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A THREE PART SPERNER THEOREM

by

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Introduction

Let X be a finite set of n elements, and let $\mathscr{A}=\{A_1, \ldots, A_m\}$ be a family of different subsets of X such that $A_i \not\supset A_j$ for $i \neq j$. SPERNER [1] proved that in this case

(1) $$m \leqq \binom{n}{\left[\frac{n}{2}\right]}.$$

KLEITMAN [2] and KATONA [3] independently proved, that if we divide X into two disjoint parts ($X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$) and the family $\mathscr{A}$ satisfies the property that for any different i and j

$$A_i \cap X_1 = A_j \cap X_1 \quad \text{implies} \quad A_i \cap X_2 \not\supset A_j \cap X_2$$

and

$$A_i \cap X_2 = A_j \cap X_2 \quad \text{implies} \quad A_i \cap X_1 \not\supset A_j \cap X_1,$$

then (1) holds again. Here, the conditions are weaker, because $A_i \supset A_j$ is excluded only in some particular cases, when A_i and A_j are equal in X_1 or in X_2, respectively. However, the maximal m under this weaker condition is the same as in the Sperner theorem.

The question arises if the conditions of this result can be still weakened. The natural way would be to divide X into 3 disjoint parts and to exclude the inclusion $A_i \supset A_j$ only if A_i and A_j are equal in two of the parts. However, under this condition it is possible to choose $m > \binom{n}{\left[\frac{n}{2}\right]}$ as it is shown by an example in [3]. The main aim of this paper is to give an additional condition which ensures (1).

We treat the problem in a more general context which is described in [4] and [5], but the definitions and the basic idea are briefly restated here.

Using the method of these papers Sperner type theorems can be reduced to problems of determining a set of points maximal with respect to certain conditions in a two or three dimensional lattice-configuration. We think that these problems are of independent interest.

Definitions and the Theorem

We say that the finite set is a *partially ordered set* if a relation $<$ is defined on G with the following properties: a) at most one of the relations $g_1 < g_2$, $g_1 = g_2$, $g_2 < g_1$ holds; (We say that g_1 and g_2 are *comparable* if one of them is true.) b) if $g_1 < g_2$ and $g_2 < g_3$ then $g_1 < g_3$.

g_2 covers g_1 if $g_1 < g_2$ and there is no g_3 satisfying $g_1 < g_3 < g_2$, that is, if g_2 is the immediate successor of g_1. Assume that there is a rank function $r(g)$ which makes correspond a nonnegative integer to every element of G, so that if g_2 covers g_1 then $r(g_2) = r(g_1)+1$ and there is at least one element $g \in G$ for which $r(g)=0$. We say in this case that G is a *partially ordered set with a rank function.*

A *chain of length* l is a sequence $g_1, \ldots, g_l \in G$, where g_l covers g_{l-1}, g_{l-1} covers $g_{l-2}, \ldots, g_2$ covers g_1. A chain is *symmetrical* if $r(g_1)+r(g_2) = n$, where $n = \max_{g \in G} r(g)$. A partially ordered set is a *symmetrical chain set* if it can be divided into disjoint symmetrical chains.

If G and H are partially ordered sets, then the *direct sum* $G+H$ is the set of ordered pairs (g, h), $g \in G$, $h \in H$ with the ordering $(g_1, h_1) \leqq (g_2, h_2)$ iff $g_1 \leqq g_2$ and $h_1 \leqq h_2$.

If the rank functions of G and H are r and s, respectively then we can define a rank function on $G+H$ as follows:

$$t((g, h)) = r(g)+s(h).$$

For example, the subsets of a finite set X of n elements form a partially ordered set with rank function $r(A)=|A|$, (that is the number of elements) if we order them by inclusion. A covers B $(A, B \subset X)$ iff $A \supset B$ and $|A-B|=1$. By a theorem of DE BRUIJN, KRUSYSWIJK and TENGEBERGEN [6] we know that this is a symmetrical chain set. (More generally this is proved for the set of points of an n-dimensional rectangle with integer coordinates.)

If G is the partially ordered set of the subsets of a set X_1 and H is the same for X_2 $(X_1 \cap X_2 = \emptyset)$, then $G+H$ is the partially ordered set of the subsets of $X_1 \cup X_2$. The same is true for the disjoint sets X_1, X_2, X_3.

