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EXTREMAL PROBLEMS FOR HYPERGRAPHS

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By a *hypergraph* we mean a pair $(V,\mathcal{A})$, where V is a finite set, and $\mathcal{A}=\{A_1,\ldots,A_m\}$ is a family of its different subsets. $|V|$ means the number of elements of V; this is usually denoted simply by n. Similarly, $|\mathcal{A}|=m$. The elements of V are called *vertices*, the elements of $\mathcal{A}$ are the *edges*.

We use the term hypergraph, because it becomes more and more familiar, but the questions concerned here did not develop directly from the theory of graphs (with some exceptions); the particular cases of these theorems give usually trivialities for graphs.

A hypergraph is a k-*graph* if $|A|=k$ holds for all $A\in\mathcal{A}$. $(V,\mathcal{A})$ is a *complete* k-*graph* if $\mathcal{A}$ consists of all the k-tuples of V.

In this paper we try to give a survey of some extremal problems of hypergraphs, namely, the problems developed from SPERNER's [74] theorem. We shall mention briefly some other areas, too. On the other hand we give some remarks on the possible generalizations for more general structures.

We have the feeling, that the classification of the problems in this paper is not good. However, the various questions are connected in many ways, thus the only proper way of classification would be a graph whose vertices are the problems and the "connected" problems are connected. (The most interesting question concerning this graph would be "how to get nice new vertices?")

For the interested readers it is suggested to read the survey paper of ERDÖS & KLEITMAN [21] on this subject, since our paper contains it only partly.

1. $|V|$ IS FIXED, MAXIMIZE $|\mathcal{A}|$

The typical problem of this type: A set of conditions is given on $\mathcal{A}$, and we are interested in determining the maximum (minimum) of $m=|\mathcal{A}|$ if

$n = |V|$ is fixed and (V,A) runs over all the possible A's satisfying the given conditions.

The origin of these theorems is the well-known theorem of SPERNER [74].

THEOREM 1. *If* (V,A) *satisfies* $A_i \not\subset A_j$ $(i \neq j)$, *then*

$$(1) \qquad m \le \binom{n}{[n/2]} ,$$

where equality holds for the complete $[n/2]$*-graph.*

The following beautiful proof is due to LUBELL.

PROOF. $C = \{C_0, \dots, C_n\}$ is called a *complete chain*, if $C_0 \subset C_1 \subset \dots \subset C_n$. ($\subset$ denotes inclusion without $=$); ($|C_i| = i$ follows). Let us count in two different ways the number of pairs (C, A_i), where $A_i \in A$ and $A_i = C_j \in C$ for some j. For a given A_i, C_j must be equal to $C_{|A_i|}$, we have $|A_i|!$ possibilities in choosing $C_0, C_1, \dots, C_{|A_i|-1}$, and $(n - |A_i|)!$ possibilities for $C_{|A_i|+1}, \dots, C_n$. The number of possible C's is $|A_i|!\,(n - |A_i|)!$, and the total number of pairs (C, A_i) is $\sum_{i=1}^m |A_i|!\,(n - |A_i|)!$. On the other hand, fixing C, there is at most one A_i since $A_i = C_j \subset C_k = A_l$ would contradict the condition given on A. Thus, the number of pairs (C, A_i) is at most $n!$, the total number of C's. We obtain the inequality

$$\sum_{i=1}^{m} |A_i|!\,(n - |A_i|)! \le n!$$

or

$$(2) \qquad \sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \le 1 .$$

(1) follows from (2) easily, using

$$\binom{n}{|A_i|} \le \binom{n}{[n/2]} .$$

The proof is completed. □

Equation (2) (which was discovered by LUBELL [67], MESHALKIN [68] and YAMAMOTO [77]) is perhaps more important than (1) itself. If $\sum_{i=1}^m f(|A_i|)$, where f is an arbitrary function, is maximized, then the maximum is attained by the complete k-graph, where k is defined by

$$f(k)\binom{n}{k} = \max_{0\le i\le n} f(i)\binom{n}{i} .$$

The proof of this statement (cf. [58,45]) easily follows from (2)

$$1 \ge \sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} = \sum_{i=1}^{m} \frac{f(|A_i|)}{f(|A_i|)\binom{n}{|A_i|}} \ge \sum_{i=1}^{m} \frac{f(|A_i|)}{f(k)\binom{n}{k}} ,$$

that is,

$$\sum_{i=1}^{m} f(|A_i|) \le \binom{n}{k} f(k) .$$

In some other cases LUBELL's method works again. In order to show, what properties of $\mathcal{C}$ are used in general, (may be) it is worthwhile to formulate the method as a separate lemma. $(W',\mathcal{B}')$ is called a *sub-hypergraph* of $(W,\mathcal{B})$ if $W'\subseteq W$ and $\mathcal{B}'\subseteq\mathcal{B}$. $(W',\mathcal{B}')$ is a *spanned sub-hypergraph* if $\mathcal{B}'=\{B:\ B\subseteq W',\ B\in\mathcal{B}\}$. We say that U is an *independent set* in $(W,\mathcal{B})$ if $U\subseteq W$, and there is no $B\in\mathcal{B}$ such that $B\subseteq U$.

LEMMA 1. *Let* $(W_1,\mathcal{B}_1),\dots,(W_z,\mathcal{B}_z)$ *be spanned sub-hypergraphs of* $(W,\mathcal{B})$, *the maximal number of independent elements being* $f_1,\dots,f_z$ *and* f, *respectively. Then*

$$(3)\qquad f \le \frac{\sum_{i=1}^{z} f_i}{\min_{w\in W} |\{i:\ w\in W_i\}|} .$$

If, additionally, $|W_1| = \dots = |W_z|$, $(W_1,\mathcal{B}_1),\dots,(W_z,\mathcal{B}_z)$ *are isomorphic, and* $|\{i:\ w\in W_i\}|$ *does not depend on* w, *then*

$$(4)\qquad \frac{f}{|W|} \le \frac{f_i}{|W_i|} .$$

PROOF. Let $F\subseteq W$ $(|F|=f)$ be an independent set in $(W,\mathcal{B})$. Let us count in two different ways the number of pairs $((W_i,\mathcal{B}_i),w)$ where $w\in F$ and $w\in W_i$. For a given $w\in F$ there are $|\{i:\ w\in W_i\}|$ sub-hypergraphs, thus the total number is $\sum_{w\in F} |\{i:\ w\in W_i\}|$. On the other hand, fixing a sub-hypergraph $(W_i,\mathcal{B}_i)$, the maximal number of w's satisfying $w \in W_i$ can be f_i. Thus the number of pairs is at most $\sum_{i=1}^{z} f_i$. The resulting inequality

$$(5) \qquad \sum_{w \in F} |\{i: w \in W_i\}| \le \sum_{i=1}^{z} f_i .$$

However, since

$$f \min_{w \in W} |\{i: w \in W_i\}| \le \sum_{w \in F} |\{i: w \in W_i\}| ,$$

the inequality (3) follows from (5).

