The number of triangles is more when they have no common vertex

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Abstract By the theorem of Mantel [5] it is known that a graph with n vertices and $\lfloor \frac{n^2}{4} \rfloor + 1$ edges must contain a triangle. A theorem of Erdős gives a strengthening: there are not only one, but at least $\lfloor \frac{n}{2} \rfloor$ triangles. We give a further improvement: if there is no vertex contained by all triangles then there are at least $n-2$ of them. There are some natural generalizations when (a) complete graphs are considered (rather than triangles), (b) the graph has t extra edges (not only one) or (c) it is supposed that there are no s vertices such that every triangle contains one of them. We were not able to prove these generalizations, they are posed as conjectures.

1 Introduction

All graphs considered in this paper are finite and simple. Let G be such a graph, the vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, the number of vertices in G is $v(G)$ and the number of edges in G is $e(G)$. We denote the degree of a vertex v by $d(v)$, the neighborhood of v by $N(v)$, the number of edges between vertex sets A and B by $e(A, B)$ and the number of triangles in G by $T(G)$. A *triangle covering set* in $V(G)$ is a vertex set that contains at least one vertex of every triangle in G. The *triangle covering number*, denoted by $\tau_{\triangle}(G)$, is the size of the smallest triangle covering set. Let $S \subset V(G)$ be any subset of $V(G)$, then $G[S]$ is the subgraph induced by S.

Mantel [5] proved that an n-vertex graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ $(t \geq 1)$ edges must contain a

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triangle. In 1941, Rademacher (unpublished, see [1]) showed that for even n, every graph G on n vertices and $\frac{n^2}{4}+1$ edges contains at least $\frac{n}{2}$ triangles and $\frac{n}{2}$ is the best possible. Later on, the problem was revived by Erdős, see [1], which is now known as the Erdős-Rademacher problem, Erdős simplified Rademacher's proof and proved more generally that for $t \leq 3$ and $n > 2t$ case. Seven years later, he [2] conjectured that a graph with $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges contains at least $t\left\lfloor \frac{n}{2} \right\rfloor$ triangles if $t < \frac{n}{2}$, which was proved by Lovász and Simonovits [4]. Motivated by earlier results, we give a further improvement for the case $t = 1$: if there is no vertex contained by all triangles then there are at least $n-2$ of them in G.

Theorem 1 (**Mantel** [5])**.** *The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$. Furthermore, the only triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

Theorem 2 (**Erdős** [1])**.** *Let G be a graph with n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, $t \leq 3$, $n > 2t$, then every G contains at least $t\left\lfloor \frac{n}{2} \right\rfloor$ triangles.*

Before presenting our main result, the following definitions, a theorem and a lemma are needed.

Definition 1. *Let $K_{i,n-i}$ denote a the complete bipartite graph on the vertex classes $|X| = i$, $|Y| = n - i$.*

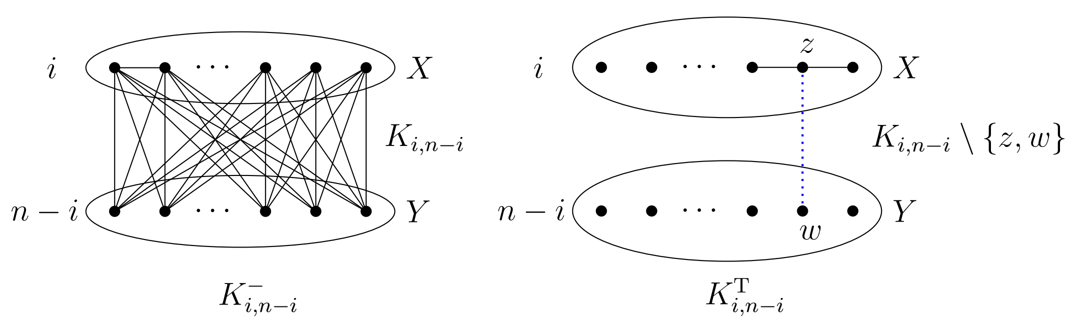

Figure 1: Graphs $K^-_{i,n-i}$ and $K^{\mathrm{T}}_{i,n-i}$

Definition 2. *Let $K^-_{i,n-i}$ denote a graph obtained from a complete bipartite graph $K_{i,n-i}$ plus an edge in the class X with i vertices, see Figure 1.*

Definition 3. *Let $K^{\mathrm{T}}_{i,n-i}$ denote a graph obtained from a complete bipartite graph $K_{i,n-i}$*

minus an edge plus two adjacent edges in the class X with i vertices, one end point of the missing edge is the shared vertex of these two adjacent edges and the other one is in the class Y, see Figure 1.