Now, we can formulate our theorem.

THEOREM. *Let F, G and H be symmetrical chain sets with respective rank functions p, r and s*

$$\max_{f \in F} p(f)+\max_{g \in G} r(g)+\max_{h \in H} s(h) = n.$$

Assume $(f_1, g_1, h_1), \ldots, (f_m, g_m, h_m)$ are different elements of $F+G+H$, such that there are no two different triples among them for which

$$(f_i, g_i, h_i) < (f_j, g_j, h_j) \tag{2}$$

and two of the corresponding components are equal and there are no four different triples for which

$$f_i = f_j, \quad f_k = f_l$$

$$\begin{array}{ll} g_i, g_k > g_j, g_l & g_i, g_k \text{ are comparable} \\ & g_j, g_l \text{ are comparable} \\ h_i, h_l > h_j, h_k & h_i, h_l \text{ are comparable} \\ & h_j, h_k \text{ are comparable} \end{array} \tag{3}$$

holds for any permutation of the letters f, g, h. Then m is maximal if we choose all the elements of $F+G+H$ with rank $\left[\frac{n}{2}\right]$.

Applying the theorem for the partially ordered sets of the subsets of the disjoint sets X_1, X_2, X_3 we obtain a Sperner type theorem, where we exclude some pairs and 4-tuples of subsets and the maximal size of the family under these conditions is $\binom{n}{\left[\frac{n}{2}\right]}$.

We formulate another particular case of the theorem as a separate lemma. The proof of the theorem is based on it.

LEMMA. *Let T be the direct sum of the chains $0<1<\ldots<a_1$, $0<1<\ldots<a_2$ and $0<1<\ldots<a_3$ ($a_3\geqq a_2\geqq a_1$ are integers). Assume $(x_1, y_1, z_1), \ldots, (x_m, y_m, z_m)$ are different elements of T ($0\leqq x_i\leqq a_1, 0\leqq y_i\leqq a_2$, $0\leqq z_i\leqq a_3$ integers; $(1\leqq i\leqq m)$ such that there are no two different triples with the property,*

(4) *two corresponding components are equal,*

and there are no 4 different triples for which

$$x_i = x_j, \quad x_k = x_l$$
$$y_i, y_k > y_j, y_l, \tag{5}$$
$$z_i, z_l > z_j, z_k$$

holds.

Then m is maximal if we choose all the elements of rank

$$x+y+z = \left[\frac{a_1+a_2+a_3}{2}\right].$$

Here it is not necessary to assume (5) for the other permutations of x, y and z.

PROOF of the theorem. From the fact that F, G and H can be split into disjoint symmetrical chains it follows that $F+G+H$ can be split into disjoint symmetrical 3-dimensional parallelotopes of type T. The word "symmetrical" means that the sum of the minimal and maximal rank in T is n. Assuming the lemma is true, we obtain the maximal number of elements of T satisfying conditions (2) and (3) ((4) and (5) follow from them) by choosing all the elements of "middle rank". By the symmetry of T they have rank $\left[\frac{n}{2}\right]$. So, under conditions (2) and (3) the maximal set of points consists of all the elements of rank $\left[\frac{n}{2}\right]$. The proof is completed. We have to prove only our lemma.

The proof of the lemma

Before starting the proof let us consider a simple example: $a_1=a_2=a_3=1$. Here $\left[\frac{a_1+a_2+a_3}{2}\right] = 1$, the number of points with $x+y+z = 1$ is 3: (0, 0, 1), (0, 1, 0), (1, 0, 0). If we add the point (1, 1, 1), we have 4 points and there are no two of them satisfying (4). Thus we really need some additional conditions excluding the 4-tuples similar to (0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1). Similar examples show that we have to exclude somewhat more general 4-tuples. One possibility is (5).

An *x-plane* is the subset of T containing the points with first component equal to x. In (5) $x_i=x_j$ means that the points $P_i=(x_i, y_i, z_i)$ and $P_j=(x_j, y_j, z_j)$ are in the same x-plane. The same is true for $P_k=(x_k, y_k, z_k)$ and $P_l=(x_l, y_l, z_l)$.