Using the additional suppositions

$$\sum_{w \in W} |\{i: w \in W_i\}| = \sum_{i=1}^{z} |W_i| = z|W_i| ,$$

and

$$|\{i: w \in W_i\}| = \frac{z|W_i|}{|W|} .$$

On the other hand $\sum_{i=1}^{z} f_i = zf_i$. Substituting this result into (3) the inequality (4) is obtained, which completes the proof. □

How to apply this lemma to our problems? W equals 2^V (the power set of V) and $\mathcal{B}$ consists of the subsets of 2^V which are excluded by the given condition. If the conditions exclude only elements and pairs of elements of 2^V, then $(W,\mathcal{B})$ is a simple graph. For instance, in the case of SPERNER's theorem: two vertices $A_i, A_j \in W$ are connected iff $A_i \subset A_j$ or $A_j \subset A_i$. For $W_1, \dots, W_z$ we choose all possible chains $\mathcal{C}$ given in LUBELL's proof. In this case (5) leads to (2), and (3) leads to (1).

The next natural condition (see [19]) for $\mathcal{A}$ is

$$(6) \qquad A_i \cap A_j \ne \emptyset .$$

This question is, however, trivial: $\mathcal{A}$ can contain at most one of the sets A, V-A, thus $|\mathcal{A}| \le$ half the number of all subsets of V:

$$(7) \qquad |\mathcal{A}| \le \frac{2^n}{2} = 2^{n-1} .$$

(The application of lemma 1 gives the same if we take $W_i = \{A_i, V-A_i\}$ for all $A_i \in 2^V$; then (4) gives (7).) This is the best possible bound: $\mathcal{A} = \{A\colon v \in A, v \in V, v \text{ fixed}\}$ gives equality in (7).

The other classical theorem (ERDÖS, KO & RADO [19]) solves the problem for a combined condition, with a small modification.

THEOREM 2. *If* $(V, \mathcal{A})$ *is a hypergraph satisfying the condition*

$$\text{(8)} \qquad A_i \not\subset A_j, A_i \cap A_j \neq \emptyset, \ |A_i| \leq k \ \text{ if } A_i, A_j \in \mathcal{A} \ (i \neq j),$$

where $k \leq \frac{n}{2}$, *then*

$$\text{(9)} \qquad m \leq \binom{n-1}{k-1},$$

and this is the best possible bound.

PROOF. First the constructions concerning (9):

$$\mathcal{A} = \{A\colon |A| = k, v \in A, v \in V, v \text{ fixed}\}.$$

In the proof lemma 1 is used again. W consists of all elements of 2^V having at least k elements. $(W, \mathcal{B})$ is a simple graph. Two different vertices A, A' are connected iff $A \subset A'$, $A \supset A'$ or $A \cap A' = \emptyset$. W_i's are defined in the following way. Let us consider all possible cyclic orderings of V. W_i consists of all subsets of V with size $\leq k$, and with consecutive elements according to the i-th ordering. The $(W_i, \mathcal{B}_i)$'s are isomorphic, f_i does not depend on i.

We shall show that $f_i \leq k$ if $k \leq n/2$. Fix the i-th cyclic ordering $v_1, \ldots, v_n$ (the indices are mod n), and suppose $w_1, \ldots, w_{f_i}$ are independent vertices in $(W_i, \mathcal{B}_i)$. By the symmetry we can suppose $w_1 = \{v_1, \ldots, v_n\}$. If the first and last elements of a w_j are outside w_1 then either $w_j \supset w_1$, or $w_j \cap w_1 = \emptyset$ holds. Then the first or last element of each w_j is in w_1. Fix an l, $(1 \leq l < r)$, and consider all sets $A \in W_i$, the last element of which is v_l or the first element of which is v_{l+1}. These vertices are all connected in $(W, \mathcal{B})$ (or in $(W_i, \mathcal{B}_i)$), thus there is at most one w_j among them. Altogether, we have at most $(r-1)$ w_j's with last element from $v_1, \ldots, v_{r-1}$ or with first element from $v_2, \ldots, v_r$. v_1 can be the first element of w_1, only. (Other $A \in W_i$ with this property either contain or are contained in w_1.)

The same holds for the w_j's having v_1 as a last element. We obtained $f_i \le r \le k$.

We need $|\{i: A \in W_i\}| = |A|!\,(n-|A|)!$. This is simply the number of cyclic orderings in which A has consecutive members. (5) gives the following inequality:

$$\text{(10)} \qquad \sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \le \frac{k}{n} \qquad \text{if } k \le \frac{n}{2}$$

and hence, using that in the case $|A_i| \le k \le n/2$,

$$\binom{n}{|A_i|} \le \binom{n}{k}$$

holds, we obtain (9), and the proof is completed. □

This proof is a stronger version of the proof given in [42]. By (10) it is also easy to determine max $\sum_{i=1}^{m} f(A_i)$ under (8).

An obvious question: what happens if the condition $|A_i| \le k$ is omitted (or more generally, $n/2 < k \le n$). If n is odd, then theorem 1 gives the estimation $\binom{n}{(n+1)/2}$, and the complete $\frac{n+1}{2}$-graph satisfies the conditions. The case of even n is solved by BRACE & DAYKIN [2].

Another type of conditions is $A_i \cup A_j \ne V$. This does not seem to be a new condition, since it is equivalent to $(V-A_i) \cap (V-A_j) \ne \emptyset$. However, in some combinations of conditions we can not use the complement sets. For instance if

$$A_i \cap A_j \ne \emptyset \quad \text{and} \quad A_i \cup A_j \ne V,$$

this is the case. Under this condition $m \le 2^{n-2}$, as DAYKIN & LOVÁSZ [12] proved; equality holds with $\mathcal{A} = \{A: v \in A,\ w \notin A$, where $v \ne w$ are fixed elements of $V\}$.

