FAMILIES OF SUBSETS HAVING NO SUBSET CONTAINING AN OTHER ONE WITH SMALL DIFFERENCE

by G.O.H. KATONA

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Introduction

Sperner proved the following theorem [1]: ***Let*** $A = \{A_1, \ldots, A_m\}$ *be a family of different subsets of a set* S *of* n *elements. If no two of them possess the property*

$$A_i \subset A_j \qquad (i \neq j),$$

then $m \leq \binom{n}{[\frac{n}{2}]}$, *and this is the best possible estimation.*
Erdös [2] answered the question what is the maximum of m if no two subsets satisfy $A_i \subset A_j$, $|A_j - A_i| \geq h$ $(i \neq j)$.
The answer is the sum of the h largest binomial coefficients of order n.
We give now the solution of the contrary problem.

Theorem A. *Let* $A = \{A_1, \ldots, A_m\}$ *be a family of different subsets of a set* S *of* n *elements. If no two of the subsets satisfy* $A_i \subset A_j$, $|A_j - A_i| < k$, *where* k *is a given positive integer, then*

$$(1) \qquad m \leq \sum_{i \equiv [\frac{n}{2}] (\mathrm{mod}\ k)} \binom{n}{i}$$

and this is the best possible estimation.

Kleitman [3] and Katona [4] independently proved a sharpening of Sperner's theorem: Let $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$ be a partition of S. If

* This work was done while the author was at the Department of Statistics of the University of North Carolina at Chapel Hill.

$A = \{A_1, \dots, A_m\}$ is a family of subsets of S and no two different A_i, A_j satisfy the properties

$$A_i \cap S_1 = A_j \cap S_1 \quad \text{and} \quad A_i \cap S_2 \subset A_j \cap S_2$$

or

$$A_i \cap S_1 \subset A_j \cap S_1 \quad \text{and} \quad A_i \cap S_2 = A_j \cap S_2,$$

then $m \le \binom{n}{[\frac{n}{2}]}$ remains true. We give here also a sharpening of Theorem A in this direction.

<u>Theorem</u> B. *Let* $A = \{A_1, \dots, A_m\}$ *be a family of different subsets of a set* S *of* n *elements, where* $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$. *If no two of the subsets satisfy either*

$$A_i \cap S_1 = A_j \cap S_1,\ A_i \cap S_2 \subset A_j \cap S_2 \ \ \text{and}\ |A_j - A_i| < k$$

or

$$A_i \cap S_1 \subset A_j \cap S_1,\ A_i \cap S_2 = A_j \cap S_2 \ \ \text{and}\ |A_j - A_i| < k,$$

then

$$m \le \sum_{i \equiv [\frac{n}{2}] \pmod k} \binom{n}{i} .$$

The next generalization of Theorem A is an analogon of an Erdös [2] generalization of Sperner's theorem.

<u>Theorem</u> C. *Let* $A = \{A_1, \dots, A_m\}$ *be a family of different subsets of a set* S *of* n *elements, let further* k *and* h *be integers* $(1 \le h \le k)$. *If no* h+1 *different members of the family satisfy*

$$A_{i_1} \subset \dots \subset A_{i_{h+1}},\ |A_{i_{h+1}} - A_{i_1}| < k,$$

then

$$m \le \sum_{j=0}^{h-1} \sum_{i \equiv [\frac{n-h+1}{2}] + j \pmod k} \binom{n}{i}, \tag{2}$$

and the estimation is the best possible.

We will prove Theorems B and C in a more general language, which is valid for example for integer-valued functions f ($0 \le f(x_k) \le \alpha_k$, $x_k \in S$) instead of subsets (see [5]).

An interesting application of Theorem A is the following one:

Theorem D. *Let* $a_1, \ldots, a_n$, a, b *be positive integers with the property* $1 \le a_i \le a$ $(1 \le i \le n)$. *The number of sums* $\sum_{i=1}^{n} \varepsilon_i a_i$ ($\varepsilon_i = 0$ *or* 1) *which may be congruent* mod ab *is at most*

$$\sum_{i \equiv [\frac{n}{2}] (\mathrm{mod}\ b)} \binom{n}{i}, \tag{3}$$

and the estimation is the best possible.

