1} \cap H_1, a_{i_2} \cap H_1, \ldots, a_{i_p} \cap H_1$ is a Sperner system. Further let $b_1 \subset H_2, b_2 \subset H_2, \ldots. b_l \subset H_2$ and $b_1 \supset b_2 \supset \ldots \supset b_l$. Now

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consider all those sets a_j for which $a_j \cap H_2 = b_i$, and denote the system of $a_j \cap H_1$ for the above-mentioned a_j's by $A(b_i)$.

Thus $A(b_i)$ $(1 \leq i \leq l)$ is a Sperner system of subsets of the set H_1, further if $i \neq j$, then $A(b_i) \cap A(b_j) = \emptyset$, since from $c \in A(b_i)$, $c \in A(b_j)$ it follows, that (1) holds for $c \cup b_i$ and $c \cup b_j$, which contradicts our supposition.

Clearly the system $B = \bigcup_{i=1}^{l} A(b_i)$ can not include a „chain of length $l+1$", that is there is no system $c_1, c_2, \ldots, c_{l+1} \in B$, such that $c_1 \supset c_2 \supset \ldots \supset c_{l+1}$. Otherwise one of the $A(b_i)$'s would contain at least two of the c_j's, which is a contradiction, since $A(b_i)$ must be a Sperner system. (It is easy to see, that the converse of this statement holds too, namely if B is a system of subsets not containing chains of length $l+1$ then it can be divided into l disjoint Sperner system. But this is not necessary here.) We now apply a theorem of ERDŐS ([1] Theorem 5.) asserting, that if B is a system of subsets of a set having n elements, and B does not include any chain of length $l+1$, then the cardinal number of B less then or equal to the sum of the l largest binomial coefficients (belonging to n). That is, denoting the cardinal number of B by $|B|$, and assuming for the sake of simplicity that $n_1 = 2m_1$ is even, but $l = 2j_l + 1$ is odd,

$$|B| \leq \sum_{i=-j_l}^{j_l} \binom{n_1}{m_1+i}. \tag{3}$$

The following lemma, guaranteeing many long chains, makes as efficient utilization of (3) as possible.

LEMMA. Let $a_1, a_2, \ldots, a_{2^n}$ be all subsets of a set H of n elements, and G_n a directed graph with the vertices $a_1, a_2, \ldots, a_{2^n}$. Two vertices a_i and a_j are joined if and only if $a_i \supset a_j$ or $a_j \supset a_i$ $(i \neq j)$, and the edge is oriented from a_j toward a_i, when $a_i \supset a_j$. Then there exist $\binom{n}{\left[\frac{n}{2}\right]}$ disjoint directed paths, one of length $n+1$ and $\binom{n}{\frac{n+1-l}{2}} - \binom{n}{\frac{n-1-l}{2}}$ of length l $(l = n-1, n-3, \ldots l > 0)$. (The length of a path is the number of its vertices, that is the number of the edges plus one.)

If $n=1$, then the statement is trivial. Now let $n>1$ and use induction over n. Namely, suppose that there exist the directed paths $U_1^n, U_2^n \ldots, U^n_{\binom{n}{[\frac{n}{2}]}}$. It is easy to see, that all the vertices are included exactly in one of these paths, since the sum of lengths is exactly 2^n. Now consider the case of $n+1$ elements. The paths $U_1^n, U_2^n, \ldots, U^n_{\binom{n}{[\frac{n}{2}]}}$ exist in G_{n+1} too. From any of these we form new paths in two different ways:

1. Consider for $1 \leq i \leq \binom{n}{\left[\frac{n}{2}\right]}$ the vertex corresponding to the largest subset

in U_i^n. We add to this subset the new $(n+1)$-st element, and let the vertex corresponding to this new subset be placed at the end of U_i^n. Thus we have a path U_i^{n+1} in G_{n+1}, and U_1^{n+1} is longer by one than U_1^n for $1 \leqq i \leqq \binom{n}{\left[\frac{n}{2}\right]}$.