The next type of conditions is the constraint on the sizes of $A_i \cap A_j$ or $A_i \cup A_j$ $(i \ne j)$ (perhaps of $A_i \cap A_j \cap A_l$, and so on). An example: in [19] the following condition is considered

$$\text{(11)} \qquad |A_i| = k, \quad |A_i \cap A_j| \ge l, \quad (k \ge l).$$

The result [19]: if n is large enough (relatively to k and l), then

(12) $$m \le \binom{n-1}{k-1},$$

where equality holds for $\mathcal{A} = \{A\colon L \subset A \text{ where } |L| = l,\ L \text{ a fixed subset of } V\}$. The result does not hold for small n, as the following example shows (given by MIN): $n=8$, $k=4$, $l=2$, $\mathcal{A}=\{A\colon |A|=4,\ |A\cap\{1,2,3,4\}|=3\}$, $m = 16 > \binom{6}{2}$. This result gives a good example for the case, that sometimes the exact formulas are valid only for large values.

There is a large class of problems, where the solution (the extremal hypergraph) can be constructed by finite geometries or block designs. We shall not consider these problems, because their methods are completely different from the problems treated here. Thus, we do not investigate (with some exceptions) the conditions of such type, where $|A_i \cap A_j|$ has to be small, or $|A_i - A_j|$ has to be large. However, the questions (11)-(12) give an opportunity for a glimpse at the connections between the two areas. Consider the case $k=3$, $l=2$ (in this simple case (12) holds if $n \ge 6$ [39]). A Steiner triple system is a 3-graph $(V,\mathcal{C})$ with the property, that each pair $v,w \in V$ $(v \ne w)$ is contained by exactly one $C \in \mathcal{C}$. It is well known [71], that such a system exists iff $n \equiv 1$ or $3 \pmod 6$. Use lemma 1; W consists of all the triples of V; w_1 and w_2 are connected in $\mathcal{B}$ iff $|w_1 \cap w_2| < 2$. W_i consists of the triples arising from a fixed Steiner triple system by the i-th permutation of V. It is easy to see, that $(W_i, \mathcal{B}_i)$ is a complete graph, so $f_i = 1$. Trivially, $|W_i| = \binom{n}{2}/3$, $|W| = \binom{n}{3}$, thus (4) gives $f \le n-2$, and this is (12) for $k=3$, $l=2$.

By the combinations of the above conditions we obtain a lot of problems. We try to list some of them.

If

(13) $$A_i \not\subset A_j,\quad A_i \cap A_j \ne \emptyset,\quad A_i \cup A_j \ne V,$$

then [2] (see also [45,59]) gives

(14) $$m \le \binom{n-1}{[(n-2)/2]}.$$

If

(15) $$|A_i \cap A_j| \ge 1,$$

then [39] gives

(16) $$m \le \sum_{i=\frac{l+n}{2}}^{n} \binom{n}{i} \quad \text{if } n+l \text{ is even}$$

and

(17) $$m \le \binom{n-1}{\frac{n+l-1}{2}} + \sum_{i=\frac{n+l+1}{2}}^{n} \binom{n}{i} \quad \text{if } n+l \text{ is odd.}$$

If

(18) $$A_i \not\subset A_j, \ |A_i \cap A_j| \ge l ,$$

then [69] gives

(19) $$m \le \binom{n}{[(n+l+1)/2]} .$$

Let $1 \le k \le n$ and $1 \le h \le \min(k,n-k)$, and suppose

(20) $$A_i \cap A_j \ne \emptyset, \quad h \le |A_i| \le k,$$

then [36] gives

(21) $$m \le \sum_{i=h}^{k} \binom{m-1}{i-1}.$$

If $1 \le k \le n$, and there is no pair $i \ne j$ such that

(22) $$A_i \supset A_j \quad \text{and} \quad |A_i - A_j| \ge k,$$

then [17] gives

(23) $m \le$ (the sum of k largest binomial coefficients of order n).

Conversely, if there is no pair satisfying

(24) $$A_i \supset A_j \quad \text{and} \quad |A_i - A_j| < k ,$$

then [43] gives

(25) $$m \le \sum_{i \equiv [n/2] \pmod{k}} \binom{n}{i} .$$

Concerning the combinations which are missing, three cases can happen.

1) It is an easy consequence of another one.

2) The author of this paper does not know the result.

3) It is a nice open problem.

An example for case 3):

If $|A_i \cap A_j| \ge 1$ but there is no pair with $A_i \cup A_j = V$, then probably the inequalities (16) and (17) hold with $n-1$ rather than n. (We can not give examples for case 2).)

2. CONDITIONS VARYING ON A WIDER SCALE

In this section we consider the same kind of problems as in section 1, but the *conditions vary on a wider scale.*

The most general form of theorem 2 (and (11)-(12)) is the following theorem of HAJNAL & ROTHSCHILD [29].

If

(26) $$\begin{cases} |A_i| = k, \text{ and for any } i_1,\dots,i_{r+1} \\ \text{there are } i_j \text{ and } i_h \text{ with } |A_{i_j} \cap A_{i_h}| \ge l, \end{cases}$$

then

(27) $$m \le \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \binom{n-il}{k-il},$$

provided n *is large enough* $(n \ge n(k,r,l))$.

What are the best values for $n(k,r,l)$? By theorem 2, $n(k,1,1) = 2k$. For the cases of $n(k,1,l)$ we can not expect a nice smallest value. The estimations of [19] are improved in [37]. The hopeful case is $n(k,r,1)$. For instance, $n(k,2,1) = 3k+1$ might be true.

The same question without $|A_i| = k$, and only for $l = 1$ is solved by KLEITMAN [55]. So, if for any $i_1,\dots,i_{r+1}$ there is a pair i_j, i_h such that

(28) $$A_{i_j} \cap A_{i_h} \neq \emptyset$$

and $n = (r+1)q$, then

(29) $$m \le \sum_{i=q+1}^{(r+1)q} \binom{(r+1)q}{i} + \binom{(r+1)q}{q} \frac{r}{r+1}.$$

If $n = (r+1)q-1$, another exact estimation is given. For other n's there is a small gap between the estimations and the constructions [55].

An obvious open question is the case $l > 1$ ($|A_{i_j} \cap A_{i_h}| \ge l$). This is solved only for $r = 1$ (see (15)-(17)).