Definitions and theorems

We say that the finite set G is a *partially ordered set* if a relation < is defined on G with the following properties: a) at most one of the relations $g_1 < g_2$, $g_1 = g_2$, $g_2 < g_1$ holds; b) if $g_1 < g_2$ and $g_2 < g_3$, then $g_1 < g_3$.

g_2 *covers* g_1 if $g_1 < g_2$ and there is no g_3 satisfying $g_1 < g_3 < g_2$, that is, if g_2 is "immediately greater" than g_1. Assume that there is a rank function r(g) which corresponds a non-negative integer to every element of G, so that the statement g_2 covers g_1 results in $r(g_2) = r(g_1)+1$ and there is at least one element $g \in G$ for which $r(g) = 0$. We say in this case that G *is a partially ordered set with a rank function.*

A chain L *of length* h is a sequence $g_1, \ldots, g_h \in G$, where g_h covers g_{h-1}, g_{h-1} covers g_{h-2}, ..., g_2 covers g_1 ($|L| = h$). A chain is *symmetrical* if $r(g_1) + r(g_h) = n$, where $n = \max_{g \in G} r(g)$.

We say that a partially ordered set is a *symmetrical chain set* if we can split G into disjoint symmetrical chains. (It is defined in [6] under a different name.) We say, further, that a partially ordered set G with a rank function is a mod k *symmetrical chain set* if we can split G into disjoint chains $L_1, \ldots, L_c$ of length at most k such that either

$$|L_i| = k$$

or

$$h = |L_i| < k, \qquad r(g_1) + r(g_h) = n^*$$

for $1 \leq i \leq c$. (That is they are either of length k or symmetrical for the fixed "axis" $[\frac{n^*}{2}]$.) n^* is called *axis*, and it is not uniquely determined.

For example, the subsets of a finite set S on n elements form a partially ordered set if we order them by inclusion. A covers B if $A \supset B$ and $|A-B| = 1$. There is also a rank function $r(A) = |A|$, that is, the number of elements of A.

More generally, let us consider the integer valued functions f defined on $S = \{x_1,\dots,x_q\}$ with the property $0 \leq f(x_k) \leq \alpha_k$, where α_k is a fixed positive integer $(1 \leq k \leq q)$. We define the ordering as follows. $f < g$ if $f(x_k) \leq g(x_k)$ for all x_k and for at least one x_k $f(x_k) < g(x_k)$ holds. g covers f if $g(x_k) = f(x_k)$ for all but one x_k for which $g(x_k) = f(x_k) + 1$ holds. The rank function is $r(f) = \sum_{k=1}^{q} f(x_k)$. It is trivial that this set of functions is a partially ordered set with rank function. Only for the identically zero function is $r(f) = 0$ and $\max_{f\in F} r(f) = \sum_{k=1}^{q} \alpha_k$. ϕ denotes the set of partially ordered sets of this type. If we choose $\alpha_k = 1$ $(1 \leq k \leq q)$ then we obtain the zero-one valued functions which are equivalent to the subsets. This means the partially ordered set of subsets of a set S is an element of ϕ, that is, it is sufficient to consider the partially ordered sets belonging to ϕ. It is proved in [5] that G is a symmetrical chain set, if $G \in \phi$.

Theorem 1. *If* $G \in \phi$, G *is a mod k symmetrical chain set.*

Theorem A is a simple consequence of this theorem, but we will deduce it from the more general Theorem 3.

If G and H are partially ordered sets, then the *direct sum* G + H is the set of ordered pairs (g,h), $g \in G$, $h \in H$ with the ordering $(g_1,h_1) < (g_2,h_2)$ iff $g_1 < g_2$ and $h_1 < h_2$, or $g_1 = g_2$ and $h_1 < h_2$, or $g_1 < g_2$ and $h_1 = h_2$. It follows from this definition, that (g_2,h_2) covers (g_1,h_1) if $g_2 = g_1$ and h_2 covers h_1, or g_2 covers g_1 and $h_2 = h_1$. If the rank function of G and H is r and s, respectively, then we can define a rank function on G + H as follows:

$$t((g,h)) = r(g) + s(h).$$

If G is the partially ordered set of the subsets of a set S_1 and H is the same for S_2 (S_1 and S_2 are disjoint), then G + H is the partially ordered set of the subsets of $S_1 \cup S_2$. The situation is similar in the case of the integer-valued functions; the direct sum of two sets of this type is again a partially ordered set of integer-valued functions defined on the union of the sets.

Now we can formulate the general theorems.