Lemma 3. *Let G be a graph with n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, such that $\tau_{\triangle}(G) = 1$ and $T(G) \le n-3$. Then G is one of the following graphs: $K^-_{\frac{n}{2},\frac{n}{2}}$, $K^-_{\frac{n-1}{2},\frac{n+1}{2}}$, $K^-_{\frac{n+1}{2},\frac{n-1}{2}}$ or $K^{\mathrm{T}}_{\frac{n+1}{2},\frac{n-1}{2}}$.*

Theorem 4. *Let G be a graph with n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edges, then either $\tau_{\triangle}(G) = 1$ or $T(G) \ge n-2$.*

2 Proofs of the main results

Proof of Lemma 3. Let v_0 be such a vertex that $G \setminus v_0$ contains no triangle. We distinguish two cases.

Case 1. $G \setminus v_0$ contains at least one odd cycle. Let C_{2k+1} ($k \ge 2$) be the shortest odd cycle in $G \setminus v_0$ and G' be the graph obtained from G by removing the vertices of C_{2k+1} and v_0, so $v(G') = n - 2k - 2$. Since C_{2k+1} is the shortest cycle in $G \setminus v_0$, each vertex in G' can be adjacent to at most 2 vertices in the C_{2k+1}, otherwise, we can find a shorter odd cycle. Since G' is an $(n-2k-2)$-vertex triangle-free graph, by Theorem 1, $e(G') \le \left\lfloor \left(\frac{n-2k-2}{2}\right)^2 \right\rfloor$. Obviously, any two vertices of C_{2k+1} are not adjacent, therefore

$$\begin{aligned} e(G \setminus v_0) &\le 2k + 1 + 2(n-2k-2) + \left\lfloor \left(\frac{n-2k-2}{2}\right)^2 \right\rfloor \\ &= k^2 - nk + \left\lfloor \frac{n^2}{4} \right\rfloor + n - 2 \\ &\le \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2 \ (k \ge 2). \end{aligned}$$

Since $e(G) = d(v_0) + e(G') \le (n-1) + (\left\lfloor \frac{n^2}{4} \right\rfloor - n + 2) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$, the only possibility for $e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor + 1$ is that $d(v_0) = n-1$ and $e(G \setminus v_0) = \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2$. In this case, we get $T(G) = \left\lfloor \frac{n^2}{4} \right\rfloor - n + 2$, which contradicts $T(G) \le n-3$.

Case 2. $G \setminus v_o$ has no odd cycles, then $G \setminus v_0$ is a bipartite graph and $e(G \setminus v_0) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil$. There are two subcases.

Case 2.1. $e(G \setminus v_0) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil$. Then $G \setminus v_0$ is $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$ and $d(v_0) = e(G) - e(G \setminus v_0) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. Let d_1 and d_2 be the numbers of neighbors of v_0 in classes X and Y of $K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$, respectively, then $d(v_0) = d_1 + d_2$ and $T(G) = d_1 d_2$. So we need $d_1 + d_2 = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and $d_1 d_2 \leq n - 3$ hold true at the same time. When n is even, we can see that the only solution is when $d_1 = 1$ and $d_2 = \frac{n}{2}$. The symmetric solution, $d_1 = \frac{n}{2}$, $d_2 = 1$ is not possible, since $d_1 \leq \frac{n}{2} - 1$ in this case. Therefore, we get that G is $K^-_{\frac{n}{2}, \frac{n}{2}}$. Assume now that n is odd, there are two possibilities,

(i) $d_1 = 1$ and $d_2 = \frac{n-1}{2}$, in the same way as in the even case, we get $T(G) = \frac{n-1}{2}$ and G is $K^-_{\frac{n+1}{2}, \frac{n-1}{2}}$. When $d_1 = \frac{n-1}{2}$ and $d_2 = 1$, we also get $T(G) = \frac{n-1}{2}$ and G is $K^-_{\frac{n+1}{2}, \frac{n-1}{2}}$.