2. Omit the vertex corresponding to the largest subset from U_i^n, if the length of U_i^n is larger than $1 \left(1 \leqq i \leqq \binom{n}{\left[\frac{n}{2}\right]}\right)$. Consider the remaining vertices, and add to the corresponding subsets the $(n+1)$-st element. The corresponding vertices form a path in G_{n+1}. Thus we have the paths $U^{n+1}_{\binom{n}{\left[\frac{n}{2}\right]}+1}, \dots, U^{n+1}_{\binom{n+1}{\left[\frac{n+1}{2}\right]}}$.

These paths $U_1^{n+1}, \dots, U^{n+1}_{\binom{n+1}{\left[\frac{n+1}{2}\right]}}$ are again disjoint, since in the first case we add the $(n+1)$-st element to the subset representing the last vertex of $U_i^n \left(1 \leqq i \leqq \binom{n}{\left[\frac{n}{2}\right]}\right)$, but in the second case we add the $(n+1)$-st element to all the other subsets. Clearly the number of the paths of length l is $(l>0,\ l=n+2,\ n,\ n-2, \dots)$

$$\left[\binom{n}{\frac{n+1-(l-1)}{2}} - \binom{n}{\frac{n-1-(l-1)}{2}}\right] + \left[\binom{n}{\frac{n+1-(l+1)}{2}} - \binom{n}{\frac{n-1-(l+1)}{2}}\right] =$$

$$= \binom{n+1}{\frac{n+2-l}{2}} - \binom{n+1}{\frac{n-l}{2}},$$

since we obtain paths of length l in the first case from the ones of length $l-1$, and in the second case from the ones of length $l+1$. This completes the proof of our lemma.

Let us return to the proof of Theorem 1, and apply the lemma for the set H_2. Suppose for the sake of simplicity, that $n_2 = 2m_2$ is even. Obviously $s = \sum_{b \subset H_2} |A(b)|$, thus supposing that the subset of H_2 are ordered in chains according to the lemma, and using (3) for all the paths,

$$s = \sum_{b \subset H_2} |A(b)| \leqq \binom{n_2}{0} \sum_{i=-m_2}^{m_2} \binom{n_1}{m_1+i} + \left[\binom{n_2}{1} - \binom{n_2}{0}\right] \sum_{i=-m_2+1}^{m_2-1} \binom{n_1}{m_1+i} + \dots +$$

$$+ \left[\binom{n_2}{m_2} - \binom{n_2}{m_2-1}\right]\binom{n_1}{m_1} = \binom{n_2}{0}\binom{n_1}{m_1-m_2} + \binom{n_2}{0}\binom{n_1}{m_1+m_2} + \binom{n_2}{1}\binom{n_1}{m_1-m_2+1} +$$

$$+ \binom{n_2}{1}\binom{n_1}{m_1+m_2-1} + \dots + \binom{n_2}{m_2}\binom{n_1}{m_1} = \sum_{i=0}^{n_2} \binom{n_2}{i}\binom{n_1}{m_1-m_2+i} = \binom{n_1+n_2}{\frac{n_1+n_2}{2}}.$$

If n_1 and n_2 are not both even, the proof is similar. This completes the proof of Theorem 1.

THEOREM 2. *If* $x_1, x_2, \ldots, x_n$ *are complex numbers,* $|x_i| \geqq 1$ *and* $\varepsilon_i = \pm 1$, *then the number of sums* $\sum_{i=1}^{n} \varepsilon_i x_i$ *lying in the interior of an arbitrary circle of radius* 1 *is at most* $\left(\begin{array}{c} n \\ \left[\frac{n}{2} \right] \end{array} \right)$.

PROOF. We may assume $\mathrm{Re}\, x_i \geqq 0$ $(1 \leqq i \leqq n)$, since multiplying by (-1) one of x_i's, the set of sums does not change. Let $x_1, x_2, \ldots, x_n$ be ordered in such way, that $\mathrm{Im}\, x_i \geqq 0$ $(1 \leqq i \leqq n_1)$ and $\mathrm{Im}\, x_i < 0$ $(n_1 < i \leqq n)$ and $n_2 = n - n_1$. We may assume in addition $n_1 \geqq n_2$ since the set of sums $\sum_{i=1}^{n} \varepsilon_i \bar{x}_i$ is equal to the set of $\sum_{i=1}^{n} \varepsilon_i x_i$'s.