Theorem 2. *Let* G *and* H *be mod k symmetrical chain sets with axes* n_1^* *and* n_2^*, *respectively. If we have a set* $(p_1,q_1), \ldots, (p_m,q_m)$ *of the elements of* G + H, *such that no two different ones of them satisfy the conditions*

$$p_i = p_j,\ q_i < q_j \quad \text{and} \quad t((p_j,q_j)) - t((p_i,q_i)) < k,$$

or

$$p_i < p_j,\ q_i = q_j \quad \text{and} \quad t((p_j,q_j)) - t((p_i,q_i)) < k,$$

then

$$m \leq \sum_{i \equiv \left[\frac{n_1^*+n_2^*}{2}\right] \pmod k} M_i,$$

where M_i *denotes the number of elements of* G + H *with rank* $t((g,h)) = i$. *The estimation is the best possible.*

Theorem 3. *Let* G *be a mod k symmetrical chain set with axis* n^*. *If we have a set* $p_1, \ldots, p_m$ *of the elements of* G *such that no* h + 1 *different ones of them satisfy the conditions*

$$p_{i_1} < \ldots < p_{i_{h+1}},$$

$$r(p_{i_{h+1}}) - r(p_{i_1}) < k,$$

then

$$m \leq \sum_{j=0}^{h-1} \sum_{i \equiv \left[\frac{n^*-h+1}{2}\right]+j \pmod k} K_i,$$

where K_i *denotes the number of elements of* G *with rank* $r(g) = i$. *The estimation is the best possible.*

Proofs

The proofs of Theorems 2 and 3 follow the proof of Theorem 2 of [6].

Proof of Theorem 2. By the definition of the mod k symmetrical sets, G and H are divisible into disjoint chains of length at most k which are either symmetrical or of length exactly k. Denote by G' and H' the partially ordered sets which have ordering relations only along these chains, that is $g_1 < g_2$ can hold only if g_1 and g_2 lie on the same chain. Thus, the set of relations in G'(H') is a part of that in G(H). It follows that the set of relations in G' + H' is a part of that in G + H. So, it is sufficient to prove the statement of the theorem for G' + H' instead of G + H. However, the direct sum of two chains $g_0, \ldots, g_a$ and $h_0, \ldots, h_b$ is a *rectangular lattice* of pairs (g_i,h_j), where (g_i,h_j) covered only by (g_{i+1},h_j) and (g_i,h_{j+1}) (Fig. 1). We say that a rectangular sublattice is *symmetrical* for n^* if the sum of its minimal and maximal rank is equal to n^*.

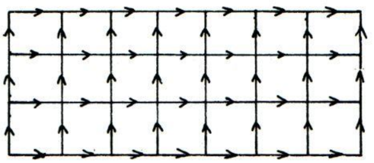

Fig. 1.

So G' + H' consists of rectangular lattices, where either (i) both $g_0, \ldots, g_a$ and $h_0, \ldots, h_b$ are symmetrical for n_1^* and n_2^*, respectively, $(a+1 < k,\ b+1 < k)$, or (ii) one of the numbers a+1, b+1 $(\leq k)$ is equal to k.

By supposition of the theorem in the case (i) the rectangle can contain at most one of $(p_1,q_1), \ldots, (p_m,q_m)$ in each row or column. The maximal number of them is min(a+1, b+1). It is easy to see that if we choose the points of the "middle diagonal" of the rectangle, that is, all the points with rank $\left[\frac{t((g_0,h_0))+t((g_a,h_b))}{2}\right]$, then the number of points is min(a+1, b+1), that is maximal.

In the case (ii) we may assume that the number of rows **is** k. By the supposition of the theorem the rectangle may have at most one of (p_1,q_1), ..., (p_m,q_m) in each column. It is easy to see if we choose all the points of the rectangle with rank $\equiv \left[\frac{n_1^*+n_2^*}{2}\right]$ (mod k) then we choose the maximal number of them. This set of points satisfies the supposition of the theorem, since it contains exactly one point in each column and the distance of the consecutive points in a row is k.

We have to verify only that the union of the sets of points chosen in the manner described above gives the set of points with rank $\equiv \left[\frac{n_1^*+n_2^*}{2}\right]$ (mod k) in G + H.