A third variant of these questions was posed by D. PETZ and solved by P. FRANKL [27] (students in Budapest):

If

(30) $$A_i \not\supseteq A_j \quad \text{and} \quad |A_{i_1} \cup \ldots \cup A_{i_r}| \le qr+s$$

where $0 \le s < r$, then

(31) $$m \le \sum_{i=0}^{\min(q,s/2)} \binom{n-s}{q-i} \binom{s}{[s/2]+i},$$

provided n is large enough depending on r and qr+s. The construction: let $C \subset V$, $|C| = s$, then $\mathcal{A} = \{A: |A| = q+[s/2],\ |A \cap (V-C)| \le q\}$. The cases $s = 0$ and $s = 1$ are solved independently by E. BOROS. Observe that (30)-(31) is a generalization of (18)-(19) using the complement set. In (18)-(19) $r = 2$, $s = 0$ or 1.

It seems that in (9) equality can hold ($k < n/2$) only for the given extremal hypergraph (all the A's containing a given $v \in V$). In [19] it is asked, what happens, if we exclude this extremal hypergraph, or suppose $\bigcap_{i=1}^{m} A_i = \emptyset$. HILTON & MILNER [33] have given the answer:

(32) $$m \le 1 + \binom{n-1}{k-1} + \binom{m-k-1}{k-1}.$$

They have more general theorems, too: If $1 \le \min(3,s+1) \le k \le n/2$, and $|A_i| \le k$, $A_i \not\subseteq A_j$ $(i \neq j)$, $A_i \cap A_j \neq \emptyset$,

(33) $$A_{i_1} \cap \ldots \cap A_{i_{m-s+1}} = \emptyset$$

for any different indices $i_1,\dots,i_{m-s+1}$, then

(34) $$m \le \begin{cases} \binom{n-1}{k-1} - \binom{n-k}{k-1} + n-k & \text{if } 2 < k \le s+2, \\ s + \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-s}{k-s-1} & \text{if } k \le 2 \text{ or } k \ge s+2. \end{cases}$$

A combination of (32) and (20)-(21) is given in [36]:

Let $1 \le k \le n-1$, $1 \le h \le \min(k,n-k)$. If

(35) $$A_i \cap A_j \ne \emptyset,\ h \le |A_i| \le k \text{ and } \bigcap_{i=1}^{m} A_i = \emptyset,$$

then

(36) $$m \le 1 + \sum_{i=h}^{k}\left[\binom{n-1}{i-1} + \binom{n-k-1}{i-1}\right].$$

The following three results [36] are modifications of the above ones, when besides $A_1,\dots,A_m$ there is an additional edge B of our hypergraph with slightly different conditions. Let h and k satisfy $1 \le k \le n/2$, $1 \le h \le n-1$. If $A_i \cap A_j \ne \emptyset$, $A_i \cap B \ne \emptyset$, $|A_i| \le k$, $|B| = h$, $A_i \not\subseteq A_j$, $A_i \subsetneq B$ ($B \subset A_i$ is not excluded), then

$$m \le \begin{cases} \binom{n-1}{k-1} - \binom{n-h-1}{k-1} & \text{if } h \ge n/2, \\ \binom{n-1}{k-1} - \binom{h-1}{k-1} & \text{if } h < n/2. \end{cases}$$

If $A_i \cap A_j \ne \emptyset$, $A_i \cap B \ne \emptyset$, $|A_i| \le k$, $|B| = h$, $A_i \not\subseteq A_j$, $B \cap A_1 \cap \dots \cap A_m = \emptyset$, then

$$m \le \begin{cases} \binom{n-1}{k-1} - \binom{n-h-1}{k-1} & \text{if } k \le h, \\ 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1} & \text{if } h < k. \end{cases}$$

Finally, if $A_i \cap A_j \ne \emptyset$, $A_i \cap B \ne \emptyset$, $g \le |A_i| \le k$, $|B| = h$, $B \cap A_1 \cap \dots \cap A_n = \emptyset$, then

$$m \le \begin{cases} \sum_{i=g}^{k} \left[\binom{n-1}{i-1} - \binom{n-h-1}{i-1} \right] & \text{if } k \le h, \\ 1 + \sum_{i=g}^{k} \left[\binom{m-1}{i-1} - \binom{m-k-1}{i-1} \right] & \text{if } h < k. \end{cases}$$

In a paper of HILTON [35] the concept of the *simultaneously disjoint pairs of edges* is defined by

$$A_{i_1} \cap A_{j_1} = \dots = A_{i_s} \cap A_{j_s} = \emptyset .$$

Let $2 \le 2k \le m$ and $s \le \binom{n-k-1}{k-1} - 1$. If

(37) $\begin{cases} |A_i| \le k,\ A_i \not\subset A_j \text{ and there are no } s+1 \\ \text{simultaneously disjoint pairs of edges,} \end{cases}$

then (cf. [35])

(38) $$m \le \binom{n-1}{k-1} + s.$$

If $A_i \subset A_j$ is allowed, then we obtain (cf.[35])

$$m \le \sum_{i=1}^{k} \binom{n-1}{i-1} + s.$$

Or more generally [32], if $h \le |A_i| \le k$, then

$$m \le \sum_{i=h}^{k} \binom{n-1}{i-1} + s.$$

(For a recent result of this type see [11].)

Similar results are obtained in [32] in case we exclude the existence of s+1 simultaneously disjoint r-tuples of edges. For s = 0 it was solved earlier by ERDÖS & GALLAI [18] for 2-graphs and later by ERDÖS [20] for k-graphs. ERDÖS' case is also included by (26)-(27), but the common generalization of HAJNAL & ROTHSCHILD [29] and HILTON [32] is still open.

3. WEAKENING THE CONDITIONS

Could we weaken the conditions of our theorems with the same conclusions? In this section we give examples for that.

First KLEITMAN [49] and KATONA [40] independently observed, that if we fix a partition $V_0 \cup V_1 = V$, $(V_0 \cap V_1 = \emptyset)$ of V and we exclude the edges satisfying

(39) $$A_i \cap V_\delta = A_j \cap V_\delta \quad \text{and} \quad A_i \cap V_{1-\delta} \subset A_j \cap V_{1-\delta}$$

(instead of $A_i \subset A_j$), then under this weaker condition the conclusion

(40) $$m \le \binom{n}{[n/2]}$$

remains the same.