Let K be a fixed circle of radius 1, and consider the subsets $a = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ of the set $\{x_1, x_2 \ldots x_n\}$, for which the sum $\sum_{x_j \in a} x_j - \sum_{x_j \notin a} x_j$ is in the interior of K. The system of these subsets, of $\{x_1, x_2, \ldots, x_n\}$ satisfies the condition of Theorem 1, if $H_1 = \{x_1, x_2, \ldots, x_{n_1}\}$ and $H_2 = \{x_{n_1+1}, \ldots, x_n\}$. Indeed, if we have two sums $\sum_{k=1}^{n} \varepsilon_k x_k$ and $\sum_{k=1}^{n} \varepsilon_k' x_k$ such that $\varepsilon = \varepsilon_k'$ $(1 \leqq k \leqq n_1)$ and $\varepsilon_k' = 1$ always holds if $\varepsilon_k = 1$, then

$$\sum_{k=1}^{n} \varepsilon_k' x_k - \sum_{k=1}^{n} \varepsilon_k x_k = \sum_{k=1}^{n} (\varepsilon_k' - \varepsilon_k) x_k = \sum_{n_1+1}^{n} (\varepsilon_k' - \varepsilon_k) x_k, \tag{4}$$

where $\varepsilon_k' - \varepsilon_k$ is 0 or 2. Moreover, if u and v are complex numbers, such that $\mathrm{Re}\, u \geqq 0$, $\mathrm{Re}\, v \geqq 0$, $\mathrm{Im}\, u < 0$, $\mathrm{Im}\, v < 0$, then obviously

$$|u + v| \geqq \min \{|u|, |v|\}.$$

Thus applying (4)

$$\left| \sum_{k=1}^{n} \varepsilon_k' x_k - \sum_{k=1}^{n} \varepsilon_k x_k \right| = \left| \sum_{n_1+1}^{n} (\varepsilon_k' - \varepsilon_k) x_k \right| \geqq \min_{k = n_1+1, \ldots, n} \{|2x_k|\} \geqq 2,$$

that is $\sum_{k=1}^{n} \varepsilon_k' x_k$ and $\sum_{k=1}^{n} \varepsilon_k x_k$ can not be in K. Consequently we may use Theorem1, and this completes our proof.

It is easy to see, that Theorem 1 does not remain true if we divide the set H into three or more parts. For example, if $H_1 = \{1, 2, \ldots n-2\}$ $H_2 = \{n-1\}$, $H_3 = \{n\}$, and n is odd, let the system A consist of the subsets a's, b's, c's, d's such that

$$|a \cap H_1| = \frac{n-3}{2} \qquad |a \cap H_2| = 0 \qquad |a \cap H_3| = 0$$

$$|b \cap H_1| = \frac{n-3}{2} \qquad |b \cap H_2| = 1 \qquad |b \cap H_3| = 1$$

$$|c \cap H_1| = \frac{n-1}{2} \qquad |c \cap H_2| = 0 \qquad |c \cap H_3| = 1$$

$$|d \cap H_1| = \frac{n-1}{2} \qquad |d \cap H_2| = 1 \qquad |d \cap H_3| = 0.$$

This system does not have two elements e and f, satisfying

$$e \cap H_1 = f \cap H_1 \quad e \cap H_2 = f \cap H_2 \quad e \cap H_3 \supset f \cap H_3$$

or

$$e \cap H_1 = f \cap H_1 \quad e \cap H_2 \supset f \cap H_2 \quad e \cap H_3 = f \cap H_3$$

or

$$e \cap H_1 \supset f \cap H_1 \quad e \cap H_2 = f \cap H_2 \quad e \cap H_3 = f \cap H_3,$$

moreover

$$|A| = 4 \binom{n-2}{\frac{n-3}{2}} = 2 \binom{n-1}{\frac{n-1}{2}} > \binom{n}{\frac{n-1}{2}}.$$

Thus A is an example, that the generalization of Theorem 1. is not true, and this explains why it is not possible to prove ERDŐS's conjecture for the n-dimensional space in the same way, as for $n=2$.

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REFERENCES

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