On strengthenings of the intersecting shadow theorem

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Abstract

Let $n > k > t \geq j \geq 1$ be integers. Let X be an n-element set, $\binom{X}{k}$ the collection of its k-subsets. A family $\mathcal{F} \subset \binom{X}{k}$ is called t-intersecting if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. The j'th shadow $\partial^j \mathcal{F}$ is the collection of all $(k-j)$-subsets that are contained in some member of $\mathcal{F}$. Estimating $|\partial^j \mathcal{F}|$ as a function of $|\mathcal{F}|$ is a widely used tool in extremal set theory. A classical result of the second author (Theorem 1.3) provides such a bound for t-intersecting families. It is best possible for $|\mathcal{F}| = \binom{2k-t}{k}$.

Our main result is Theorem 1.4 which gives an asymptotically optimal bound on $|\partial^j \mathcal{F}|/|\mathcal{F}|$ for $|\mathcal{F}|$ slightly larger, e.g., $|\mathcal{F}| > \frac{3}{2}\binom{2k-t}{k}$. We provide further improvements for $|\mathcal{F}|$ very large as well.

1 Introduction

Throughout the paper n, k, t are positive integers, $n > k > t$. Let $[n] = \{1, 2, \dots, n\}$ be the standard n-element set and $\binom{[n]}{k}$ the collection of all its k-subsets. For a family $\mathcal{F} \subset \binom{[n]}{k}$ and $0 < j < k$ define the *j'th shadow* $\partial^j \mathcal{F} = \left\{ G \in \binom{[n]}{k-j} : \exists F \in \mathcal{F}, G \subset F \right\}$.

Estimating the minimum possible size, $|\partial^j \mathcal{F}|$ in function of $|\mathcal{F}|$ has proved to be one of the most important tools of extremal set theory. As a matter of fact, the first paper written on this subject, due to Sperner, is heavily relying on such a bound.

Proposition 1.1 (Sperner [S]). *Suppose that* $\emptyset \neq \mathcal{F} \subset \binom{[n]}{k}$, $0 < j < k$. *Then*

$$|\partial^j \mathcal{F}|/|\mathcal{F}| \geq \binom{n}{k-j} \Big/ \binom{n}{k} \tag{1.1}$$

with equality holding iff $\mathcal{F} = \binom{[n]}{k}$.

The classical Kruskal–Katona Theorem ([Kr], [Ka2]) determines the minimum of $|\partial^j \mathcal{F}|$, given $|\mathcal{F}|$.

For $j = 1$ the notation $\partial\mathcal{F}$ is common and $\partial\mathcal{F}$ is called the *immediate shadow.*

Definition 1.2. Let $0 \leq \ell < k$, $\mathcal{F} \subset \binom{[n]}{k}$. Define the ℓ*-shadow* $\sigma_\ell(\mathcal{F})$ by

$$\sigma_\ell(\mathcal{F}) = \left\{ G \in \binom{[n]}{\ell} : \exists F \in \mathcal{F},\, G \subset F \right\}.$$

Note that $\partial\mathcal{F} = \sigma_{k-1}(\mathcal{F})$ and $\partial^{k-\ell}\mathcal{F} = \sigma_\ell(\mathcal{F})$.

One of the most widely investigated properties in extremal set theory is the t-intersecting property. For $t \geq 1$, $\mathcal{F}$ is said to be t*-intersecting* if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. For $t = 1$, the term *intersecting* is used as well.

A widely used result of the second author shows that $|\partial^j \mathcal{F}| \geq |\mathcal{F}|$ for $0 < j \leq t$ provided that $\mathcal{F}$ is t-intersecting.

Theorem 1.3 (Intersecting Shadow Theorem [Ka1]). *Suppose that* $\emptyset \neq \mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{F}$ *is* t*-intersecting,* $k - t \leq \ell < k$. *Then*

$$\big|\sigma_\ell(\mathcal{F})\big| \Big/ |\mathcal{F}| \geq \binom{2k-t}{\ell} \Big/ \binom{2k-t}{k} \tag{1.2}$$

with strict inequality unless $\mathcal{F} = \binom{Y}{k}$ *for some* $2k - t$*-element set* Y.

Note that for $n \leq 2k - t$ the inequality (1.2) can be deduced from Sperner's bound (1.1). However for fixed k and n tending to infinity the RHS of (1.1) tends to 0 while the RHS of (1.2) is at least 1. To be more exact, for $\ell = k-1$ its value is $k \big/ (k - t + 1)$. For $t \geq 2$ this is strictly larger than 1. Our first result gives a further improvement provided that $|\mathcal{F}| \geq \left(1 + \frac{t-1}{k+t}\right)\binom{2k-t}{k}$.

Theorem 1.4. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$, $\mathcal{F}$ is t-intersecting, $1 \leq j < t < k$, $|\mathcal{F}| \geq \binom{2k-t}{k}\left(1 + \frac{t-j}{k+t+1-j}\right)$. Then*

$$\left|\partial^j \mathcal{F}\right| / |\mathcal{F}| \geq \binom{2(k-1)-t}{k-1-j} \Big/ \binom{2(k-1)-t}{k-1}. \tag{1.3}$$

Let us mention that the requirement on $|\mathcal{F}|$ is relatively weak, e.g., it is weaker than $|\mathcal{F}| \geq \frac{3}{2}\binom{2k-t}{k}$. For $j = 1$, the most widely used case, (1.3) reduces to

$$|\partial \mathcal{F}|/|\mathcal{F}| \geq \frac{k-1}{k-t}.$$

At first sight it might appear to be only a small improvement with respect to $\frac{k}{k-t+1}$, coming from (1.2). However, for k and t fixed the difference is substantial. Most importantly, the new bound is essentially best possible.

Example 1.5. Fix $k > t > 2$ and an integer s, $0 \leq s < k - t - 1$. Define $\mathcal{A} = \left\{A \in \binom{[2k-t]}{k} : |A \cap [k-1+s]| \geq t + s\right\}$, $\mathcal{B} = \left\{B \in \binom{[n]}{k}, B_0 \cup \{x\}, B_0 \in \binom{[k-1+s]}{k-1}, x \in [2k-t+1, n]\right\}$. Set $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$. Then $\mathcal{F}$ is t-intersecting.

Proposition 1.6. *For a proper choice of s and n, Example 1.5 shows that (1.3) does not hold for $k > k_0(j)$ even if*

$$|\mathcal{F}| = \left(1 + \frac{j(t-j)s(s-1)\cdot \ldots \cdot (s-j+1)}{(k-1)^{j+2}} - o(1)\right)\binom{2k-t}{k}.$$

The paper is organized as follows. In Section 2 we review some results concerning shifting and shifted families. Then we prove Theorem 2.10 concerning shadows. In Section 3 we prove Theorem 1.4, in the very short Section 4 the proof of Proposition 1.6 is provided.

In Section 5 we introduce the notion of a semistar and prove a best possible lower bound on the shadow of t-intersecting semistars (Theorem 5.5). In Section 6 along with some structural results we prove the best possible bound $\left|\partial^j \mathcal{F}\right| > \binom{t}{j}|\mathcal{F}|$ for families satisfying $|\mathcal{F}| > (t+2)\binom{n-t-1}{k-t-1}$, $n > n_0(k,t)$ in a more precise form.

Section 7 contains some more general results.