A natural question: what happens for the partition $V_0 \cup V_1 \cup V_2 = V$ $(V_0, V_1, V_2$ are pairwise disjoint), if we exclude edges equal in two V_i's and containing each other in the third? The answer is disappointing: m can be larger than $\binom{n}{[n/2]}$. In [47] an additional condition is given, under which (40) remains true. This additional condition is rather complicated. It excludes some 4-tuples of edges of the hypergraph. Recently, GREENE & KLEITMAN [28] determined weak conditions from the symmetric chain method (see [3]).

A combination of (39) and (22) is given in [44], and a combination of (39) and (24) in [43]. Recent generalizations of this type can be found in [60].

A question: how could we weaken the conditions of theorem 2 with the same conclusion?

4. ONE CONDITION CONTAINING MORE OPERATIONS OR RELATIONS

In this section we treat the problems where *one condition contains more operations or relations*.

Probably the oldest result of this type is due to KLEITMAN [56]. If there is no triple satisfying

$$A_i \cap A_j = \emptyset \quad \text{and} \quad A_i \cup A_j = A_h$$

simultaneously, then

$$m \leq \sum_{i=r+1}^{2r+1} \binom{n}{i},$$

provided $n = 3r+1$, and this is the best estimation. For $n = 3r$ and $n = 3r+2$ the results are near best possible.

Another problem: there are no 4 different edges in the hypergraph satisfying both

$$A_i \cup A_j = A_k \quad \text{and} \quad A_i \cap A_j = A_l.$$

ERDÖS & KLEITMAN [24] have constructed $c_1 \frac{2^n}{n^{\frac{1}{4}}}$ edges with this condition and they proved that

$$m \leq c_2 \frac{2^n}{n^{\frac{1}{4}}}$$

but $c_1 < c_2$.

Many obvious general questions can be asked.

In the next problem $|V|$ is not fixed, but we list it here, because its character is similar to the other problems treated here. Now $|A| = m$ is fixed and $f(m)$ is the largest number such that there are always $f(m)$ edges in the hypergraph no three different ones of them having the property

$$A_i \cup B_i = C_i.$$

The first result is given by KLEITMAN [50]:

$$f(m) \leq cm\sqrt{\log m}\ .$$

J. RIDDEL proved $\sqrt{m} < f(m)$, and finally ERDŐS & KOMLÓS [22] determined

$$f(m) < 2\sqrt{2n}+4.$$

BOLLOBÁS proved for 3-graphs that if

(41) there are no three different edges $A_h \supseteq (A_i - A_j) \cup (A_j - A_i)$,

then

$$m \leq (\frac{n}{3})^3$$

if $3|n$. The hypergraph with equality: V is divided into 3 equal parts, and we choose the edges having exactly one vertex from each part.

It is conjectured also by BOLLOBÁS, that a similar for k-graphs. For 2-graphs it is a particular case of TURÁN's graph theorem [76]. A conjecture of ERDÖS & KATONA: Under the condition (41) (without size restrictions) the best hypergraph can be constructed in the following way. Divide V into $[\frac{n}{3}]$ classes of 3 and 2 elements, and choose those edges which contain exactly one vertex from each class.

5. MISCELLANY

We will treat three further problems which do not really fit into any of these sections. The first question was proposed by RÉNYI [70]. The edges of the hypergraph are called *qualitatively independent* if

(42) $A_i \cap A_j,\ A_i \cap \bar{A}_j,\ \bar{A}_i \cap A_j,\ \bar{A}_i \cap \bar{A}_j$

are all non-empty. What is the maximum of m under this condition? The answer is

$$m \leq \binom{n-1}{[(n-2)/2]}.$$

This is an easy consequence of theorems 1 and 2, as it is pointed out by KLEITMAN & SPENCER [59] and independently in [45]. (Observe, that (13) and (42) are equivalent, thus [2] also gives the solution.) In [59] a harder problem is also considered. We say, the edges are k-*qualitatively independent* if

$$A_{i_1}^{\delta_1} \cap \ldots \cap A_{i_k}^{\delta_k} \neq \emptyset$$

for any different $i_1,\ldots,i_k$, where A^δ is either A or $\bar{A} = V-A$. Under this condition

$$m \le 2^{c\frac{n}{2^k}}$$

and a hypergraph is constructed with

$$2^{d\frac{n}{k2^k}}$$

edges, where c and d are constants, k fixed and $n \to \infty$.

An unsolved question: maximize m under the condition that any of (42) has a size $\ge r$.

The density of a hypergraph was defined by ERDÖS. It is the largest s such that there is a $U \subseteq V$ such that $|U| = s$ and $|A \cap U| = 2^s$. SAUER [72] proved, that supposing

$$s \le k,$$

we obtain

$$m \le \sum_{i=0}^{k} \binom{n}{i}.$$

A similar problem of ERDÖS & KATONA: what is the maximum of m under the condition that $|A_i \cap A_j|$ are all different $(1 \le i < j \le m)$?

A new area of problems is considered in [2]. The valency $v = v((V,A))$ of a hypergraph is the minimal valency of its vertices. In [2] the maximum of v is asked for under several conditions.

If $A_i \cap A_j \ne 0$, then

$$v \le \begin{cases} 2^{n-2} + \frac{1}{2}\binom{n-1}{(n-1)/2} & \text{if n is odd,} \\ 2^{n-2} + [\frac{1}{4}\binom{n}{n/2}] & \text{if n is even.} \end{cases}$$

If $A_i \not\subseteq A_j$, then

$$v \le \binom{n-1}{[(n-1)/2]}$$

and if $A_i \not\subset A_j$, $A_i \cap A_j = \emptyset$, then the same holds.

6. THE PROBELMS WE SHALL NOT CONSIDER HERE

These problems -although they have many points in common with our subject- require different methods, and are approached from various points of view. These problems are also extremal problems for hypergraphs, but this concept is too wide.

1) If $A_i \cap A_j$ is small, $A_i - A_j$ or $(A_i - A_j) \cup (A_j - A_i)$ are large, the problems are usually *coding problems*. Their methods are closer to block designs and finite geometries.

2) *Covering problems*. Usually a smallest family of edges is sought under some conditions, covering all the edges of a given hypergraph. In 1) and 2) the solutions give hypergraphs where the edges are "far" from each other, in our cases they are "close".