$y_i>y_j$ and $z_i>z_j$ mean that these two points form an increasing function (as a function $z(y)$). On the other hand $y_k>y_l$, $z_l>z_k$ mean that these points form a decreasing function. (5) contains the further $(y_i>y_l,\ y_k>y_l,\ z_i>z_k,\ z_l>z_j)$ conditions, which mean that the intervals (y_j, y_i) and (y_l, y_k) are not disjoint, and similarly, (z_j, z_i) and (z_k, z_l) are not disjoint.

1. First let us investigate the points in one x-plane. A *row* is a set of points in an x-plane with a fixed z-component. (The *column* is defined similarly.) By (4) there are no two points in one row or column. So we can consider the points as a function. We shall prove that this *function is essentially monotonically increasing or decreasing.*

Define p by $y_p=\min\{y_j:x_j=x\}$ and q by $y_q=\max\{y_j:x_j=x\}$. If $p=q$, there is only one point in the plane, we have no problem. Otherwise $y_p<y_q$ holds. Assume $z_p<z_q$.

1a. There are no two points (x_i, y_i, z_i) and (x_l, y_l, z_l) satisfying $x_i=x_l=x$, $y_l<y_i$, $z_i, z_l>z_q$, because otherwise (5) holds with $p=j$ and $q=k$. Similarly, there are no two points (x_i, y_i, z_i) and (x_l, y_l, z_l) with $x_i=x_l=x$, $y_l>y_i$, $z_i, z_l>z_q$, changing the role of l and i. Summarizing, there are no two different points (x_i, y_i, z_i) and (x_l, y_l, z_l) with $x_i=x_l=x$ and $z_i, z_l>z_q$. It means, the function $z(y)$ has no two different values $>z_q$. We can see in a similar way that there are no two different values $<z_p$.

1b. On the other hand, assume that we have the different points (x_k, y_k, z_k) and (x_l, y_l, z_l) satisfying the conditions $k\neq p, q$, $l\neq p, q$, $x_k=x_l=x$, $y_k>y_l$, $z_l>z_k$. If $z_k<z_l<z_p$ (or $z_q<z_k<z_l$) this case is settled above. If $z_l>z_p$ (or $z_k<z_q$) then we obtain a contradiction. The points with $i=q$, $j=p$ satisfy (5). That means, there are no two values of $z(y)$ different from z_p and z_q which are in decreasing position.

1c. Finally, if (x_l, y_l, z_l) satisfies $x_l=x$ and $z_l>z_q$, then there is no point (x_k, y_k, z_k) with $x_k=x$ and $y_l<y_k<y_q$, because otherwise these points satisfy (5) choosing $i=p$, $j=q$. (If $z_k>z_l$, it belongs to the case 1a, thus we can assume $z_k<z_l$).

Summarizing our results, if we assume $z_p<z_q$, then the function $z(y)$ almost monotonically increasing; the exceptional pairs have either (x_p, y_p, z_p) or (x_q, y_q, z_q) as a member (by 1b.).

To both (x_p, y_p, z_p) and (x_q, y_q, z_q) we can have at most one exceptional pair (by 1a). If (x_p, y_p, z_p) and (x_i, y_i, z_i) form such an exceptional pair, then y_i is the second value of the domain of $z(y)$ (That is, there is no (x_k, y_k, z_k) with $x_k=x$, $y_p<y_k<y_i$). Similarly, if (x_i, y_i, z_i) and (x_q, y_q, z_q) is an exceptional pair, then

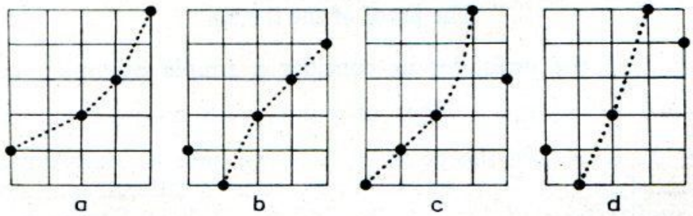

Fig. 1

y_i is the last but one element of the domain of $z(y)$. We call such a function *essentially monotonically increasing function.* Figure 1 gives the typical diagrams of such functions.