2 Preliminaries

Let $(a_1, \ldots, a_k)$ denote the k-element set $\{a_1, \ldots, a_k\}$ where we know that $a_1 < \ldots < a_k$. Let us define $\prec$, the *shifting partial order* by setting

$$(a_1, \ldots, a_k) \prec (b_1, \ldots, b_k) \quad \text{iff} \quad a_i \le b_i \quad \text{for} \quad 1 \le i \le k.$$

Definition 2.1. The family $\mathcal{F}$ is called *shifted* if $(a_1, \ldots, a_k) \prec (b_1, \ldots, b_k)$ and $(b_1, \ldots, b_k) \in \mathcal{F}$ always imply $(a_1, \ldots, a_k) \in \mathcal{F}$.

In their seminal paper [EKR], Erdős, Ko and Rado defined a simple operation on families of sets called shifting. Repeated application of this operation eventually transforms a family into a shifted family. Erdős, Ko and Rado showed that shifting maintains the t-intersecting property. In [Ka1] it is shown that shifting never increases the ℓ-shadow. Consequently, it is sufficient to prove Theorem 1.4 for shifted families.

On the other hand, shifted t-intersecting families have some nice properties.

Proposition 2.2 ([F78]). *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is shifted and t-intersecting. Then for every $F \in \mathcal{F}$ there exists an integer h, $0 \le h \le k - t$ such that*

$$\big|F \cap [t + 2h]\big| \ge h + t. \tag{2.1}$$

In [F78] the following families were defined:

$$\mathcal{A}_h(n, k, t) = \left\{ A \in \binom{[n]}{k} : \big|A \cap [t + 2h]\big| \ge h + t \right\}.$$

It is easy to see that $\mathcal{A}_h(n, k, t)$ is always t-intersecting.

In [F78] it was conjectured that for $n \ge 2k - t$,

$$|\mathcal{F}| \le \max\left\{\big|\mathcal{A}_h(n, k, t)\big| : 0 \le h \le k - t\right\}. \tag{2.2}$$

In [FF2] (2.2) was proved for a wide range. However, it was not before the seminal paper of Ahlswede and Khachatrian [AK1] that (2.2) was established in its integrity.

It is easy to check that for k and t fixed

$$\lim_{n \to \infty} \big|\partial^j \mathcal{A}_{k-t-1}(n, k, t)\big| \,/\, \big|\mathcal{A}_{k-t-1}(n, k, t)\big| = \binom{2(k-1)-t}{k-1-j} \Big/ \binom{2(k-1)-t}{k-1}$$

which shows that (1.3) is essentially best possible.

Based on Proposition 2.2 one can define the following relaxation of the t-intersecting property.

Definition 2.3. The family $\mathcal{F} \subset \binom{[n]}{k}$ is said to be *pseudo t-intersecting* if for every $F \in \mathcal{F}$ and some h, $0 \le h \le k-t$, (2.1) holds.

It was shown in [F91] that (1.2) holds for pseudo t-intersecting families as well.

We need some more definitions.

Let $\mathcal{F} \subset \binom{[n]}{k}$ be pseudo t-intersecting. Define the *width* $w = w_t(\mathcal{F})$ as the minimum integer such that for every $F \in \mathcal{F}$ (2.1) holds for some h, $0 \le h \le w$. From Definition 2.3 it is clear that $w_t(\mathcal{F})$ exists and $w_t(\mathcal{F}) \le k-t$. However, in certain situations it needs to be smaller. For example, define $\mathcal{F}_{\text{out}} = \mathcal{F} \setminus \binom{[2k-t]}{k}$. For $F \in \mathcal{F}_{\text{out}}$, $|F \cap [2k-t]| < k$ implies $w_t(\mathcal{F}_{\text{out}}) \le k-t-1$. This will be very important for our proofs.

Definition 2.4. Let $\mathcal{F} \subset \binom{X}{k}$ be pseudo t-intersecting and $w = w_t(\mathcal{F})$. For $F \in \mathcal{F}$ define its *height* $h(F)$ as

$$h(F) = \max \left\{ h : 0 \le h \le w, \left| F \cap [t+2h] \right| \ge t+h \right\}.$$

Claim 2.5. *If $h(F) < w$ then*

$$\left| F \cap [t+2h(F)] \right| = t + h(F). \tag{2.3}$$

Proof. Should $\left| F \cap [t+2h(F)] \right| \ge t+h(F)+1$ hold, we conclude

$$\left| F \cap [t+2(h(F)+1)] \right| \ge t+h(F)+1,$$

contradicting the maximal choice of $h(F)$. □

Let us define the tail $T = T(F)$ for $F \in \mathcal{F}$ by $T(F) = F \setminus [t+2h(F)]$. In view of (2.3),

$$|T(F)| = k-t-h(F) \quad \text{holds if} \quad h(F) < w_t(\mathcal{F}). \tag{2.4}$$

If $h(F) = w_t(\mathcal{F})$ then either (2.4) holds or

$$|T| < k-t-h(F).$$

Definition 2.6. For $0 < j \le t$ and $F \in \mathcal{F}$ let us define the *restricted j'th shadow* $\partial_R^j F = \left\{ G \in \binom{F}{k-j} : T \subset G \right\}$. In human language G is obtained from F by arbitrarily deleting j vertices from $F \setminus T$.

Claim 2.7. *If $h(F) < w_t(\mathcal{F})$ and $G \in \partial^j_R F$ then* (i) *and* (ii) *hold.*

(i) $\big|G \cap [t+2h(F)]\big| = t - j + h(F)$,

(ii) $\big|G \cap [t+2h]\big| < t - j + h$ *for* $h(F) < h \leq w_t(\mathcal{F})$. □

Applying this claim we infer

Corollary 2.8. *Suppose that $F, F' \in \mathcal{F}$, $h(F) < h(F')$. Then*

$$\partial^j_R F \cap \partial^j_R F' = \emptyset. \tag{2.5}$$

Proof. Using (i) and (ii)

$$\big|G \cap [t+2h(F')]\big| < \big|G' \cap [t+2h(F')]\big|$$

follows for $G \in \partial^j_R F$ and $G' \in \partial^j_R F'$. □

Note that (2.5) is immediate also if $h(F) = h(F')$ but $T(F) \neq T(F')$. Define $\mathcal{T} = \big\{T \subset [n] : \exists F \in \mathcal{F}, T(F) = T\big\}$. For $T \in \mathcal{T}$ define $\mathcal{F}_T = \{F \in \mathcal{F} : T(F) = T\}$ and $\overline{\mathcal{F}}_T = \big\{F \setminus T : F \in \mathcal{F}_T\big\}$. This permits to define the restricted j'th shadow of $\mathcal{F}_T$:

$$\partial^j_R \mathcal{F}_T = \bigcup_{F \in \mathcal{F}_T} \partial^j_R F.$$

The next lemma is the core of the proofs.

Lemma 2.9. *Suppose that $\mathcal{F}$ is pseudo t-intersecting, $0 < j \leq t$. Then $\mathcal{F} = \bigcup\limits_{T \in \mathcal{T}} \mathcal{F}_T$ is a partition, and*

$$\big|\partial^j \mathcal{F}\big| \geq \sum_{T \in \mathcal{T}} \big|\partial^j_R \mathcal{F}_T\big|. \tag{2.6}$$

Proof. The first part is trivial. To show the second one we need to prove for $T, T' \in \mathcal{T}$, $T \neq T'$,

$$\partial^j_R \mathcal{F}_T \cap \partial^j_R \mathcal{F}_{T'} = \emptyset.$$

This follows from (2.5) unless both F and F' with $T(F) = T$ and $T(F') = T'$ satisfy $h(F) = h(F') = w = w_t(\mathcal{F})$. (Actually, by (2.3) these are equivalent to $|T|, |T'| \leq k - t - w$.) In this case $T = F \setminus [t+2w]$, $T' = F' \setminus [t+2w]$ imply $\partial^j_R \mathcal{F}_T \cap \partial^j_R \mathcal{F}_{T'} = \emptyset$. □

With this preparation the next theorem is easy to prove.