To the first part it is sufficient to prove that

$$\left[\frac{t((g_0,h_0))+t((g_a,h_b))}{2}\right] \equiv \left[\frac{n_1^*+n_2^*}{2}\right] \pmod{k}.$$

We may show that exact equality holds. Indeed, by the symmetricity

$$t((g_0,h_0)) + t((g_a,h_b)) = r(g_0) + s(h_0) + r(g_a) + s(h_b).$$

Conversely, every point of G + H with rank $\equiv \left[\frac{n_1^*+n_2^*}{2}\right]$ (mod k) is contained in a rectangular. If the rectangular is of type (ii) the point is chosen above. If the rectangular is of type (i) then the considered point has a rank $\left[\frac{n_1^*+n_2^*}{2}\right]$ (which is chosen above) because a rectangular of type (i) cannot contain points with rank $\equiv \left[\frac{n_1^*+n_2^*}{2}\right]$ (mod k) except which have rank $\left[\frac{n_1^*+n_2^*}{2}\right]$. Indeed, $t((g_0,h_0))$ and $t((g_a,h_b))$ differ from $\left[\frac{t((g_0,h_0))+t((g_a,h_b))}{2}\right] = \left[\frac{n_1^*+n_2^*}{2}\right]$ with at most $\{\frac{a+b}{2}\}$ (where $\{x\}$ denotes the least integer greater than or equal to x), and $\{\frac{a+b}{2}\} \leq k-1$. The proof is completed.

Proof of Theorem 3. Let us divide G into disjoint chains of form $g_0, \ldots, g_a$ which are either (i) symmetrical ($r(g_0) + r(g_a) = n^*$) or (ii) of length k ($a+1 = k$). By the supposition of the theorem each chain can contain at most min(h,a+1) points from $p_1, \ldots, p_m$. Let us choose the optimal set in each chain in the following manner:

Case (i) : points g_j where $[\frac{a-h+1}{2}] \leq j \leq [\frac{a-h+1}{2}] + h - 1$, if $a+1 > h$,

and all the points if $a+1 \leq h$.

Case (ii): points with rank $\equiv [\frac{n^*-h+1}{2}] + i$, where $0 \leq i \leq h-1$.

We have to verify only that the union of the sets of points chosen in the above described manner gives the set of all points with rank $\equiv [\frac{n^*-h+1}{2}] + i$ $(0 \leq i \leq h-1)$.

To the first part of this statement it is sufficient to prove that $r(g_j)$ runs from $[\frac{n^*-h+1}{2}]$ to $[\frac{n^*-h+1}{2}] + h - 1$ if j runs from $[\frac{a-h+1}{2}]$ to $[\frac{a-h+1}{2}] + h - 1$. But this follows from the symmetricity, since

$$r(g_j) = r(g_0) + j = r(g_0) + [\frac{a-h+1}{2}] + i = r(g_0) + \left[\frac{r(g_a)-r(g_0)-h+1}{2}\right] + i =$$
$$= \left[\frac{r(g_0)+r(g_a)-h+1}{2}\right] + i = [\frac{n^*-h+1}{2}] + i.$$

Conversely, every point of G with rank $\equiv [\frac{n^*-h+1}{2}] + i$ $(0 \leq i \leq h-1)$ is contained in a chain of type (i) or (ii) and it is chosen there.

The proof is completed.

In order to use Theorems 2 and 3 for finite functions (elements of ϕ) we have to prove Theorem 1.

<u>Lemma</u>. *If* G *and* H *are* mod k *symmetrical chain sets then* G + H *is a* mod k *symmetrical chain set.*

<u>Proof</u>. We proved at the beginning of the proof of Theorem 2 that G + H is divisible into disjoint union of rectangular lattices $R = \{g_0,\dots,g_a\} + \{h_0,\dots,h_b\}$ where either (i) both $\{g_0,\dots,g_a\}$ and $\{h_0,\dots,h_b\}$ are symmetrical for some n_1^*, n_2^* $(a+1 < k,\ b+1 < k)$ or (ii) one of the numbers $a+1$, $b+1$ $(\leq k)$ is equal to k.

Now, we have to divide these rectangular lattices into chains of length at most k which are either symmetrical for $n_1^* + n_2^*$ or of length k. In the case (ii) it is very easy. Put e.g. $a+1 = k$. The lengths of the chains (g_0,h_i), ..., (g_a,h_i) $(0 \leq i \leq b)$ are k, and their union is R.

