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TWO APPLICATIONS (FOR SEARCH THEORY AND TRUTH FUNCTIONS) OF SPERNER TYPE THEOREMS

by

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To the memory of A. RÉNYI

> *"I am constantly pondering what kind of knowledge I should try to acquire. Recently, Theaitetos told me that certainty exists only in mathematics and suggested that I learn mathematics from his master, Theodoros who is the leading expert on numbers and geometry in Athens."*
> From RÉNYI's "Dialogues on Mathematics"

§ 1.

Assume a finite set $X = \{x_1, \ldots, x_n\}$ is given and we are looking for an unknown $x \in X$. We have informations of type

$$x \in A_i \quad \text{or} \quad x \notin A_i$$

where A_i's are subsets of X. If one of the sets

$$BC,\ \bar{B}C,\ B\bar{C},\ \bar{B}\bar{C} \tag{1}$$

is empty, then after knowing $x \in B$ or $x \notin B$ it may occur that $x \in C$ or $x \notin C$ does not contain any new information. For example, if $\bar{B}C = \emptyset$, then $x \notin B$ contains the information $x \notin C$. In the contrary case, if none of the sets (1) is $\emptyset$, then we need the information "$x \in C$ or $x \notin C$", independently of the answer of the question "$x \in B$ or $x \notin B$". We say, following MARCZEWSKI [1] that B and C are *qualitatively independent*, if none of the sets (1) is $\emptyset$. RÉNYI [2] asked what is the maximal number of pairwise qualitatively independent subsets $B_1, \ldots, B_m$ of an n-element set X. He solved in [2] the question for even n in the following way: The statement "none of $BC, \bar{B}C, B\bar{C}, \bar{B}\bar{C}$ is empty" is equivalent to the statement "none of $B, \bar{B}, C, \bar{C}$ is contained in an other one". That means, if $B_1, \ldots, B_m$ are pairwise qualitatively independent, then none of $B_1, \ldots, B_m, \bar{B}_1, \ldots, \bar{B}_m$ is contained in another one. The well-known theorem of SPERNER [3] says that the maximal number of such subsets is $\binom{n}{\left[\frac{n}{2}\right]}$. It follows $2m \le \binom{n}{\left[\frac{n}{2}\right]}$ and

$$m \le \frac{\binom{n}{\left[\frac{n}{2}\right]}}{2}.$$

If n is even, this is the best possible upper estimation, since we can choose $\binom{n}{\frac{n}{2}}/2$ qualitatively independent sets, taking arbitrary one of each complementary pair of $\frac{n}{2}$-tuples.

In this paper we solve the case of odd n.

Theorem 1. *If $B_1, \ldots, B_m$ are pairwise qualitatively independent subsets of a set of n elements, then*

$$m \leq \binom{n-1}{\left[\frac{n}{2}\right]-1}$$

and this is the best possible estimation.

Proof. 1. If B and C are qualitatively independent, then B and $\bar{C}$ are qualitatively independent, too. If $|B_i| > \frac{n}{2}$ we may change B_i for $\bar{B}_i$; $B_1, \ldots, \bar{B}_1, \ldots, B_m$ are qualitatively independent. Thus we may assume $B_1, \ldots, B_m$ are chosen in such a way that

$$|B_i| \leq \left[\frac{n}{2}\right] \quad (1 \leq i \leq m). \tag{2}$$

2. Define $k = \min_{1 \leq i \leq m} |B_i|$. Assume B_i's are indexed in such a way that for some p

$$k = |B_1| = \ldots = |B_p| < |B_i| \qquad (p < i \leq m).$$

Denote by $c(B_1, \ldots, B_p)$ the family $\{C_1, \ldots, C_r\}$ of sets C satisfying $|C| = k + 1$ and $C \supset B_i$ for some $1 \leq i \leq p$. If n, k and p are given the minimum of r is determined in [4] and [5]. However we do not need this exact minimum here, we need only a simple estimation for r, which is determined by Sperner [3]:

$$p\frac{n-k}{k+1} \leq r. \tag{3}$$

The number of pairs (B_i, C), where $1 \leq i \leq p$, $B_i \subset C$, $|C| = k+1$ is $p(n-k)$. On the other hand, a fixed C can contain $k+1$ B_i:

$$p(n-k) \leq r(k+1)$$

which is equivalent to (3).

3. $C_1, \ldots, C_r, B_{p+1}, \ldots, B_m$ *are pairwise qualitatively independent, if* $k < 2\left[\frac{n}{2}\right]$.

It is trivial for two of B's.