Of course, if we assume $z_p > z_q$ we obtain *essentially monotonically decreasing functions.*

2. If (x_i, y_i, z_i) and (x_j, y_j, z_j) are two points with $x_i = x_j$, $y_i < y_j$, $z_i < z_j$, then the rectangle $y_i \leqq y \leqq y_j$, $z_i \leqq z \leqq z_j$ is called a *red rectangle.* On the other hand, if $x_i = x_j$, $y_i < y_j$ and $z_i > z_j$ hold, then the rectangle $y_i \leqq y \leqq y_j$, $z_j \leqq z \leqq z_i$ is called a *blue rectangle. A red rectangle of a plane cannot contain an inner point of a blue rectangle of another plane* (and vice versa), because otherwise the 4 points determining them satisfy (5).

3. We say that an x-plane is *increasing (decreasing)* if $z_p < z_q$ $(z_q < z_p)$. We now prove that *if a plane is increasing, then the union of the red rectangles (determined by its $c(x)$ points) covers a $c(x) \times c(x)$ square possibly without its two opposite corners.*

If the points of the plane are of form a, then the rectangle determined by (x_p, y_p, z_p) and (x_q, y_q, z_q) has sizes $\geqq c(x)$, thus the statement is proved in this case.

If the points are of form b, then (x_p, y_p, z_p) and (x_q, y_q, z_q) determine a rectangle with at least $c(x)$ columns and at least $c(x)-1$ rows. In this case, however, there is a point (x_r, y_r, z_r) $(x_r = x)$ satisfying $z_r < z_p$. Thus, (x_r, y_r, z_r) and (x_q, y_q, z_q) determine a rectangle with at least $c(x)-1$ columns and at least $c(x)$ rows. The union of these two rectangles cover a $c(x) \times c(x)$ square possibly except one corner.

The same holds for the case c.

In the case d, by similar arguments, we obtain the $c(x) \times c(x)$ square without two corners. In the case b the lowest point of the first column can be missing from the square, in the case c the topmost point of the last column, and in the case d both of them. We call these figures *incomplete squares.*

The same holds, of course, with blue rectangles for decreasing planes.

4. *If an increasing x-plane contains $c(x) \geqq 0$ points and a decreasing x'-plane contains $c(x') \geqq 0$ points then*

$$c(x) + c(x') \leqq a_3 + 3 \tag{6}$$

with equality only if $a_3 = a_2$.

It is easy to see that any inner point of a blue incomplete square is an inner point of some blue rectangle of the same plane. Using this fact and the statement of section 2 we obtain that the red incomplete square in the plane x cannot contain any inner point of the blue incomplete square in the plane x'. Of course, they can touch each other.

If none of the two squares contains an inner point of the other one, including now the missing corners, then one of the sizes of the rectangle $(a_2+1) \times (a_3+1)$ is $\geqq c(x) + c(x') - 1$. Using $a_3 \geqq a_2$, (6) follows.

On the other hand, if a complete and an incomplete corners of the two squares are touching each other, then both

$$a_3 + 1 \geqq c(x) + c(x') - 2 \tag{7}$$

and

$$a_2 + 1 \geqq c(x) + c(x') - 2 \tag{8}$$

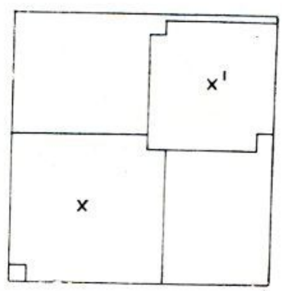

Fig. 2

hold. If (7) is satisfied with equality, then (8) results in $a_2 \geqq a_3$. Using $a_3 \geqq a_2$ the statement of this section is proved.

5. We try now count the number of points contained in the plane x from the optimal system given by the lemma. In other words, what is the number of solutions in y, z $(0 \leqq y \leqq a_2, 0 \leqq z \leqq a_3)$ of the equation

$$x+y+z=\left[\frac{a_1+a_2+a_3}{2}\right]$$

for a fixed x?