Theorem 2.10. *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted pseudo t-intersecting of width $w = w_t(\mathcal{F})$. Then for every $0 < j \leq t$,*

$$\left|\partial_R^j \mathcal{F}\right| \geq |\mathcal{F}| \binom{t+2w}{t-j+w} \Big/ \binom{t+2w}{t+w}. \tag{2.7}$$

Proof. Let $\mathcal{T}$ be the family of possible tails for $\mathcal{F}$. In view of Lemma 2.9 it is sufficient to show

$$\left|\partial_R^j \mathcal{F}_T\right| \geq |\mathcal{F}_T| \binom{t+2w}{t-j+w} \Big/ \binom{t+2w}{t+w}. \tag{2.8}$$

Recall that $\overline{\mathcal{F}}_T = \{F \setminus T : F \in \mathcal{F}_T\}$. If $|T| \geq k-t-w$ then $\overline{\mathcal{F}}_T \subset \binom{[t+2(k-t-|T|)]}{k-|T|}$ and $\left|\partial_R^j \mathcal{F}_T\right| = \left|\partial^j \overline{\mathcal{F}}_T\right|$.

If $|T| < k-t-w$ then $\overline{\mathcal{F}}_T \subset \binom{[t+2w]}{k-|T|}$ and again $\left|\partial_R^j \mathcal{F}_T\right| = \left|\partial_j \overline{\mathcal{F}}_T\right|$. In the first case $t+2(k-t-|T|) = 2(k-|T|)-t$, showing that $\overline{\mathcal{F}}_T$ is t-intersecting. In the second case $t+2w > 2(k-|T|)-t$ by $w+|T| < k-t$, that is $\overline{\mathcal{F}}_T$ is $(t+1)$-intersecting. However, the desired bound readily follows using (1.1) and the next proposition.

Proposition 2.11. *Let $0 < j < t$, $0 \leq h < w$ and $1 \leq r \leq w$, then the following two inequalities hold.*

(i) $\displaystyle \binom{t+2h}{t+h-j} \Big/ \binom{t+2h}{t+h} > \binom{t+2w}{t+w-j} \Big/ \binom{t+2w}{t+w}$,

(ii) $\displaystyle \binom{t+2w}{t+w-j+r} \Big/ \binom{t+2w}{t+w+r} > \binom{t+2w}{t+w-j} \Big/ \binom{t+2w}{t+w}$.

Proof. Let $f(h)$ denote the LHS of (i). That is, $f(h) = \prod_{1\leq i\leq j} \frac{t+h-j+i}{h+i} = \prod_{1\leq i\leq j} \left(1 + \frac{t-j}{h+i}\right)$. Since $1 + \frac{t-j}{h+i}$ is a strictly monotone decreasing function of h, $f(h) > f(w)$ follows.

To prove (ii) let $g(r)$ be the LHS, i.e.,

$$g(r) = \prod_{1\leq i\leq j} \frac{t-j+w+i+r}{w+i-r}.$$

Since $\frac{a+r}{b-r}$ is a strictly monotone increasing function of r (for $a > 0$, $b > r$), $g(r) > g(0)$ and thereby (ii) follows. □

This concludes the proof of Theorem 2.10 as well. □

3 The proof of Theorem 1.4

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a shifted t-intersecting family, $t \geq 2$. If $w_t(\mathcal{F}) \leq k-t-1$ then for every $1 \leq j < t$, from Theorem 2.10 we infer

$$\left|\partial^j \mathcal{F}\right| \geq |\mathcal{F}| \binom{t+2(k-t-1)}{k-1-j} \Big/ \binom{t+2(k-t-1)}{k-1}$$

proving (1.3).

From now on we suppose $w_t(\mathcal{F}) = k-t$ and fix an $A = (a_1, \ldots, a_k) \in \mathcal{F}$ such that

$$\left|A \cap [t+2h]\right| \leq t+h-1 \quad \text{for} \quad 0 \leq h < k-t. \tag{3.1}$$

Applying (2.1) to A yields $\left|A \cap [t+2(k-t)]\right| = k$, i.e., $A \in \binom{[2k-t]}{k}$. Our plan for proving (1.3) is the following. We partition $\mathcal{F}$ into two families $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$ where $\mathcal{F}_{\text{in}} = \mathcal{F} \cap \binom{[2k-t]}{k}$, $\mathcal{F}_{\text{out}} = \mathcal{F} \setminus \mathcal{F}_{\text{in}}$. Then we show that

$$\partial^j \mathcal{F}_{\text{in}} \cap \partial^j_R \mathcal{F}_{\text{out}} = \emptyset \tag{3.2}$$

and thereby

$$\left|\partial^j \mathcal{F}\right| \geq \left|\partial^j \mathcal{F}_{\text{in}}\right| + \left|\partial^j_R \mathcal{F}_{\text{out}}\right|. \tag{3.3}$$

For the first term on the RHS we use (1.2) with $\ell = k-j$. As for the second, we prove a stronger inequality

$$\left|\partial^j_R \mathcal{F}_{\text{out}}\right| \geq \left|\mathcal{F}_{\text{out}}\right| \binom{t+1+2(k-t-2)}{k-1-j} \Big/ \binom{t+1+2(k-t-2)}{k-1}. \tag{3.4}$$

Defining $\alpha = \alpha(k,t,j)$ and $\beta = \beta(k,t,j)$ by

$$\alpha = \frac{\binom{t+2(k-t-1)}{k-1-j}}{\binom{t+2(k-t-1)}{k-1}} - \frac{\binom{t+2(k-t)}{k-j}}{\binom{t+2(k-t)}{k}}, \qquad \beta = \frac{\binom{t+1+2(k-t-2)}{k-1-j}}{\binom{t+1+2(k-t-2)}{k-1}} - \frac{\binom{t+2(k-t)}{k-j}}{\binom{t+2(k-t)}{k}},$$

(3.3) and (3.4) imply

$$\left|\partial^j \mathcal{F}\right| \geq \left(\left|\mathcal{F}_{\text{in}}\right| + \left|\mathcal{F}_{\text{out}}\right|\right) \frac{\binom{t+2(k-t)}{k-j}}{\binom{t+2(k-t)}{k}} + \beta\left|\mathcal{F}_{\text{out}}\right|. \tag{3.5}$$

Finally we show that the assumption on $|\mathcal{F}|$ implies

$$\left|\mathcal{F}_{\text{out}}\right| \geq |\mathcal{F}| - \binom{2k-t}{k} \geq \frac{\alpha}{\beta}|\mathcal{F}|.$$

Plugging this into (3.5) yields

$$\left|\partial^j \mathcal{F}\right| \geq \left(\frac{\binom{t+2(k-t)}{k-j}}{\binom{t+2(k-t)}{k}} + \alpha \right) |\mathcal{F}| = \frac{\binom{t+2(k-t-1)}{k-1-j}}{\binom{t+2(k-t-1)}{k-1}} |\mathcal{F}|, \quad \text{as desired.}$$

Let us now execute this plan. (3.2) is essentially trivial. If $G \in \partial^j \mathcal{F}_{\text{in}}$ then $G \subset [2k-t]$. For $F \in \mathcal{F}_{\text{out}}$, $F \not\subset [2k-t]$ and the definition of the restricted shadow imply that $G' \not\subset [2k-t]$ for each $G' \in \partial^j_R F$. Thus $G \neq G'$.