3) *Ramsey type theorems*. See the paper of GRAHAM & ROTHSCHILD in this tract (pp. 61-76).

4) *Turán type theorems*. Certain generalizations are very near (see [46]).

5) *Combinatorial search problems*. They are closely related to the coding problems (see [46]).

6) We did not touch the question of the number of optimal hypergraphs. In many cases there is only one. In some other cases it is an open problem how many of them exist. A closely related problem: how many hypergraphs do we have under several conditions? For these questions see [21].

7) *Δ-systems and* B-*property*. A hypergraph is a *strong Δ-system* if $A_i \cap A_j$ $(i \neq j)$ does not depend on i and j. In the case of a *weak Δ-system* $|A_i \cap A_j|$ $(i \neq j)$ is independent of i and j. $f_s(k,l)$ denotes the minimum of $|A|$ with the property that in the case $|A_i| = k$, $(1 \le i \le m)$, there are always l A_i's forming a strong Δ-system. $f_w(k,l)$ denotes the same for weak Δ-systems. There are lower and upper estimations for $f_s(k,l)$ and $f_w(k,l)$. We say that (V,A) has *property* B, if there is a set $B \subset V$ such that $|B \cap A_i| \ge 1$ but $B \supset A_i$ $(1 \le i \le m)$. The questions concerning Δ-systems and the B-property are closely related to our problems; however, ERDÖS [25] has recently published a survey paper on this subject.

7. $|\mathcal{A}|$ IS FIXED

Perhaps, the main feature of the problems in this section is not $|\mathcal{A}|$ being fixed, because in many cases we obtain an inequality and in an inequality usually it is not important, which variable is fixed and which one is not. However, the problems treated here -as we shall see- have a definitely different character.

SPERNER's theorem says, if we have $\binom{n}{[n/2]}+1$ edges in a hypergraph (with $|V|=n$), then there is a pair of different edges $A_i \subset A_j$. Observe, however, that adding one edge to the complete $[n/2]$-graph there are always more pairs with $A_i \subset A_j$. What is the minimum? More generally, if m and n are fixed, what is the minimal number of pairs $A_i \subset A_j$? The solution is given by KLEITMAN [51]. The optimal hypergraph is constructed easily. Order all subsets of V, first take all $[n/2]$-tuples, then all $[n/2]+1$-tuples, all $[n/2]-1$-tuples, all $[n/2]+2$-tuples, and so on. The edges of the optimal hypergraph are the first m subsets according to this order.

The corresponding question is not solved yet, not even for the case of (15)-(16). This latter one can not be too hard for $l=1$. The optimal hypergraph could be constructed by taking the subsets of V according to their sizes, starting from n. (For the case of theorem 2 see later in this section.)

Let $(V,\mathcal{A})$ now be a k-graph, and let $\mathcal{C}(\mathcal{A})$ denote the family of subsets $C: C \subset A$ for an $A \in \mathcal{A}$ and $|C| = k-1$. SPERNER [74] used in his proof the easy fact

$$|\mathcal{C}(\mathcal{A})| \geq \frac{|\mathcal{A}|\cdot k}{n-k+1} .$$

The question arises, what is $\min|\mathcal{C}(\mathcal{A})|$ if n,k,m are fixed ($m \leq \binom{n}{k}$). The construction of the optimal k-graph is as follows. Fix an order $v_1,\dots,v_n$ of the vertices in V. Form a sequence of 0's and 1's in the usual way from each k-set of V. The first m sequences in the lexicographic order give the optimal k-graph. A formula can also be given for $\min|\mathcal{C}(\mathcal{A})|$. There is a unique expression of the form

$$m = \binom{a_k}{k}+\binom{a_{k-1}}{k-1}+\dots+\binom{a_t}{t} , \tag{43}$$

where $t \geq 1$, $a_k > a_{k-1} > \dots > a_t$ and $a_i \geq i$. Then

(44) $$\min|\mathcal{C}(\mathcal{A})| = \binom{a_k}{k-1}+\dots+\binom{a_t}{t-1} = f_k(m).$$

The result is more clear if m has the form $\binom{a_k}{k}$. Then we have for the optimum a complete k-graph in $V' \subset V$ where $|V'| = a_k$. An interesting thing: (44) does not depend on n. (44) was first proved by KRUSKAL [63]. Some years later it was rediscovered in [41]. Then CLEMENTS & LINDSTRÖM [4] proved a more general theorem by a different method. They also proved the theorem independently, but they found [41] and [63] before publishing it. HANSEL [30] also has a paper, and recently DAYKIN [13] found a relatively short proof.

A similar result was found earlier by KLEITMAN [52]: If $(V,\mathcal{A})$ is a hypergraph with $A_i \not\supseteq A_j$ and $|\mathcal{A}| = \binom{n}{k}$ then the number of different sets C for which there exists an $A_i \in \mathcal{A}$ with $C \subseteq A_i$ is at least

(45) $$\sum_{i=0}^{k} \binom{n}{i} .$$

This question was solved for any m by CLEMENTS [9], using (44). In this solution only an algorithm is given determining the optimal $\mathcal{A}$, no formula of type (45) is given for the minimum in general. This remains open. [9] also contains useful inequalities concerning (44).

There are a lot of other consequences of (44). E.g., recently DAYKIN [14] observed that theorem 2 (ERDÖS, KO & RADO) follows from (44). Now we give some examples, where (44) is used in the proof.

Let $(V,\mathcal{A})$ be a k-graph. A (k-1)-*representation* of $(V,\mathcal{A})$ is a set $\{B_1,\dots,B_m\}$ of (k-1)-tuples such that $B_i \subset A_i$ $(1 \le i \le m)$. ERDÖS asked what is the maximal m for which any $(V,\mathcal{A})$ with $|\mathcal{A}| = m$ has a (k-1)-representation. The answer [41] is

$$m = \binom{2k-1}{k}+\binom{2(k-1)-1}{k-1}+\dots+\binom{1}{1}.$$

From inequality (2) it is trivial that if we modify the conditions of theorem 1 in such a way that $|A_i| = n/2$ (let n be even) is excluded, then $m \le \binom{n}{(n/2)-1}$ and this is the best. However, if we describe the number of edges A_i with $|A_i| = n/2$ (and this number is > 0, but $< \binom{n}{n/2}$), then usually we do not obtain an exact estimation for m. This question was solved in

[10] more generally: $p_j = |\{A: A\in\mathcal{A}, |A|=j\}|$ are called the *parameters* of a hypergraph. Let $0 \le i_0 < n$ and the parameters $p_{i_0} \ne 0, p_{i_0+1},\dots,p_n$ be fixed. $\max|\mathcal{A}|$ is determined under this condition, provided $A_i \not\supset A_j$.