Assume

$$0 \leqq x \leqq \left[\frac{a_1+a_2-a_3}{2}\right].$$

If $y<\left[\frac{a_1+a_2-a_3}{2}\right]-x$, then obviously

$$z=\left[\frac{a_1+a_2+a_3}{2}\right]-x-y>\left[\frac{a_1+a_2+a_3}{2}\right]-\left[\frac{a_1+a_2-a_3}{2}\right]=a_3$$

holds, y, z is not a good solution. We may assume

$$\left[\frac{a_1+a_2-a_3}{2}\right]-x \leqq y \leqq a_2.$$

In this case we have always exactly one z satisfying $0 \leqq z \leqq a_3$:

$$0 \leqq a_3-a_2=\left[\frac{a_1+a_2+a_3}{2}\right]-\left[\frac{a_1+a_2-a_3}{2}\right]-a_2 \leqq z=$$

$$=\left[\frac{a_1+a_2+a_3}{2}\right]-x-y \leqq \left[\frac{a_1+a_2+a_3}{2}\right]-\left[\frac{a_1+a_2+a_3}{2}\right]=a_3.$$

The number of points is

$$a_2-\left[\frac{a_1+a_2+a_3}{2}\right]+x+1=\left\{\frac{-a_1+a_2+a_3}{2}\right\}+x+1$$

($\{a\}$ denotes the smallest integer $\geqq a$).

It is easy to determine the number of points in the plane x for the remaining values of x:

$$(9) \qquad c^*(x)=\begin{cases}\left\{\frac{-a_1+a_2+a_3}{2}\right\}+x+1 & \text{if } 0 \leqq x \leqq \left[\frac{a_1+a_2-a_3}{2}\right] \\ a_2+1 & \text{if } \left[\frac{a_1+a_2+a_3}{2}\right] \leqq x \leqq \left[\frac{a_1-a_2+a_3}{2}\right] \\ \left[\frac{a_1+a_2+a_3}{2}\right]-x+1 & \text{if } \left[\frac{a_1-a_2+a_3}{2}\right] \leqq x \leqq a_1.\end{cases}$$

Obviously

$$\min_{0 \leq x \leq a_1} c^*(x) = \left[\frac{-a_1+a_2+a_3}{2}\right]+1 \tag{10}$$

holds if $a_3 \leq a_1+a_2$. If $a_3 \geq a_1+a_2$, then $c^*(x) = a_2+1$ $(0 \leq x \leq a_1)$.

6. Let us prove the lemma for the *case* $a_1+a_2 \leq a_3$. In a fixed x-plane every column contains at most one point, hence $c(x) \leq a_2+1$. However, in this case $c^*(x) = a_2+1$. The construction given in the lemma is the best possible in every x-plane.

7. Now, let us investigate the *case* $a_2 < a_3$ $(a_1+a_2 > a_3)$. We fix a_2 and a_3, and prove the statement by induction over a_1. If $a_1 = a_3 - a_2$, then the statement holds by section 6. Assume now $a_1 > a_3 - a_2$, but the statement is true for smaller a_1. We have to prove

$$\sum_{x=0}^{a_1} c(x) \leq \sum_{x=0}^{a_1} c^*(x). \tag{11}$$

7a. Assume there exist both increasing (say x') and decreasing (say x) x-planes. By section 4 and by $a_3 > a_2$ we have

$$c(x)+c(x') \leq a_3+2.$$

Hence either $c(x)$ or $c(x') \leq \left[\frac{a_3+2}{2}\right]$ holds, that is, a term (say $c(x')$) in the left hand side of (11) $\leq \left[\frac{a_3+2}{2}\right]$. Omitting this plane we obtain a new 3-dimensional parallelotope with sizes a_1-1, a_2, a_3. We can use the induction hypothesis:

$$\sum_{\substack{x=0 \\ x \neq x'}}^{a_1} c(x) = \sum_{x=0}^{a_1-1} c'(x) \leq \sum_{x=0}^{a_1-1} c^{*\prime}(x), \tag{12}$$

where $c'(x)$ is the number of points in the plane x after omitting the x'-plane, and $c^{*\prime}(x)$ is the same number for the optimal system in the new parallelotope. What is the connection between the numbers $c^*(x)$ and $c^{*\prime}(x)$? It is easy to see from (9) that $c^*(0), \ldots, c^*(a_1)$ is a sequence of integers starting with $\left\{\frac{-a_1+a_2+a_3}{2}\right\}+1$, increasing 1 by 1 up to a_2+1 (it is really larger, since $a_3 < a_1+a_2$), there are a_3-a_2+1 (≥ 2) numbers a_2+1 and decreasing 1 by 1 up to $\left[\frac{-a_1+a_2+a_3}{2}\right]+1$. Hence, it is clear, that the sequence $c^{*\prime}(0), \ldots, c^{*\prime}(a_1-1)$ consists of the same numbers with the only difference that either the first or the last number is missing, depending on the parity of $-a_1+a_2+a_3$. If $-a_1+a_2+a_3$ is even, then the missing first term is $\frac{-a_1+a_2+a_3}{2}+1$ if it is odd, then the missing last term is $\left[\frac{-a_1+a_2+a_3}{2}\right]+1$. In both cases the missing term is a minimal one.