To prove (3.4) let us show:

Proposition 3.1. *The family $\mathcal{F}_{\text{out}}$ is pseudo $(t{+}1)$-intersecting and $w_{t+1}(\mathcal{F}_{\text{out}}) \leq k-t-2$.*

Proof. Define the two sets E and D as follows.

$$\begin{aligned} E &= (1, 2, \ldots, t-1, t+1, t+3, \ldots, 2k-t-3, 2k-t-1, 2k-t), \\ D &= (1, 2, \ldots, t, t+2, t+4, \ldots, 2k-t-2, 2k-t+1). \end{aligned}$$

Note that $E \cap D = [t-1]$.

Let us show that $E \prec A$, implying $E \in \mathcal{F}$. $i \leq a_i$ is trivial for $1 \leq i < t$. As to a_{t+h}, $0 \leq h < k-t$, (3.1) implies $t+2h+1 \leq a_{t+h}$. Finally, using this inequality for $h = k-t-1$ gives $2k-t-1 \leq a_{k-1}$ implying $2k-t \leq a_k$. By shiftedness $E \in \mathcal{F}$. On the other hand the t-intersecting property and $|D \cap E| = t-1$ imply $D \notin \mathcal{F}$.

Choose an arbitrary $B = (b_1, \ldots, b_k) \in \mathcal{F}_{\text{out}}$. As $\mathcal{F}$ is shifted, $D \prec B$ cannot hold. Note that for $1 \leq i \leq t$, $i \leq b_i$. Also, $B \notin \mathcal{F}_{\text{in}}$ implies $2k-t+1 \leq b_k$. Therefore there exists a g, $0 \leq g \leq k-t-2$ such that b_{t+1+g} is strictly smaller than the corresponding element of D. That is,

$$b_{t+1+g} \leq t+1+2g.$$

Equivalently

$$\left|B \cap [t+1+2g]\right| \geq t+1+g$$

proving the pseudo $(t+1)$-intersecting property. Also, $g \leq k-t-2$ implies $w_{t+1}(\mathcal{F}_{\text{out}}) \leq k-t-2$ as well. □

Now (3.4) follows by applying Theorem 2.10 with t replaced by $t+1$. Let us compute α and β.

$$\frac{\binom{2k-t-2}{k-1-j}}{\binom{2k-t-2}{k-1}} \Bigg/ \frac{\binom{2k-t}{k-j}}{\binom{2k-t}{k}} = \frac{(k-j)(k-t+j)}{k(k-t)} = 1 + \frac{j(t-j)}{k(k-t)}.$$

Thus

$$\alpha = \frac{j(t-j)}{k(k-t)} \cdot \frac{\binom{2k-t}{k-j}}{\binom{2k-t}{k}}. \tag{3.6}$$

$$\frac{\binom{2k-t-3}{k-1-j}}{\binom{2k-t-3}{k-1}} \Bigg/ \frac{\binom{2k-t}{k-j}}{\binom{2k-t}{k}} = \frac{(k-j)(k-t+j)(k-t+j-1)}{k(k-t)(k-t-1)}$$
$$= 1 + \frac{j(k^2-t^2-t) - j^2(k-2t-1) - j^3}{k(k-t)(k-t-1)}.$$

Thus

$$\beta = \frac{j(k^2-t^2-t) - j^2(k-2t-1) - j^3}{k(k-t)(k-t-1)} \cdot \frac{\binom{2k-t}{k-j}}{\binom{2k-t}{k}}.$$

Consequently,

$$\frac{\beta}{\alpha} = \frac{k^2-t^2-t-j(k-2t+1)-j^2}{(t-j)(k-t-1)}$$
$$= \frac{k+t+1-j}{t-j} + \frac{t+1+(t-j)j}{(t-j)(k-t-1)} > \frac{k+t+1-j}{t-j}.$$

We proved

$$\frac{\alpha}{\beta} < \frac{t-j}{k+t+1-j}.$$

On the other hand the assumption of Theorem 1.4 was

$$|\mathcal{F}| \geq \binom{2k-t}{k}\left(1 + \frac{t-j}{k+t+1-j}\right)$$

implying

$$\left|\mathcal{F}_{\text{out}}\right| / |\mathcal{F}| \geq \frac{t-j}{k+t+1-j} > \frac{\alpha}{\beta},$$

concluding the proof. □

4 The proof of Proposition 1.6

First of all note that

$$\left|\binom{[2k-t]}{k} \setminus |\mathcal{A}|\right| = \sum_{0 \le i < t+s} \binom{k-1+s}{i}\binom{k+1-s-t}{k-i} = o\left(\binom{2k-t}{k}\right)$$

for fixed s, t as $k \to \infty$.

Let us compute the size of $\partial^j \mathcal{B} \setminus \binom{[2k-t]}{k-j}$. For a fixed $x \in [2k-t+1, n]$, $\{x\} \cup B_0 \in \mathcal{B}$ iff $B_0 \in \binom{[k-1+s]}{k-1}$. Thus the sets $D \in \left(\partial^j B \setminus \binom{[2k-t]}{k}\right)$ are of the form $\{x\} \cup D_0$ with $D_0 \in \binom{[k-1+s]}{k-1-j}$. Thus

$$\left|\partial^j \mathcal{F}\right| \le \binom{2k-t}{k-j} + (n-2k+t)\binom{k-1+s}{s+j}.$$

Comparing this with

$$|\mathcal{F}| = (1-o(1))\binom{2k-t}{k} + (n-2k+t)\binom{k-1+s}{k-1}$$

and recalling the definition of α (cf. Section 3), we see that $\left|\partial^j \mathcal{F}\right| / |\mathcal{F}| < (1+\alpha)\binom{2k-t}{k-t}$ as long as

$$|\mathcal{B}| < \frac{\alpha\binom{k-1+s}{s}}{\binom{k-1+s}{s+j}}\binom{2k-t}{k}(1-o(1)).$$

Noting $\binom{k-1+s}{s} \;/\; \binom{k-1+s}{s+j} = \prod_{0 \le i < j} \frac{k-1-i}{s-i} < \frac{(k-1)^j}{s(s-1)\cdot \ldots \cdot (s-j+1)}$ and $\alpha > \frac{j(t-j)}{k(k-t)}$ we see that

$$|\mathcal{B}| < \frac{j(t-j)s(s-1)\cdot \ldots \cdot (s-j+1)}{(k-1)^{j+2}}\binom{2k-t}{k}(1-o(1))$$

is fine. Setting $\varepsilon(k) = \frac{j(t-j)s(s-1)\cdot \ldots \cdot (s-j+1)}{(k-1)^{j+2}}$ we get $|\mathcal{F}| = (1+\varepsilon(k)-o(1))\binom{2k-t}{k}$.
□

5 The shadow of stars and semistars

The most important result concerning intersecting families is the Erdős–Ko–Rado Theorem.

Theorem 5.1 ([EKR]). *Suppose that* $n \ge n_0(k,t)$, $\mathcal{F} \subset \binom{[n]}{k}$ *is* t*-intersecting,* $k > t > 0$. *Then*

$$|\mathcal{F}| \le \binom{n-t}{k-t}. \tag{5.1}$$

As to the bound $n_0(k,t)$, its exact value is $(k-t+1)(t+1)$. For $t = 1$ it was proved already by Erdős, Ko and Rado. For $t \ge 15$ it was proved by the first author ([F78]). Finally Wilson [W] showed it by a proof using eigenvalues for $2 \le t \le 14$ (the proof is valid for all t).