$C_i \cap C_j$ is not empty since $C_i \supset B_u, C_j \supset B_v$ for some u, v $(1 \leq u, v \leq p)$ and $C_j \cap C_j \supset B_u \cap B_v$ is not empty. $C_i \cap \bar{C}_j$ can not be empty because C_i has $k+1$ elements, $\bar{C}_j$ has $n-k-1$ elements and they can be complementer sets only if $C_j = C_i$, that is if $j = i$. The total number of elements in $\bar{C}_i$ and $\bar{C}_j$ is $2n - 2k - 2$. They can be disjoint only if

$$2n - 2k - 2 \leq n - 1 \tag{4}$$

as there is an element of $\overline{\bar{C}_i \cap \bar{C}_j} = C_i \cap C_j$. From (4) it follows $\frac{n-1}{2} \leq k$ which contradicts our supposition. $\bar{C}_i \cap \bar{C}_j$ can not be empty. $C_i \cap B_j$ $(1 \leq i \leq p,\ p < j \leq m)$ is not empty since $C_i \supset B_u$ for some u $(1 \leq u \leq p)$ and $C_i \cap B_j \supset B_u \cap B_j$ is not empty. C_i has $k+1$, $\bar{B}_j$ has $n-k$ elements. Thus they can not be complementer sets as $k + 1 + n - k > n$. $C_i \cap \bar{B}_j \neq \emptyset$. We have similarly $\bar{C}_i \cap B_j \neq 0$. Finally let us verify that $\bar{C}_i$ and $\bar{C}_j$ have also a common element. The total number of their elements is $2n - 2k - 1$. $\bar{C}_i \cap \bar{B}_j = C_i \cap B_j$ has at least one element. Thus, if $\bar{C}_i$ and $\bar{B}_j$ are disjoint, we have

$$2n - 2k - 1 \leq n - 1 .$$

This inequality contradicts our supposition $k \leq \left[\frac{n}{2}\right]$.

4. Now we prove *if* $B_1, \ldots, B_m$ *are pairwise independent and* m *is maximal, then* $|B_1| = \ldots = |B_m| = \left[\frac{n}{2}\right]$.

Suppose the contrary, $k = \min\limits_{1 \leq i \leq m} |B_i| < \left[\frac{n}{2}\right]$. We may apply the result of Section 3: $C_1, \ldots, C_r, B_{p+1}, \ldots, B_m$ are pairwise independent. However, by (3) $p < r$ since $\frac{n-k}{k+1} > \frac{n - \left[\frac{n}{2}\right]}{\left[\frac{n}{2}\right] + 1} \geq 1$. $C_1, \ldots, C_r, B_{p+1}, \ldots, B_m$ has more members than $B_1, \ldots, B_m$ in contradiction with the maximality of $B_1, \ldots, B_m$. Thus, $k \geq \left[\frac{n}{2}\right]$ and (2) ensure the validity of the statement.

5. $B_1, \ldots, B_m$ have the same number of elements $\left(\left[\frac{n}{2}\right]\right)$ and $B_i \cap B_j \neq 0$ $(1 \leq i, j \leq m)$. We may apply the next theorem of ERDŐS—CHAO KO—RADO [6]:

If $|B_1| = \ldots = |B_m| = l$, *where* $B_1, \ldots, B_m$ *are pairwise non-disjoint subsets of a set of* n *elements, then*

$$m \leq \binom{n-1}{l-1}.$$

In our case

$$m \leq \binom{n-1}{\left[\frac{n}{2}\right]-1}.$$

The proof is completed.

OPEN PROBLEMS. 1. Determine the maximal m for which there exists a family $B_1, B_2, \ldots, B_m$ satisfying

$$|B_i \cap B_j| \geq r, \quad |B_i \cap \bar{B}_j| \geq r, \quad |\bar{B}_i \cap B_j| \geq r, \quad |\bar{B}_i \cap \bar{B}_j| \geq r$$
$$(1 \leq i, j \leq m),$$

where $r \geq 1$ is a fixed integer.

2. Determine the maximal m for which there exists a family $B_1, B_2, \ldots, B_m$ satisfying

$$H(B_i, B_j) \geq r \qquad (1 \leq i, j \leq m),$$

where $H(B_i, B_j) = -|B_i \cap B_j| \log |B_i \cap B_j| - |\bar{B}_i \cap B_j| \log |\bar{B}_i \cap B_j| - |B_i \cap \bar{B}_j| \log |B_i \cap \bar{B}_j| - |\bar{B}_i \cap \bar{B}_j| \log |\bar{B}_i \cap \bar{B}_j|$ and r is a positive real number.

The first problem is solved for $r = 1$ in Theorem 1. The second problem is also solved by Theorem 1 for $r = -3\left(\frac{1}{n}\log\frac{1}{n}\right) - \frac{n-3}{n}\log\frac{n-3}{n}$.