$$\sum_{x=0}^{a_1} c^*(x) = \sum_{x=0}^{a_1-1} c^{*\prime}(x) + \left[\frac{-a_1+a_2+a_3}{2}\right]+1. \tag{13}$$

However, $a_2 \geqq a_1$ results in

$$\left[\frac{a_3+2}{2}\right] \leqq \left[\frac{-a_1+a_2+a_3}{2}\right]+1, \tag{14}$$

and (11) follows from (12), $c(x') \leqq \left[\frac{a_3+2}{2}\right]$, (13) and (14).

7b. If all the x-planes are increasing or if all of them are decreasing we have to find in a different way an x' for which

$$c(x') \leqq \left[\frac{-a_1+a_2+a_3}{2}\right]+1 \tag{15}$$

holds. For sake of simplicity, let us assume, all the x-planes are decreasing.

7ba. Assume, there is a point (x_i, y_i, z_i) satisfying

$$y_i+z_i \leqq \left[\frac{-a_1+a_2+a_3}{2}\right]-1. \tag{16}$$

There are at most y_i points (x_j, y_j, z_j) such that $x_j=x_i$, $y_j<y_i$, since every column contains at most one point. On the other hand, the number of points (x_j, y_j, z_j) satisfying $x_j=x_i$, $y_j>y_i$ is at most z_i+1, because the plane x_i is decreasing, thus $z_j<z_i$ holds for all but possibly one points. Hence we have

$$c(x_i) \leqq 1+y_i+z_i+1 \leqq \left[\frac{-a_1+a_2+a_3}{2}\right]+1 \tag{17}$$

by (16). (15) is satisfied with $x'=x_i$, the proof of section 7a can be repeated here, too.

7bb. Assume, now there is a point (x_i, y_i, z_i) satisfying

$$y_i+z_i \geqq \left\{\frac{a_1+a_2+a_3}{2}\right\}. \tag{18}$$

The number of points (x_i, y_i, z_i) such that $x_j=x_i$, $y_j>y_i$ is at most a_2-y_i. On the other hand $x_j=x_i$, $y_j<y_i$ results in $z_j>z_i$ with at most one exception $(z_j<z_i)$.

7bba. If there is no exception, then the number of points of the latter type is at most a_3-z_i; the total number of points in the plane is at most $1+a_2-y_i+a_3-z_i$, or

$$c(x_i) \leqq 1+a_2-y_1+a_3-z_i \leqq \left[\frac{-a_1+a_2+a_3}{2}\right]+1 \tag{19}$$

by (18). (15) is satisfied by $x_i=x'$; this case is settled.

7bbb. If there is an exceptional point (x_j, y_j, z_j) with $x_j=x_i$, $y_j<y_i$, $z_j<z_i$, then, by section 1b, either $j=p$ or $i=q$ holds (but not both of them, since $z_p>z_q$).

7bbba. $j=p$. Then by section 1a (x_i, y_i, z_i) is the only point in the plane x_i such that $z_i>z_p$. We distinguish two cases.

7bbbaa. $z_i<a_3$. If there is at least one $z>z_i$ for which we cannot find (x_l, y_l, z_l) such that $x_l=x_i$, $z_l=z$, then the number of points satisfying $y_j<y_i$ is at most, a_3-z_i; (19) holds, again. On the other hand, if for any z there exists such an (x_l, y_l, z_l) then $z_p = a_3-1$, and $z_p<z_i=a_3$. This contradicts our supposition.