The *full* t*-star*, $\mathcal{A}_0(n,k,t) = \left\{A \in \binom{[n]}{k} : [t] \subset A\right\}$ shows that (5.1) is best possible. Let us note that for $n = (k-t+1)(t+1)$, $\left|\mathcal{A}_0(n,k,t)\right| = \left|\mathcal{A}_1(n,k,t)\right|$ and for $t \ge 2$ up to isomorphism these are the only families achieving equality in (5.1).

Let us mention that the Intersecting Shadow Theorem implies $|\mathcal{F}| \le \left|\partial^t \mathcal{F}\right| \le \binom{n}{k-t}$ for all $n \ge 2k-t$. Very recently the first author [F20] showed the slightly stronger universal bound

$$|\mathcal{F}| \le \binom{n-1}{k-t} \quad \text{for all} \quad n > 2k-t, \quad \mathcal{F} \text{ is } t\text{-intersecting.} \tag{5.2}$$

Definition 5.2. If $C \subset F$ holds for all $F \in \mathcal{F}$ with a t-set C then $\mathcal{F}$ is called a t*-star*. If for some $(t+1)$-element set D, $|F \cap D| \ge t$ holds for all $F \in \mathcal{F}$ then $\mathcal{F}$ is called a $t+1$*-semistar*. When the value of t is clear from the context, we say for short that $\mathcal{F}$ is a *star* or *semistar*.

Let us note that the family $\mathcal{A}_0(n,k,t) \cup \mathcal{A}_1(n,k,t)$ is a semistar with $D = [t+1]$.

Let us fix n, k, t, $t \ge 2$ and use the shorthand notation $\mathcal{A}_0$, $\mathcal{A}_1$.

Proposition 5.3. *If* $\emptyset \ne \mathcal{F} \subset \mathcal{A}_0 \cup \mathcal{A}_1$ *then*

$$\left|\partial^j \mathcal{F}\right| / |\mathcal{F}| \ge \binom{t+2}{j+1} \Big/ (t+2) \quad \textit{for} \quad 1 < j < t. \tag{5.3}$$

If $\emptyset \ne \mathcal{F} \subset \mathcal{A}_0$ *then*

$$\left|\partial^j \mathcal{F}\right| \,/\, |\mathcal{F}| \ge \left|\partial^j \mathcal{A}_0\right| \,/\, |\mathcal{A}_0| > \binom{t}{j}. \tag{5.4}$$

Proof. To prove (5.3) just note that $w_t(\mathcal{F}) \leq w_t(\mathcal{A}_0 \cup \mathcal{A}_1) = 1$. Now the inequality follows from Theorem 2.10.

To prove (5.4) we are going to use Proposition 1.1.

Set $\overline{\mathcal{F}} = \{F \setminus [t]; F \in \mathcal{F}\}$. Since $\mathcal{F} \subset \mathcal{A}_0$, $|\overline{\mathcal{F}}| = |\mathcal{F}|$. For convenience let us introduce the notation $\partial^0 \overline{\mathcal{F}} = \overline{\mathcal{F}}$, $\partial^1 \overline{\mathcal{F}} = \partial \overline{\mathcal{F}}$.

Claim 5.4.

$$|\partial^j \mathcal{F}| = \sum_{0 \leq i \leq j} \binom{t}{j-i} |\partial^i \overline{\mathcal{F}}|. \tag{5.5}$$

Proof. For $0 \leq i \leq j$ define

$$\mathcal{H}_i = \{H \in \partial^j \mathcal{F} : |H \cap [t]| = i\}.$$

That is, $\mathcal{H}_i$ consists of the j'th shadows where we omit $j-i$ elements from $[t]$ and i elements from $F \setminus [t]$. Then $|\mathcal{H}_i| = \binom{t}{j-i} |\partial^i \overline{\mathcal{F}}|$. Since $\partial^j \mathcal{F} = \mathcal{H}_0 \sqcup \ldots \sqcup \mathcal{H}_j$ is a partition, (5.5) follows. □

Applying (1.1) to $\overline{\mathcal{F}}$ and using (5.5) we infer

$$|\partial^j \mathcal{F}| \,/\, |\mathcal{F}| \geq \sum_{0 \leq i \leq j} \binom{t}{j-i} \binom{n-t}{k-t-i} \Big/ \binom{n-t}{k-t}. \tag{5.6}$$

For the family $\mathcal{A}_0$, $\overline{\mathcal{A}_0} = \binom{[t+1,n]}{k-t}$. Thus $|\partial^i \overline{\mathcal{A}}_0| = \binom{n-t}{k-t-i}$. Consequently, $|\partial^j \mathcal{A}_0| \,/\, |\mathcal{A}_0| = \sum_{0 \leq i \leq j} \binom{t}{j-i} \binom{n-t}{k-t-i} \Big/ \binom{n-t}{k-t}$. Comparing with (5.6) the inequality (5.4) follows. □

The main result of the present section is the following.

Theorem 5.5. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting $(t+1)$-semistar. Then for all $1 < j < t$, (5.3) holds.*

Since $\mathcal{A}_0 \cup \mathcal{A}_1$ is a semistar with $D = [t+1]$, Theorem 5.5 generalizes Proposition 5.3.

Proof. Without loss of generality let $D = [t+1]$. That is, $|F \cap [t+1]| \geq t$ for all $F \in \mathcal{F}$. Since shifting maintains this property and does not increase the shadow, we may assume that $\mathcal{F}$ is shifted.

Set $\mathcal{F}_0 = \{F \in \mathcal{F} : [t+1] \subset \mathcal{F}\}$ and $\overline{\mathcal{F}}_0 = \big\{F \setminus [t+1] : F \in \mathcal{F}_0\big\}$. Define the restricted shadow $\partial_R^j \mathcal{F}_0$ by

$$\partial_R^j \mathcal{F}_0 = \left\{ S \cup T : S \in \binom{[t+1]}{t+1-j}, T \in \overline{\mathcal{F}}_0 \right\}.$$

Define next $\mathcal{T} = \left\{ T \in \binom{[t+2,n]}{k-t} : \exists G \in \binom{[t+1]}{t}, G \cup T \in \mathcal{F} \right\}$. For $T \in \mathcal{T}$ we define

$$\mathcal{G}_T = \left\{ G \in \binom{[t+1]}{t} : G \cup T \in \mathcal{F} \right\} \quad \text{and} \quad \mathcal{F}_T = \big\{ G \cup T : G \in \mathcal{G}_T \big\}.$$

Since $\mathcal{G}_T \subset \binom{[t+1]}{t}$, (1.1) yields

$$|\partial^j \mathcal{G}_T| \geq |\mathcal{G}_T| \binom{t+1}{t-j} \Big/ \binom{t+1}{t} = |\mathcal{G}_T| \binom{t+1}{j+1} \Big/ (t+1). \tag{5.7}$$

Let us note that for $T \in \mathcal{T}$ the families $\mathcal{F}_T$ partition $\mathcal{F} \setminus \mathcal{F}_0$.