Another formulation is given independently by DAYKIN, GODFREY & HILTON [15]: If $p_0 = 0, p_1,\dots,p_g > 0$ are given integers, then the least integer n such that there exists a hypergraph $(V,\mathcal{A})$ with $|V| = n$, $A_i \not\supset A_j$ and with the parameters $p_0,p_1,\dots,p_g$ is

$$n = p_1+f_2(p_2+f_3(p_3+\dots+f_g(p_g)\dots)),$$

where f_i is defined in (44).

[15] solves a conjecture of KLEITMAN & MILNER, too: If $(V,\mathcal{A})$ satisfies $A_i \not\supset A_j$ and has the parameters $p_0,p_1,\dots,p_n$, then there is an other hypergraph $(V,\mathcal{A}')$ satisfying $A_i' \not\supset A_j'$ and with parameters $0,\dots,0,p_{n/2},p_{n/2+1}+p_{n/2-1},\dots,p_n+p_0$ (if n is odd, then the middle is: $\dots,0,0,p_{\frac{1}{2}(n+1)}+p_{\frac{1}{2}(n-1)}, p_{\frac{1}{2}(n+3)}+p_{\frac{1}{2}(n-3)},\dots$).

Let the parameters $p_1,\dots,p_{l+r}$ be fixed. What is the minimal number of $(l-1)$-tuples contained in any edge $A \in \mathcal{A}$? This is answered in [8].

CLEMENTS [11] dealt with the problem what happens in theorem 2 if we take more edges than $\binom{n-1}{k-1}$. However, he did not minimize the number of disjoint pairs, but maximized the number of edges meeting all other edges of $(V,\mathcal{A})$.

As is clear from the examples, (44) is almost necessary if containment is involved and the optimal arrangement does not consist of complete i-graphs. We had to write "almost", because KLEITMAN's result in [51] is an exception.

Another type of problems where $|\mathcal{A}|$ is fixed: what is the maximal number of pairs $A_i \supset A_j$, $|A_i - A_j| = 1$? An extremal hypergraph can be constructed by choosing the first $m = |\mathcal{A}|$ edges according to the lexicographic order. This is proved in [1,31,65]. However, as CLEMENTS [6] pointed out it is an easy consequence of (44).

$\min|\mathcal{C}(\mathcal{A})|$ can be asked for under several conditions. For instance in [39] it is tried to do this supposing $|A_i \cap A_j| \ge l$ ($|A_i| = k$ remains true, $l < k$). However, only $\frac{|\mathcal{C}(\mathcal{A})|}{|\mathcal{A}|}$ is minimized. The optimal hypergraph is a complete k-graph on a $(2k-l)$-element subset of V. For fixed $|\mathcal{A}|$, the hypergraph minimizing $|\mathcal{C}(\mathcal{A})|$ seems rather complicated, but it is regular enough to have some hope for the solution.

P. FRANKL asked the following question of similar type. If $|\mathcal{A}|$ is fixed,

$|A| = k$ for $A \in \mathcal{A}$, what is the minimum of $(2k-1)$-tuples contained in a union $A_i \cup A_j$ $(A_i, A_j \in \mathcal{A})$?

8. MORE HYPERGRAPHS

In these problems we have more hypergraphs with the same vertex set. Usually it is supposed that the hypergraphs do not have common edges. The conditions and the questions are usually similar to those in the above sections.

The first result was achieved by ERDÖS [17]. If the hypergraphs $(V,\mathcal{A}_1),\dots,(V,\mathcal{A}_d)$ satisfy the condition

$$A_i \not\supset A_j, \quad A_i, A_j \in \mathcal{A}_h \quad (1 \le h \le d)$$

(and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ $(i \ne j)$), then

$$\sum_{i=1}^{d} |\mathcal{A}_i| \le \text{(the sum of the d largest binomial coefficients of order n).} \tag{46}$$

By the same proof as in the case of theorem 1 we obtain the inequality

$$\sum_{i=1}^{d} \sum_{A \in \mathcal{A}_i} 1/\binom{n}{|A|} \le d, \tag{47}$$

where simply the hypergraph $(V, \bigcup_{i=1}^{d} \mathcal{A}_i)$ was considered; thus one chain C can contain at most d A's. (47) is equivalent to $\sum_{k=0}^{n} x_k/\binom{n}{k} \le d$, where x_k denotes the number of A's with $|A| = k$. It is clear, that under this inequality $\sum_{k=0}^{n} x_k$ is maximal if we take the maximal values of the x_k's with minimal coefficients, thus $x_k = \binom{n}{k}$ for the d middle k's and 0 otherwise.

The next question, what is max $\sum_{i=1}^{d} |\mathcal{A}_i|$, if the $\mathcal{A}_i$'s are disjoint and $A_j \cap A_h \ne \emptyset$, $A_j, A_h \in \mathcal{A}_i$ $(1 \le i \le d)$. The answer was found by KLEITMAN [53]:

$$\sum_{i=1}^{d} |\mathcal{A}_i| \le 2^n - 2^{n-d}.$$

The corresponding question for theorem 2 is unsolved. A problem of KNESER [62] is the following. If $(V,\mathcal{A}_1),\dots,(V,\mathcal{A}_d)$ are k-graphs $(k < n/2)$ $A_i \cap A_j \ne \emptyset$ for $A_i, A_j \in \mathcal{A}_h$, $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ and $(V, \bigcup_{h=1}^{d} \mathcal{A}_h)$ is the complete k-graph, what is the minimum of d under these conditions?