7bbbab. $z_l=a_3$. If strict equality holds in (18) then (19) follows despite the existence of the exceptional point. Thus we may suppose $y_i+z_i=\left\{\frac{a_1+a_2+a_3}{2}\right\}$. Our point has the coordinates

$$z_i=a_3, \quad y_i=\left\{\frac{a_1+a_2+a_3}{2}\right\}-a_3=\left\{\frac{a_1+a_2-a_3}{2}\right\}.$$

7bbbaba. There is a point (x_l, y_l, z_l) with $z_l=a_3$, $y_l=\left\{\frac{a_1+a_2-a_3}{2}\right\}-1$. In this case we cannot find a point (x_k, y_k, z_k) such that $x_k=x_l$, $y_k>y_l$ $(z_k<z_l)$ because otherwise (x_i, y_i, z_i), $(x_p, y_p, z_p)=(x_j, y_j, z_j)$, (x_k, y_k, z_k) and (x_l, y_l, z_l) satisfy (5), a contradiction. On the other hand, if the point (x_k, y_k, z_k) possesses the properties $x_k=x_l$, $y_k<y_l$ and $z_k<z_l$, then using the assumption that the plane is decreasing we obtain either $k=p'$ or $l=q'$ (p' and q' are p and q of this plane). In both cases there are at most two points in the plane $x_l=x_k$, they are in increasing position, which is a contradiction. It follows $c(x_l)=1$, and (15) trivially holds for $x'=x_l$.

7bbbabb. There is no point (x_l, y_l, z_l) with $z_l=a_3$, $y_l=\left\{\frac{a_1+a_2-a_3}{2}\right\}-1$. We have proved earlier that if there is any point satisfying (18) with strict inequality (7bbbab), or with equality but with $z_i<a_3$ (7bbbaa) then the statement is true.

The same holds, if some point satisfies (16). Thus we may suppose, that all of our points (x_r, y_r, z_r) satisfy

$$\left[\frac{-a_1+a_2+a_3}{2}\right] \leqq y_r+z_r \leqq \left\{\frac{a_1+a_2+a_3}{2}\right\}-1 \tag{20}$$

with the exception of (x_i, y_i, z_i), where $z_i=a_3$, $y_i=\left\{\frac{a_1+a_2+a_3}{2}\right\}-a_3$. If we fix a pair y_r, z_r there is at most one point (x_r, y_r, z_r) with these second and third coordinates, by the supposition of the lemma. The maximal number od points is the number of solutions of (20) in $0\leqq y_r\leqq a_2$, $0\leqq z_r\leqq a_3$, since we have an additional point (x_i, y_i, z_i) but the solution $y_r=\left\{\frac{a_1+a_2-a_3}{2}\right\}-1$, $z_r=a_3$ is omitted by the suppositions of this (7bbbabb) case. However, the number of solutions of (20) is not larger than the expected optimum of the lemma. We will see this in the case 7bc.

7bbbb. $i=q$.

7bbbba. $y_i<a_2$. The number of points in the plane is $c(x_i)=2+a_3-z_i$. Using (18) and $y_i<a_2$ we obtain

$$c(x_i) \leqq 2+a_3-\left\{\frac{a_1+a_2+a_3}{2}\right\}+y_i \leqq$$

$$\leqq 2+a_3-\left\{\frac{a_1+a_2+a_3}{2}\right\}+a_2-1=\left[\frac{-a_1+a_2+a_3}{2}\right]+1,$$

that is, (15) and the statement holds in this case.

7bbbbb. $y_i = a_2$. Like in case 7bbbab we may assume

$$y_i + z_i = \left\{\frac{a_1+a_2+a_3}{2}\right\}.$$

Thus,

$$z_i = \left\{\frac{a_1+a_2+a_3}{2}\right\} - a_2 = \left\{\frac{a_1-a_2+a_3}{2}\right\}.$$

7bbbbba. There is a point (x_k, y_k, z_k) with $z = \left\{\frac{a_1-a_2+a_3}{2}\right\} - 1,\ y_k = a_2$.

In this case we cannot find a point (x_l, y_l, z_l) such that case we cannot find a point (x_l, y_l, z_l) such that $x_l = x_k,\ z_l > z_k (y_l < y_k)$, because otherwise $(x_i, y_i, z_i) = (x_q, y_q, z_q)$, (x_j, y_j, z_j), (x_k, y_k, z_k) and (x_l, y_l, z_l) satisfy (5), which contradiction (i and j defined in 7bb and 7bbb). On the other hand, if (x_l, y_l, z_l) possesses the properties $x_l = x_k,\ z_l < z_k$, then similarly to the case 7. bbbaba we have $c(x_k) = 1$, which proves the statement.