Let us divide $\mathcal{T}$ into two parts, $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ where $\mathcal{T}_1 = \big\{T \in \mathcal{T} : |\mathcal{G}_T| = 1\big\}$, $\mathcal{T}_2 = \big\{T \in \mathcal{T} : |\mathcal{G}_T| \geq 2\big\}$. For $T \in \mathcal{T}_1$ one has $|\partial^j \mathcal{G}_T| = \binom{t}{j}$. Setting $\mathcal{F}_i = \bigcup\limits_{T \in \mathcal{T}_i} \mathcal{G}_T$, $i = 1, 2$, we have

$$|\partial_R^j \mathcal{F}_1| = |\mathcal{F}_1| \binom{t}{j}, \tag{5.8}$$

and using (5.7)

$$|\partial_R^j \mathcal{F}_2| \geq |\mathcal{F}_2| \frac{\binom{t+1}{j+1}}{t+1}. \tag{5.9}$$

Note that $\binom{t}{j}$ is larger than the coefficient in (5.3). Indeed,

$$\frac{\binom{t+2}{j+1}}{t+2} = \frac{\binom{t+1}{j}}{j+1} = \frac{t+1}{(j+1)(t-j+1)} \binom{t}{j} < \binom{t}{j}.$$

From (5.8), (5.9) and the obvious formula $|\partial_R^j \mathcal{F}_0| = |\mathcal{F}_0| \binom{t+1}{j}$ we infer

$$|\partial^j \mathcal{F}| \geq \sum_{0 \leq i \leq 2} |\partial_R^j \mathcal{F}_i| \geq |\mathcal{F}_0| \binom{t+1}{j} + |\mathcal{F}_1| \frac{\binom{t+2}{j+1}}{t+2} + |\mathcal{F}_2| \frac{\binom{t+1}{j+1}}{t+1}. \tag{5.10}$$

To conclude the proof we need a relation between $\mathcal{F}_0$ and $\mathcal{F}_2$.

Claim 5.6. $(t+1)\cdot\left|\mathcal{F}_0\right| \geq \left|\mathcal{F}_2\right|$.

Proof of the Claim. First we show that $\mathcal{T}_2$ is intersecting. Indeed, if $T \in \mathcal{T}_2$ then there are at least two choices of $G \in \binom{[t+1]}{t}$, $G \in \mathcal{G}_T$. Thus for $T, T' \in \mathcal{T}_2$ we can choose *distinct* $G, G' \in \binom{[t+1]}{t}$ so that $G \cup T$, $G' \cup T' \in \mathcal{F}$. Now $\left|(G\cup T)\cap(G'\cup T')\right| = t-1+|T\cap T'|$. Since $\mathcal{F}$ is t-intersecting, $T\cap T' \neq \emptyset$.

Applying Theorem 1.3 to $\mathcal{T}_2$ yields $\left|\partial\mathcal{T}_2\right| \geq \left|\mathcal{T}_2\right|$. The inequality $\left|\mathcal{F}_2\right| \leq (t+1)\left|\mathcal{T}_2\right|$ should be obvious. To conclude the proof of the claim let us show

$$\left|\mathcal{F}_0\right| \geq \left|\partial\mathcal{T}_2\right|.$$

More is true. Namely

$$\overline{\mathcal{F}}_0 \supset \partial\mathcal{T}. \tag{5.11}$$

To prove (5.11) pick an arbitrary $V \in \partial\mathcal{T}$. Then we can choose $G \in \binom{[t+1]}{t}$, $T \in \mathcal{T}$ and $x \in T$ so that $V = T \setminus \{x\}$ and $G \cup T \in \mathcal{F}$. Let y be the unique element in $[t+1]\setminus G$. Obviously $y < x$. Thus $[t+1]\cup V \prec G \cup T$ whence $[t+1]\cup V \in \mathcal{F}$. That is, $V \in \overline{\mathcal{F}}_0$. □

Now let us rewrite (5.10):

$$\left|\partial^j\mathcal{F}\right| \geq |\mathcal{F}|\frac{\binom{t+2}{j+1}}{t+2} + \left\{\left|\mathcal{F}_0\right|\left(\binom{t+1}{j} - \frac{\binom{t+2}{j+1}}{t+2}\right) - \left|\mathcal{F}_2\right|\left(\frac{\binom{t+2}{j+1}}{t+2} - \frac{\binom{t+1}{j+1}}{t+1}\right)\right\}.$$

By Claim 5.6 the quantity in $\{\quad\}$ is at least

$$\begin{aligned}\left|\mathcal{F}_0\right|&\left(\binom{t+1}{j} - \frac{\binom{t+2}{j+1}}{t+1}\right) - (t+1)\left(\frac{\binom{t+2}{j+1}}{t+2} - \frac{\binom{t+1}{j+1}}{t+1}\right)\\ &= \left|\mathcal{F}_0\right|\left(\binom{t+1}{j} - \binom{t+2}{j+1} + \binom{t+1}{j+1}\right) = 0,\end{aligned}$$

completing the whole proof. □

6 On the structure and shadow of very large families

Throughout this section $\mathcal{F} \subset \binom{[n]}{k}$ is shifted and t-intersecting. We assume also that $n \geq (k-t+1)(t+1)$ which guarantees by Theorem 5.1 (Full Erdős–Ko–Rado Theorem) that $|\mathcal{F}| \leq \left|\mathcal{A}_0\right|$.

Since $\mathcal{A}_0$ is a t-star, it is natural to investigate the maximum of $\mathcal{F}$ assuming $\mathcal{F} \not\subset \mathcal{A}_0$, i.e., $\mathcal{F}$ is not a t-star. Of course, $\mathcal{A}_1$ is a strong candidate, but there is an other one.

Definition 6.1. Define $\mathcal{H} = \mathcal{H}(n,k,t) = \Big\{H \in \binom{[n]}{k} : [t] \subset H, H \cap [t+1, k+1] \neq \emptyset\Big\} \cup \Big\{[k+1] \setminus \{x\} : x \in [t]\Big\}$.

Theorem 6.2 (Hilton–Milner–Frankl Theorem)**.** *Let $n \geq (k-t+1)(t+1)$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting but $\mathcal{F}$ is not a t-star. Then*

$$|\mathcal{F}| \leq \max\big\{\big|\mathcal{A}_1\big|, |\mathcal{H}|\big\}. \tag{6.1}$$

Moreover, except for the case $(n,k,t) = (2k,k,1)$ equality holds only if $\mathcal{F}$ is isomorphic to $\mathcal{A}_1$ or $\mathcal{H}$.

The case $t = 1$ was proved by Hilton and Milner ([HM]). There have been various shorter proofs given cf. [FF2], [KZ], [HK] or [F19]. The case of $t \geq 15$ was proved in [F78], cf. also [F78b]. Ahlswede and Khachatrian [AK2] gave a different proof valid for the full range.

One should note that for $t+2 > k-t+1$, i.e., $k \leq 2t$, $|\mathcal{A}_1| > |\mathcal{H}|$. This implies

Corollary 6.3. *Suppose that $n \geq (k-t+1)(t+1)$, $k \leq 2t$, $t > j \geq 1$. Let $\mathcal{F} \subset \binom{[n]}{k}$ be t-intersecting and $|\mathcal{F}| > |\mathcal{A}_1|$. Then*

$$\big|\partial^j \mathcal{F}\big| \;/\; |\mathcal{F}| \geq \big|\partial^j \mathcal{A}_0\big| \;/\; \big|\mathcal{A}_0\big| > \binom{t}{j}. \tag{6.2}$$

Our aim is to prove a similar result for the case $k > 2t$ as well.

We need quite some preparation. Let us recall a structural result from [F87]. For a shifted t-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$ define its base $\mathcal{B} = \mathcal{B}(\mathcal{F})$ by

$$\mathcal{B} = \big\{F \cap [2k-t] : F \in \mathcal{F}\big\}.$$

Define $\mathcal{B}^{(\ell)} = \{B \in \mathcal{B} : |B| = \ell\}$, $b_\ell = \big|\mathcal{B}^{(\ell)}\big|$.