Another line was started by HILTON & MILNER [33]. Let $(V,\mathcal{A})$ and $(V,\mathcal{B})$ be two hypergraphs such that

$$|A_i| \le k,\ |B_i| \le k,\ A_i \cap B_j \ne \emptyset,\ A_i \not\subset A_j,\ B_i \not\subset B_j;$$

then supposing $p \le |\mathcal{A}|, |\mathcal{B}|$ and $1 \le \min(2,p) \le k \le n/2$,

$$|\mathcal{A}|+|\mathcal{B}| \le \begin{cases} \binom{n}{k}-\binom{n-k+1}{k}+n-k+1 & \text{if } 1 < k \le p+1, \\ p+\binom{n}{k}-\binom{n-k+1}{k}+\binom{n-k-p+1}{k-p} & \text{otherwise,} \end{cases}$$

holds. HILTON [34] generalized, for the case $|B_i| \le l \ne k$, KLEITMAN's result [57] on the same subject: $(V,\mathcal{A})$ and $(V,\mathcal{B})$ are hypergraphs satisfying

$$|A_i| = k,\ |B_i| = l,\ k+l \le n,\ A_i \cap B_j \ne \emptyset$$

then either

$$|\mathcal{A}| \le \binom{n-1}{k-1}$$

or

$$|\mathcal{B}| \le \binom{n-1}{l-1}-\binom{n-1-k}{l-1}.$$

EHRENFEUCHT & MYCIELSKI [16] conjectured that if the hypergraphs satisfy

$$|A_i| = k\ (A_i \in \mathcal{A}),\ |B_i| = l\ (B_i \in \mathcal{B}),\ |\mathcal{A}| = |\mathcal{B}| = m$$

and

$$A_i \cap B_j \ne \emptyset \quad \text{iff } i \ne j$$

then

(48) $$m \le \binom{k+l}{l}.$$

It is proved in [48]. T. TARJÁN [75] modified the proof yielding a stronger result:

Let $(V,\mathcal{A})$ and $(V,\mathcal{B})$ be two hypergraphs with $|\mathcal{A}| = |\mathcal{B}|$ and

(49) $$A_i \cap B_j \neq \emptyset \quad \text{iff } i \neq j.$$

Lemma 1 will be applied for the following graph. W consists of all pairs (S,T) where $S,T \subseteq V$, $S \cap T = \emptyset$, and two distinct vertices $(S_1,T_1),(S_2,T_2)$ are connected iff one of the sets $S_1 \cap T_2, S_2 \cap T_1$ is empty. Fix an order on the elements in V. Let W_i consist of those vertices (S,T) in which all elements of S precede all elements of T according to the i-th permutation of the elements of V. Observe that W_i spans a complete graph. That means $f_i = 1$. We need the number

$$|\{i: (S,T) \in W_i\}| = \binom{n}{|S|+|T|} |S|!\,|T|!\,(n-|S|-|T|)! = \frac{n!\,|S|!\,|T|!}{(|S|+|T|)!}.$$

From inequality (5) we obtain

(50) $$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

If $|A_i| = k$, $|B_i| = 1$, (48) trivially follows. Other variants follow, too. E.g. if $|A_i|+|B_i| \leq k$ then

$$m \leq \binom{k}{[k/2]}.$$

9. n-DIMENSIONAL LATTICE-POINTS

SPERNER's question can be formulated in the following way. A square-free integer $N = p_1 p_2 \cdots p_n$ is given; what is the maximal number of its divisors not dividing each other? After answering this question it is a must to answer the same for arbitrary $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, too. The divisors of N have the form $p_1^{x_1} \cdots p_n^{x_n}$, where $0 \leq x_i \leq \alpha_i$ $(1 \leq i \leq n)$. Thus, with the divisors we can associate the lattice-points of an $(\alpha_1+1)\times\cdots\times(\alpha_n+1)$ n-dimensional parallelotope. All questions can be extended to n-dimensional parallelotopes in this way. Some of these extensions are motivated by other applications.

If the character of the problem is such that in the parallelotopes there do not appear new phenomena (compared to the hypergraphs), then it is easier to start making a conjecture and proof for the 2- and 3-dimensional parallelotopes, since they are more graphic.

We briefly list the results which are generalizations of this type.

SPERNER's theorem was generalized in [28]. The bound for m is the maximal number of lattice-points with a fixed coordinate sum ($= [\sum \alpha_i/2]$).

SCHÖNHEIM [73] generalized (46) and (39)-(40). In [44] the common generalization is given. (25) is generalized in [43].

ERDÖS & SCHÖNHEIM [26], further ERDÖS, HERZOG & SCHÖNHEIM [23] have investigated the generalization of (6). The max of m is not equal to the minimal m for which there exists an m-element set of divisors such that any other divisor is coprime to one of them. Both values are determined.

An analogue of (15)-(16) is generalized in [54].

The analogue of (44) is proved in [4]. Of course, there are no formulas, but it is proved that one of the optimal sets of lattice-points gives the first m in the lexicographic order. Other results concerning this theorem can be found in [7]. [5] gives the generalization of ERDÖS' problem of (k-1)-representation of k-edges. [8] also concerns this generalization.

In [6] CLEMENTS shows, that the theorem of LINDSEY [65] (which maximizes the pairs of neighbouring lattice-points if their number is given) is an easy consequence of the generalized formula (44). Recently KLEITMAN, KRIEGER & ROTHSCHILD [61] determined the maximal number of such pairs which differ only in one coordinate.

LINDSTRÖM [66] solved an interesting question of KRUSKAL [64], which is an analogue of (44). A hypergraph can be imagined as a set of certain faces of an (n-1)-dimensional simplex. Thus, if we fix the number of (k-1)-dimensional faces, then (44) gives the minimal number of (k-2)-dimensional subfaces. LINDSTRÖM solved the same question for more-dimensional cubes.

10. FURTHER ANALOGUES AND GENERALIZATIONS

There is an attempt to put these combinatorial theorems in a more general -algebraic- form. Most results concern SPERNER's theorem and close modifications. All these papers state the theorems for certain partial orders. We do not even give the list of these papers because KLEITMAN's paper in this tract contains it. The results contain all important combinatorial analogues of SPERNER's theorem with one exception: the partitions of a finite set under refinement.

For generalizing other problems, there is only one result by HSIEH [38]. It solves an analogue of theorem 2: what is the maximal number of k-dimensional non-disjoint subspaces? And what is interesting, the harder problem, when the subspaces must have 1-dimensional common subspaces, is also solved for small n's. Compare this with (11)-(12) which is true only for large n's. The reason for the difference is, that the middle levels of the partial order of the subspaces are much larger than those of the subsets of a set.

It would be nice to have an algebraic generalization of (44). However, it seems to be hard, because besides the partial order we need an ordering in the levels of the elements of the same rank.

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