§ 2.

A *logical* or *truth function* is an n-dimensional function defined on the n-dimensional 0, 1 vectors and taking on the values 0, 1. A truth function f is said to be *monotonically increasing* if $f(x_1, \ldots, x_n) = 1$ and $x_1 \leq y_1, \ldots, x_n \leq y_n$ imply $f(y_1, \ldots, y_n) = 1$.

(5) $$(z_{i_{11}} \wedge z_{i_{12}} \wedge \ldots \wedge z_{i_{1r}}) \vee \ldots \vee (z_{i_{t1}} \wedge z_{i_{t2}} \wedge \ldots \wedge z_{i_{ts}})$$

is called a *disjunctive-normal form*, where $z_{i_{kl}} = x_{i_{kl}}$ or $1 - x_{i_{kl}}$ and $0 \wedge 0 = 0$, $0 \wedge 1 = 0$, $1 \wedge 0 = 0$, $1 \wedge 1 = 1$ ($\wedge$ = "and"), $0 \vee 0 = 0$, $0 \vee 1 = 1$, $1 \vee 0 = 1$, $1 \vee 1 = 1$ ($\vee$ = "or"). Every truth function has a disjunctive-normal form which is equivalent to it. We may produce such a form in the following way. Fix a 0, 1 vector $e = (a_1, \ldots, a_n)$ satisfying $f(e) = 1$. We correspond an expression $z_1 \wedge z_2 \wedge \dots \wedge z_n$, where $z_i = x_i$ if $a_i = 1$ and $z_i = 1 - x^i$ if $a_i = 0$. It is easy to see that $z_1 \wedge z_2 \wedge \dots \wedge z_n = 1$ if and only if $x_i = a$ $(1 \leq i \leq n)$. These expressions $z_1 \wedge \dots \wedge z_n$ stand in the place of the bracket-expressions in (5) for all e satisfying $f(e) = 1$. It is easy to see that this function is identical to f.

A disjunctive-normal form is *minimal* if it has a minimal number of variables (with multiplicity). Assume f is a monotonically increasing function. It is easy to see that we can omit the terms of the form $z = 1 - x$ from its disjunctive-normal form. Thus, a minimal disjunctive-normal form of a monotonically increasing function has the form

$$(6) \qquad (x_{i_{11}} \wedge \dots \wedge x_{i_{1r}}) \vee \dots \vee (x_{i_{t1}} \wedge \dots \wedge x_{i_{tu}}) .$$

On the other hand, if the index set of one bracket has a proper subset, which is the index set of an other bracket, it can be omitted.

Summarizing what has been said, the minimal disjunctive-normal form of a monotonically increasing function may be determined by a family of subset of the n indices not containing each other. (For the interested reader see [7].)

By this manner the question *what is the maximum of the number of variables* (with multiplicity) *in the minimal disjunctive-normal form of a truth function of n variables is reduced to the problem what is the maximum of the sum of the number of elements in a family consisting of subsets of an n element set not containing each other*. By formula: $\max \sum_{i=1}^{m} |A_i|$, where $A_i \not\subset A_j$ $(i \neq j)$.

We solve the problem in a more general form.

THEOREM 2. *Let $g(k)$ be a real function defined on natural numbers. If $A_1, \dots, A_m$ are subsets of a set of n elements with the property $A_i \not\subset A_j$ $(i \neq j)$ then*

$$\sum_{i=1}^{m} g(|A_i|)$$

attains its maximum for the family of all subsets of

$$\max_{0 \leq k \leq n} g(k) \binom{n}{k}$$

elements.

PROOF. 1. First let us prove the LUBELL—MEŠALKIN inequality ([8], [9]). A family $B_0 \subset B_1 \subset \dots \subset B_n$ of subsets with $|B_i| = i$ $(0 \leq i \leq n)$ is called a *complete chain*. The total number of complete chains is $n!$. The number of complete chains containing A_i $(B_{|A_i|} = A_i)$ is $|A_i|!\,(n - |A_i|)!$. It is easy to see that the complete chains containing A_i are different from the complete chains containing A_j $(i \neq j)$ (using $A_i \not\subset A_j$). Thus, we obtain

$$\sum_{i=1}^{m} |A_i|!\,(n - |A_i|)! \leq n!.$$

It follows the desired inequality

$$\sum_{i=1}^{m} \frac{1}{\binom{n}{|A_i|}} \leq 1. \tag{7}$$

2. We have to maximize $\sum_{i=1}^{m} g(|A_i|)$ under the condition (7). This is trivial:

$$\sum_{i=1}^{m} g(|A_i|) = \sum_{i=1}^{m} \frac{g(|A_i|)\binom{n}{|A_i|}}{\binom{n}{|A_i|}} \leq \sum_{i=1}^{m} \frac{g(z)\binom{n}{z}}{\binom{n}{|A_i|}} \leq g(z)\binom{n}{z},$$

where z is defined by $g(z)\binom{n}{z} = \max g(k)\binom{n}{k}$.