Proposition 6.4 ([F87])**.** (i) $\sim$ (iv) *hold.*

(i) *$\mathcal{B}$ is shifted and t-intersecting.*
(ii) *$b_\ell = 0$ for $\ell < t$.*
(iii) *$b_t \leq 1$ with $b_t = 1$ implying that $\mathcal{F}$ is a t-star.*
(iv) $|\mathcal{F}| \leq \sum\limits_{t \leq \ell \leq k} b_\ell \binom{n-2k+t}{k-\ell}$.

Let us mention that using Theorem 1.3 (i) implies $\left|\partial^t \mathcal{B}^{(\ell)}\right| \geq \left|\mathcal{B}^{(\ell)}\right|$. Since $\partial^t \mathcal{B}^{(\ell)} \subset \binom{[2k-t]}{\ell-t}$,

$$b_\ell \leq \binom{2k-t}{\ell-t}. \tag{6.3}$$

For $\ell = t+1$ one can analyze the possible structure of $\mathcal{B}^{(\ell)}$. Note that $[t+1] \prec [t] \cup \{t+2\}$ are the two smallest $(t+1)$-sets in the shifting partial order. The third ex aequo are $A_3 = [t+2] \setminus \{t\}$ and $D_3 = [t] \cup \{t+3\}$.

Claim 6.5. *If $A_3 \in \mathcal{B}^{(t+1)}$ then $\mathcal{F} \subset \mathcal{A}_1$.*

Proof. We must show $|F \cap [t+2]| \geq t+1$. If this fails then using shiftedness we can find F with $F \cap [t+2] = [t]$. This implies $F \cap A_3 = [t-1]$ contradicting Proposition 6.4 (i). □

From now on throughout this section we suppose $\mathcal{F} \not\subset \mathcal{A}_1$ and thereby $A_3 \notin \mathcal{B}^{(t+1)}$.

Claim 6.6. *If $A_3 \notin \mathcal{B}^{(t+1)}$ then $\mathcal{B}^{(t+1)} = \{[t] \cup \{x\} : t+1 \leq x \leq t+b_{t+1}\}$.*

Proof. The statement is trivially true by shiftedness for $b_{t+1} = 0$, 1 or 2. Suppose $b_{t+1} \geq 3$. Then $D_3 \in \mathcal{B}^{(t+1)}$. We claim that $[t] \subset B$ for all $B \in \mathcal{B}^{(t+1)}$.

Set $D_i = [t] \cup \{t+i\}$ for $i = 1, 2$. By $D_1 \prec D_2 \prec D_3$, all three are in $\mathcal{B}^{(t+1)}$. In view of Proposition 6.4 (i), $|B \cap D_i| \geq t$, $i = 1, 2, 3$, implying $[t] \subset B$. Now Claim 6.6 follows by shiftedness. □

Now we are ready to state and prove the main result of this section.

Theorem 6.7. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is shifted, t-intersecting, $\mathcal{F} \not\subset \mathcal{A}_1$ and $b^{(t+1)} \geq t+1$. Then*

$$\left|\partial^j \mathcal{F}\right| > \binom{t}{j} |\mathcal{F}|. \tag{6.4}$$

Proof. For simpler notation set $s = b_{t+1}$. If $\mathcal{F} \subset \mathcal{A}_0$, then (6.4) is evident. Suppose that $\mathcal{F} \not\subset \mathcal{A}_0$.

Claim 6.8. *If $F \in \mathcal{F} \setminus \mathcal{A}_0$ then*

$$F \cap [t+s] = [t+s] \setminus \{y\} \quad \textit{for some} \quad y \in [t]. \tag{6.5}$$

Proof. In view of Claim 6.6, $\mathcal{B}^{(t+1)} = \{[t] \cup \{x\} : t < x \le t+s\}$. By Proposition 6.4 (i) $|F \cap B| \ge t$ for all $B \in \mathcal{B}^{(t+1)}$. Since $[t] \not\subset F$, $x \in F$ for all $t < x \le t+s$ and $|F \cap [t]| = t-1$. □

Define $\mathcal{F}_1 = \{F \in \mathcal{F} : |F \cap [t+s]| = t+s-1\}$. In view of Claim 6.8, $\mathcal{F} \setminus \mathcal{A}_0 \subset \mathcal{F}_1$. Setting $\mathcal{F}_0 = \mathcal{F} \setminus \mathcal{F}_1$, $\mathcal{F}_0 \subset \mathcal{A}_0$ follows. Defining the restricted shadow with respect to $[t+s]$ as

$$\partial_R^j \mathcal{F} = \bigcup_{F \in \mathcal{F}} \partial_R^j F \quad \text{where} \quad \partial_R^j F = \left\{ S \in \binom{F}{k-j} : S \setminus [t+s] = F \setminus [t+s] \right\},$$

it should be clear that $|F \cap [t+s]| \ne |F' \cap [t+s]|$ implies $\partial_R^j F \cap \partial_R^j F' = \emptyset$. Consequently,

$$\partial_R^j \mathcal{F}_0 \cap \partial_R^j \mathcal{F}_1 = \emptyset. \tag{6.6}$$

For $\mathcal{F}_0$, $\mathcal{F}_0 \subset \mathcal{A}_0$ implies

$$|\partial_R^j \mathcal{F}_0| > \binom{t}{j} |\mathcal{F}_0|. \tag{6.7}$$

To deal with $\mathcal{F}_1$ define $\mathcal{T} \subset \binom{[t+s+1,n]}{k-t-s+1}$ by

$$\mathcal{T} = \{F \setminus [t+s] : F \in \mathcal{F}_1\}.$$

For $T \in \mathcal{T}$ define $\mathcal{G}_T = \left\{ G \in \binom{[t+s]}{t+s-1} : G \cup T \in \mathcal{F}_1 \right\}$.
Now (1.1) implies

$$|\partial^j \mathcal{G}_T| \ge |\mathcal{G}_T| \frac{\binom{t+s}{t+s-1-j}}{\binom{t+s}{1}} = |\mathcal{G}_T| \frac{\binom{t+s}{j+1}}{t+s}.$$

By definition

$$|\mathcal{F}_1| = \sum_{t \in \mathcal{T}} |\mathcal{G}_T| \quad \text{and} \quad |\partial_R^j \mathcal{F}_1| = \sum_{t \in \mathcal{T}} |\partial^j \mathcal{G}_T|.$$

Consequently,

$$|\partial_R^j \mathcal{F}_1| \ge |\mathcal{F}_1| \frac{\binom{t+s}{j+1}}{t+s}. \tag{6.8}$$

Let us show that $s \geq t+1$ implies

$$\frac{\binom{t+s}{j+1}}{t+s} = \frac{\binom{t+s-1}{j}}{j+1} \geq \frac{\binom{2t}{j}}{j+1} \geq \binom{t}{j}.$$

Indeed,

$$\frac{\binom{2t}{j}}{\binom{t}{j}} = \prod_{0 \leq i < j} \frac{2t-i}{t-i} \geq 2^j \geq j+1.$$

Thus adding (6.7) and (6.8), and using (6.6) imply (6.4). □

Remark. For $j = 1$, $2^1 = 1+1$. However for larger values of j one can considerably relax the condition $b_{t+1} \geq t+1$.