In the contrary case (i), when $a+1 < k$, $b+1 < k$ both chains are symmetrical:

$$r(g_0) + r(g_a) = n_1^*, \qquad s(h_0) + s(h_b) = n_2^*.$$

The new chains will be the following:

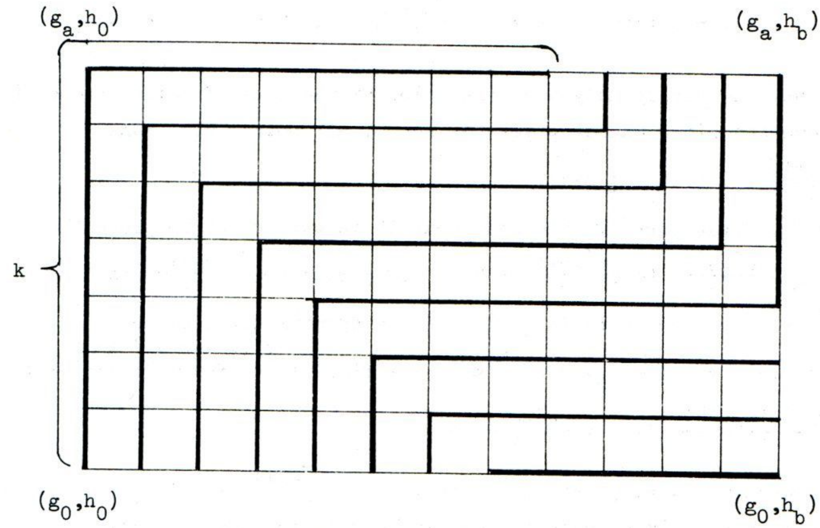

Fig. 2.

Or formally, but less clearly (assume $a \leq b$ and $a+b+1 \geq k$):

$L_1 = \{(g_0,h_0),(g_1,h_0),\ldots,(g_a,h_0),(g_a,h_1),\ldots,(g_a,h_{k-a-1})\}$,

$L_2 = \{(g_0,h_1),(g_1,h_1),\ldots,(g_{a-1},h_1),(g_{a-1},h_2),\ldots,(g_{a-1},h_{k-a}),(g_a,h_{k-a})\}$,

$L_3 = \{(g_0,h_2),(g_1,h_2),\ldots,(g_{a-2},h_2),(g_{a-2},h_3),\ldots$
$\ldots,(g_{a-2},h_{k-a+1}),(g_{a-1},h_{k-a+1}),(g_a,h_{k-a+1})\}$,

.
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$L_i = \{(g_0,h_{i-1}),(g_1,h_{i-1}),\ldots,(g_{a-i+1},h_{i-1}),(g_{a-i+1},h_i),\ldots$
$\ldots,(g_{a-i+1},h_{k-a+i-2}),\ldots,(g_{a-i+2},h_{k-a+i-2}),\ldots,(g_a,h_{k-a+i-2})\}$,

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$L_{a+b-k+2} = \{(g_0,h_{a+b-k+1}),\ldots,(g_{k-b-1},h_{a+b-k+1}),(g_{k-b-1},h_{a+b-k+2}),\ldots$
$\ldots,(g_{k-b-1},h_b),\ldots,(g_a,h_b)\}$,

$L_{a+b-k+3} = \{(g_0,h_{a+b-k+2}),\ldots,(g_{k-b-2},h_{a+b-k+2}),(g_{k-b-2},h_{a+b-k+3}),\ldots$
$\ldots,(g_{k-b-2},h_b)\}$,

.
.
.

$L_j = \{(g_0,h_{j-1}),(g_1,h_{j-1}),\ldots,(g_{a-j+1},h_{j-1}),(g_{a-j+1},h_j),\ldots,(g_{a-j+1},h_b)\}$,

.
.
.

$L_{a+1} = \{(g_0,h_a),\ldots,(g_0,h_b)\}$.

From Fig. 2 it is easy to see, that the new chains are either of length k or symmetrical. Formally: L_i is of length k if $1 \leq i \leq a+b-k+2$ because of $a - i + 2 + ((k-a+i-2) - (i-1)) + (a - (a-i+1)) = k$. L_j is symmetrical if $a+b - k+3 \leq j \leq a+1$ because $r(g_0) + s(h_{j-1}) + r(g_{a-j+1}) + s(h_b) =$
$= r(g_0) + r(g_a) - (j-1) + s(h_0) + j - 1 + s(h_b) =$
$= n_1^* + n_2^*$.

If $a+b+1 < k$, the situation is similar (even simpler) and we have only symmetrical new chains:

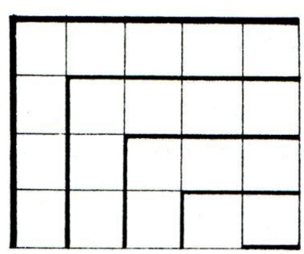


Fig. 3.