3. The estimation is the best possible as $\sum_{i=1}^{m} g(|A_i|) = g(z)\binom{n}{z}$ for the family of all the sets of z elements. The proof is completed.

Examples. 1. If $g(k) = 1$ $(0 \leq k \leq n)$, then Theorem 2 gives the original Sperner theorem.

2. If $g(k) = k$ $(0 \leq k \leq n)$ we obtain the inequality

$$\sum_{i=1}^{m} |A_i| \leq \left\{\frac{n}{2}\right\}\binom{n}{\left\{\frac{n}{2}\right\}} \tag{8}$$

since

$$\max_{0 \leq k \leq n} k\binom{n}{k} = n \max_{1 \leq k \leq n} \binom{n-1}{k-1} = n\binom{n-1}{\left[\frac{n-1}{2}\right]} = \left\{\frac{n}{2}\right\}\binom{n}{\left\{\frac{n}{2}\right\}},$$

where $\{x\}$ denotes the least integer $\geq x$. (8) gives the solution of the problem induced by the minimal disjunctive-normal form of a truth function. Let us notice that there exists a function which has not a "shorter" disjunctive-normal form: the function which has value 1 iff the number of one is $\geq \left\{\frac{n}{2}\right\}$ in the vector.

3. This example is worthy of formulation as a new theorem.

Theorem 3. (Iterated Sperner theorem.) *Let $A_1, \dots, A_m$ be subsets of a set of n elements satisfying $A_j \not\subset A_k$ $(1 \leq j, k \leq m, j \neq k)$. Let, further $B_{i1}, \dots, B_{im_i}$ be subsets of A_i $(1 \leq i \leq m)$ satisfying $B_{ij} \subset B_{ik}$ $(1 \leq j, k \leq m_i, j \neq k)$. Then the number of subsets*

$$\sum_{i=1}^{m} m_i \leq \binom{n}{\left[\frac{2n}{3}\right]} \binom{\left[\frac{2n}{3}\right]}{\left[\frac{n}{3}\right]}, \tag{9}$$

and the estimation is the best possible.

Proof. By the Sperner theorem we have

$$m_i \leq \binom{|A_i|}{\left[\frac{|A_i|}{2}\right]}. \tag{10}$$

Choose the function $g(k) = \binom{k}{\left[\frac{k}{2}\right]}$. Then, by Theorem 2

$$\sum_{i=1}^{m} \binom{k}{\left[\frac{k}{2}\right]} \leq \binom{z}{\left[\frac{z}{2}\right]} \binom{n}{z}, \tag{11}$$

where z is defined by

$$\binom{z}{\left[\frac{z}{2}\right]} \binom{n}{z} = \max_{0 \leq k \leq n} \binom{k}{\left[\frac{k}{2}\right]} \binom{n}{k}.$$

Here we have

$$\binom{n}{k} \binom{k}{\left[\frac{k}{2}\right]} = \frac{n(n-1)\dots(n-k+1)}{\left[\frac{k}{2}\right]! \, \left\{\frac{k}{2}\right\}!}$$

and

$$\binom{n}{k+1} \binom{k+1}{\left[\frac{k+1}{2}\right]} = \binom{n}{k} \binom{k}{\left[\frac{k}{2}\right]} \cdot \frac{n-k}{\left[\frac{k}{2}\right]+1}.$$

The coefficient satisfies the inequality

$$\frac{n-k}{\left[\frac{k}{2}\right]+1} \begin{cases} > 1 \text{ if } k < \frac{2n-2}{3} \text{ and } k \text{ is even or } k < \frac{2n-1}{3} \text{ and } k \text{ is odd} \\ \leq 1 \text{ if } k \geq \frac{2n-2}{3} \text{ and } k \text{ is even or } k \geq \frac{2n-1}{3} \text{ and } k \text{ is odd.} \end{cases}$$

The maximal k having a coefficient >1 is $\left[\frac{2n}{3}\right]-1$.

Hence we obtain the optimal k:

$$z=\left[\frac{2n}{3}\right]. \tag{12}$$

The theorem follows from (10), (11) and (12) using

$$\left[\frac{\left[\frac{2n}{3}\right]}{2}\right]=\left[\frac{n}{3}\right].$$

It is easy to generalize the theorem to obtain the r times iterated Sperner theorem.

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