Corollary 6.8. *Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is shifted, t-intersecting, $\mathcal{F} \not\subset \mathcal{A}_1$, $t+2 \leq k-t+1$. If*

$$|\mathcal{F}| > t\binom{n-2k+t}{k-t-1} + \sum_{t+2 \leq \ell \leq k} \binom{2k-t}{\ell-t}\binom{n}{k-\ell}$$

then

$$\left|\partial^j \mathcal{F}\right| > \binom{t}{j}|\mathcal{F}|. \tag{6.9}$$

Proof. If $\mathcal{F} \subset \mathcal{A}_0$ then (6.9) is evident. Otherwise $b_t = 0$ and thereby $b_{t+1} \geq t+1$ follow from Proposition 6.4. Now (6.9) is a consequence of Theorem 6.7. □

7 A general bound

To make notation simpler let us define $\gamma(\ell, t, j) = \binom{t+2(\ell-t)}{t+\ell-j} \Big/ \binom{t+2(\ell-t)}{t+\ell}$. Consider a shifted t-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$. Recall the definition of $w_t(\mathcal{F})$ as the minimal integer w, $0 \leq w \leq k-t$ such that for every $F \in \mathcal{F}$ there exists $\ell = \ell(F)$, $0 \leq \ell \leq w$, with

$$\left|F \cap [t+2\ell]\right| \geq t+\ell. \tag{7.1}$$

Since (7.1) holds with i for $F \in \mathcal{A}_i$, $w_t(\mathcal{F}) > i$ implies $\mathcal{F} \not\subset \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_i$.

Suppose that

$$\left|\partial^j \mathcal{F}\right| \,/\, |\mathcal{F}| < \gamma(w,t,j). \tag{7.2}$$

By Theorem 2.10, $\mathcal{F} \not\subset \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_w$. That is, we can find some $F \in \mathcal{F}$ failing (7.1) for all $0 \le \ell \le w$.

Define $E = [t-1] \cup (t+1, t+3, \ldots, t+2w+1) \cup [t+2w+2, k+w+1]$. Then $E \prec F$ and by shiftedness $E \in \mathcal{F}$.

Define $D = [t] \cup (t+2, \ldots, t+2w) \in \binom{[2k-t]}{t+w}$. Note that $|E \cap D| = t-1$. This permits to prove

Proposition 7.1. *If* $G \in \mathcal{F}$ *then either* (i) *or* (ii) *hold.*

(i) $\left|G \cap [t+1+2h]\right| \ge t+1+h$ *for some* $0 \le h < w$.

(ii) $\left|G \cap [2k-t]\right| > w+t$.

Proof. Suppose that (ii) does not hold. Let $|G \cap [2k-t]| = t+h$ for some $0 \le h \le w$. Set $D_h = D \cap [t+2h]$. Since $\left|E \cap D_h\right| = t-1$, we infer $D_h \not\prec G \cap [2k-t]$ by shiftedness and Proposition 6.4. Thus (i) follows. □

Define the partition $\mathcal{F} = \mathcal{F}_{\text{in}} \cup \mathcal{F}_{\text{out}}$ by

$$\mathcal{F}_{\text{in}} = \left\{F \in \mathcal{F} : |F \cap [2k-t]| > w+t\right\},$$
$$\mathcal{F}_{\text{out}} = \left\{F \in \mathcal{F} : |F \cap [2k-t]| \le w+t\right\}.$$

With the definition of restricted j-shadows as in Definition 2.6 we have

$$\left|\partial^j \mathcal{F}\right| \ge \left|\partial^j_R \mathcal{F}_{\text{in}}\right| + \left|\partial^j_R \mathcal{F}_{\text{out}}\right|. \tag{7.3}$$

In view of Proposition 6.4, the family $\left\{F \cap [2k-t] : F \in \mathcal{F}\right\}$ is t-intersecting. Thus by Theorem 1.3 we have

$$\left|\partial^j_R \mathcal{F}_{\text{in}}\right| \ge \gamma(k-t,t,j)\left|\mathcal{F}_{\text{in}}\right|. \tag{7.4}$$

As to $\mathcal{F}_{\text{out}}$, Proposition 7.1 (i) implies that it is pseudo $t+1$-intersecting with $w_{t+1}(\mathcal{F}_{\text{out}}) \le w-1$. By Theorem 2.10 we have

$$\left|\partial^j_R \mathcal{F}_{\text{out}}\right| \ge \gamma(w-1,t+1,j)\left|\mathcal{F}_{\text{out}}\right|. \tag{7.5}$$

Defining α, β by

$$\alpha = \gamma(w,t,j) - \gamma(k-t,t,j) \quad \text{and} \quad \beta = \gamma(w-1,t+1,j) - \gamma(w,t,j)$$

we infer from (7.3), (7.4) and (7.5)

$$\left|\partial^j \mathcal{F}\right| \ge \gamma(w,t,j)|\mathcal{F}| + \beta\left|\mathcal{F}_{\text{out}}\right| - \alpha\left|\mathcal{F}_{\text{in}}\right|.$$

Thus we proved

Proposition 7.2. *If* $\left|\mathcal{F}_{out}\right| \geq \frac{\alpha}{\beta}\left|\mathcal{F}_{in}\right|$ *then*

$$\left|\partial^j \mathcal{F}\right| \geq \gamma(w,t,j)|\mathcal{F}|. \tag{7.6}$$

Note that α and β are independent of n, that is, $\frac{\alpha}{\beta}$ is a constant. Also, to bound $\left|\mathcal{F}_{\text{in}}\right|$ we may use (6.3) and Proposition 6.4:

$$\left|\mathcal{F}_{\text{in}}\right| \leq \sum_{w<\ell\leq k-t} \binom{2k-t}{\ell}\binom{n-2k+t}{k-\ell-t} = (1+o(1))\binom{2k-t}{w+1}\binom{n-2k+t}{k-w-t-1}.$$

If (7.6) fails then

$$\left|\mathcal{F}_{\text{out}}\right| < \left(\frac{\alpha}{\beta}+o(1)\right)\binom{2k-t}{w+1}\binom{n-2k+t}{k-w-t-1}, \quad \text{i.e.,}$$

$$|\mathcal{F}| < \frac{\alpha+\beta+o(1)}{\alpha}\binom{2k-t}{w+1}\binom{n-2k+t}{k-w-t-1}.$$

That is, we proved the following

Theorem 7.3. *Suppose that* $\mathcal{F} \subset \binom{[n]}{k}$ *is* t*-intersecting,*

$$|\mathcal{F}| > \frac{\alpha+\beta+o(1)}{\alpha}\binom{2k-t}{w+1}\binom{n-2k+t}{k-w-t-1}. \tag{7.7}$$

Then

$$\left|\partial^j \mathcal{F}\right| \geq \gamma(w,t,j)|\mathcal{F}|. \tag{7.8}$$

There are many ways that Theorem 7.3 can be improved. The simplest is to replace $\binom{2k-t}{w+1}$ by $\binom{2k-t-1}{w+1}$ unless $w = k-t$ (the case that we treated in Theorem 1.4). More substantial is the improvement that except for the part of $\mathcal{F}_{\text{in}}$ contained in $\mathcal{A}_{\ell+1} \cup \mathcal{A}_{\ell+2} \cup \ldots \cup \mathcal{A}_{k-t}$ one can replace the factor $\gamma(k-t,t,j)$ in (7.4) by the larger $\gamma(\ell,t,j)$ leading to a considerably smaller value of α.

For $n \to \infty$, $\left|\mathcal{A}_{i+1}\right| = O\left(\left|\mathcal{A}_i\right|/n\right)$ showing that asymptotically only $\gamma(w+1,t,j)$ matters. That is, Theorem 7.3 holds with $\alpha = \gamma(w+1,t,j)+\varepsilon$ for any $\varepsilon > 0$ and $n > n_0(\varepsilon)$.

Let us close the paper by an open problem.

Problem 7.4. Determine or estimate the smallest value of $c = c(k,t,j)$ such that (7.8) holds whenever $n > n_0(k,t,j)$ and $|\mathcal{F}| > c\binom{n}{k-w-t-1}$.

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