7bbbbbb. There is no point (x_k, y_k, z_k) with

$$z_k = \left\{\frac{a_1-a_2+a_3}{2}\right\} - 1, \quad y_k = a_2.$$

The proof of this case is the same as in the case 7bbbabb.

7bc. All the points (x_r, y_r, z_r) satisfy (20). y_r and z_r uniquely determine x_r; it is sufficient to count the number of solutions of (20), and to see that this number $\leqq$ the number of points of the construction given in the lemma.

The number of solutions of

$$\left[\frac{-a_1+a_2+a_3}{2}\right] \leqq y_r + z_r \leqq \left[\frac{a_1+a_2+a_3}{2}\right] \tag{21}$$

is not smaller than that of (20). Define x_r' by

$$x_r' + y_r + z_r = \left[\frac{a_1+a_2+a_3}{2}\right]. \tag{22}$$

To every solution of (21) $(0 \leqq y_r \leqq a_2,\ 0 \leqq z_r \leqq a_3)$ there is a solution of (22) with $0 \leqq x_r' \leqq a_1$, and vice versa. Thus the number of solution of (20) and (21) the number of solution of (22). However, (22) defines the construction of the optimal system. The proof in case 7 is completed.

8. Let us prove the case $a_2 = a_3$. We prove this case by induction on a_1 for fixed $a_2 = a_3$. If $a_1 = 0$, the statement follows from section 6. Assume $a_1 > 0$ and suppose it is proved for smaller a_1. We can repeat the proof of case 7. The only place where we used the assumption $a_2 < a_3$ is the case 7a. There we showed that there is a plane x' satisfying $c(x') \leqq \left[\frac{a_3+2}{2}\right]$. If such a plane exists, we are done. We have to investigate, how can it happen that such a plane does not exist.

We have an increasing plane x and a decreasing plane x'. By section 4 we can state only $c(x)+c(x') \leqq a_3+3$ if $a_2=a_3$.

None of $c(x)$ and $c(x') \leqq \left[\frac{a_3+2}{2}\right]$ only if a_3+3 is even and $c(x)=c(x') = \frac{a_3+3}{2}$. It can happen only if in the proof of section 4 the incomplete squares touch each other at the missing point like on the figure. We have used $c(x') \leqq \left[\frac{a_3+2}{2}\right]$ in (14), only; thus if

$$\frac{a_3+3}{2} \leqq \left[\frac{-a_1+a_2+a_3}{2}\right]+1$$

holds, we are done. Using $a_2 \geqq a_1$, it does not hold only if $a_1=a_2$ $(=a_3)$. Hence, it is sufficient to investigate the case, when there are a_3+1 x-planes and every one contains at least $\frac{a_3+3}{2}$ points. They cannot have more points because each of them is increasing or decreasing, thus we can use it instead of x or x'. This shows that the number of points is $\leqq \frac{a_3+3}{2}$.

Hence

$$\sum_{x=0}^{a_3} c(x) = (a_3+1)\frac{a_3+3}{2}. \tag{23}$$

We are going to prove that $\sum_{x=0}^{a_3} c^*(x)$ is not smaller than (23). Using (9) we have

$$\sum_{x=0}^{a_3} c^*(x) = \frac{a_3+3}{2}+\frac{a_3+5}{2}+\ldots+(a_3+1)+a_3+\ldots+\frac{a_3+3}{2}+\frac{a_3+1}{2}. \tag{24}$$

The number of terms in (23) and (24) are the same. There is only one term in (24) smaller than $\frac{a_3+3}{2}$ by (1). Two terms are equal, all the other terms are $> \frac{a_3+3}{2}$. If there are at least 4 terms in (24) that is if $a_3 \geqq 3$, then (24) $\geqq$ (23). However, the statement of the lemma is trivial for $a_1=a_2=a_3=1$. The lemma is proved.

Problems

Probably, both the Theorem and the lemma can be strengthened. We only formulate a possible generalization of the lemma.

Conjecture. The conclusion of the lemma holds under the following condition weaker than (5)

$$x_i = x_j, \quad x_l = x_k$$
$$y_i = y_k, \quad y_j = y_l$$
$$z_i = z_l < z_j, z_k$$

provided (4) is true.

A further problem is to find an n-part Sperner theorem with weak enough conditions.

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