We have proved the lemma, which is a generalization of the basic idea of [5].

If $G \in \phi$, then $G = \sum_{i=1}^{q} L_i$ where L_i's are totally ordered sets (chains). Let $\alpha_i+1 = |L_i|$ be the length of the chain L_i. α_i is called the i-th size of G. We say that the size α_i is *even*, or *odd* if in the equation $\alpha_i+1 = \alpha k+\beta$ $(0 \leq \beta < k)$ α is even or odd, respectively. It is easy to see that L_i is a mod k symmetrical chain set and its axis may be chosen as

$$\alpha_i \quad \text{if } \alpha_i \text{ is even}$$

and

$$\alpha_i \pm k \quad \text{if } \alpha_i \text{ is odd.}$$

(See Fig. 4.)

Fig. 4.

Applying the lemma we obtain that G is a mod k symmetrical chain set, and in addition, its axis may be chosen as

$$\sum_{i=1}^{q} \alpha_i \quad \text{if the number of odd sizes is even}$$

and

$$\sum_{i=1}^{q} \alpha_i + k \quad \text{if the number of odd sizes is odd.}$$

As the maximal rank is $n = \sum_{i=1}^{q} \alpha_i$, we may also write the axis in the form n or n + k. The proof is completed.

<u>Proof of Theorem</u> B. Let G and H be the subsets of the sets S_1 and S_2, respectively. The conditions of theorem 2 and theorem B are equivalent in this case. G and H are mod k symmetrical chain sets, by theorem 1. Here $\alpha_i = 1$ so all the sizes are even for $k > 0$. Consequently, the axes n_1^*, n_2^* are given by

$$n_1^* = n_1 = |S_1|, \qquad n_2^* = n_2 = |S_2|,$$

that is, $n_1^* + n_2^* = n_1 + n_2 = n$. The proof is completed.

<u>Proof of Theorems</u> C and A. We apply theorem 3 for the subsets of S, which is a mod k symmetrical chain set by theorem 1. For theorem A we substitute h = 1 into theorem C. The proofs are completed.

Proof of Theorem D. Consider the sums $\sum_{i=1}^{n} \varepsilon_i a_i \equiv c \pmod{ab}$. Let A_j be the set of indices i where $\varepsilon_i = 1$ in the j-th sum. If $A_{j'} \supset A_j$ and $|A_{j'} - A_j| < b$, then for the corresponding sums

$$\sum_{i \in A_{j'}} a_i - \sum_{i \in A_j} a_i = \sum_{i \in A_{j'} - A_j} a_i .$$

Here

$$1 \leq |A_{j'} - A_j| \cdot 1 \leq \sum_{i \in A_{j'} - A_j} a_i \leq |A_{j'} - A_j| \cdot a < ab,$$

which is a contradiction: the sums cannot be congruent mod ab. Thus we may apply theorem A for $A_1, \ldots, A_m$:

$$m \leq \sum_{i \equiv [\frac{n}{2}] \pmod{b}} \binom{n}{i}.$$

The proof is completed. If the modulus d is not a multiple of the bound a, we obtain a similar inequality

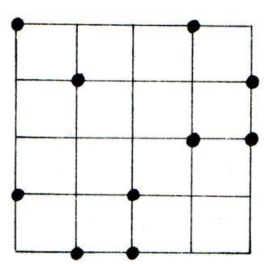
$$m \leq \sum_{i \equiv [\frac{n}{2}] \pmod{[\frac{d}{a}]}} \binom{n}{i},$$

but we cannot reach this upper bound in any case.

Further remarks

Why did we not prove theorem 3 under the weaker conditions of type given in theorem 2 (see [6])?

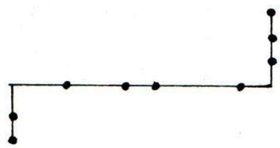
The following example shows the reason:



Here k = 6 and h = 2. This rectangle is a direct sum of two totally ordered sets. The set of points given in the figure does not contain a triple of point of form

(4) 

with length ≤ 6, but their number is 10, while the number of elements of the two largest "middle diagonal" is 9. It is easy to see that if we exclude the configurations

(5) 

with h + 1 points and with length ≤ k, then the general statement follows from the proofs. For general h the configuration

(6) 

is excluded in [6] instead of (4) (with h + 1 points). The counterexample shows that to exclude (6) with length ≤ k is too weak, however to exclude (5) is too strong. To find a good condition between (5) and (6) would be interesting.

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