(ii) $d_1 = 2$ and $d_2 = \frac{n-3}{2}$, then $T(G) = 2(\frac{n-3}{2}) = n - 3$ and G is $K^{\mathrm{T}}_{\frac{n+1}{2}, \frac{n-1}{2}}$. Similarly, when $d_1 = \frac{n-3}{2}$ and $d_2 = 2$, we get the same result.

Case 2.2 . $e(G \setminus v_0) = \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil - t$. Then $d(v_0) = \left\lfloor \frac{n}{2} \right\rfloor$+1+t, $1 \leq t \leq \left\lceil \frac{n}{2} \right\rceil - 2$. Let $G \setminus v_0$ be the bipartite graph with partitions X' and Y', where $|X'| = i'$, then we have

$$i'(n-1-i') \geq \left\lfloor \frac{n-1}{2} \right\rfloor \left\lceil \frac{n-1}{2} \right\rceil - t$$

$$\Rightarrow \begin{cases} \dfrac{n-1-\sqrt{4t+1}}{2} \leq i' \leq \dfrac{n-1+\sqrt{4t+1}}{2}, \text{n is even,} \\ \dfrac{n-1-2\sqrt{t}}{2} \leq i' \leq \dfrac{n-1+2\sqrt{t}}{2}, \text{ n is odd.} \end{cases} \tag{1}$$

Suppose v_0 has d_1 (≥ 1) neighbors in X' and d_2 (≥ 1) neighbors in Y'. Since $G \setminus v_0$ is bipartite, if $d_1 d_2 = 0$, then G contains no triangle which contradicts the fact that $\tau_\triangle(G) = 1$. In this situation, $d_1 d_2 \geq T(G) \geq d_1 d_2 - t = d_1(\left\lfloor \frac{n}{2} \right\rfloor + 1 + t - d_1) - t = -d_1^2 + (\left\lfloor \frac{n}{2} \right\rfloor + 1 + t) d_1 - t \geq -d_1^2 + (\left\lfloor \frac{n}{2} \right\rfloor + 1) d_1$.

When n is even, we know that the solutions of $n - 3 \geq T(G) = d_1(\frac{n}{2} + 1 - d_1)$ is exactly one of $d_1 = 1$ or $d_2 = 1$ holds like in Case 2.1. However, when $d_2 = 1$, since $d_1 + d_2 = \frac{n}{2} + 1 + t$, we have $d_1 = \frac{n}{2} + t$, which contradicts (1) namely $i' \leq \frac{n-1+\sqrt{4t+1}}{2}$ $(1 \leq t \leq \frac{n}{2} - 2)$ because $d_1 \leq i'$. The case $d_1 = 1$ and $d_2 = \frac{n}{2} + t$ can be settled in the same way.

When n is odd, $n-3 \geq T(G) = d_1(\left\lfloor \frac{n}{2} \right\rfloor + 1 - d_1)$ implies that one of $d_1 = 1$, $d_2 = 1$, $d_1 = 2$ or $d_2 = 2$ holds. By symmetry we can consider the cases $d_1 = 1$ and $d_1 = 2$. We check the details of the following 3 subcases.

(i) $t = 1$ and $d_1 = 1$. We get $d_2 = \frac{n+1}{2}$ because $d_1 + d_2 = \frac{n-1}{2} + 1 + t$. Since $d_2 \leq |Y'| = n - 1 - i' \leq \frac{n-1+2\sqrt{t}}{2} = \frac{n+1}{2}$, we get $|Y'| = \frac{n+1}{2}$ and $|X'| = \frac{n-3}{2}$. Since $e(G \setminus v_0) = \frac{n-1}{2}\frac{n-1}{2} - 1$, we see that $G \setminus v_0$ is $K_{\frac{n-3}{2},\frac{n+1}{2}}$. Thus, G is $K^{-}_{\frac{n-1}{2},\frac{n+1}{2}}$ and $T(G) \leq d_1 d_2 = \frac{n+1}{2}$.

(ii) $t \geq 2$ and $d_1 = 1$. By $d_1 + d_2 = \frac{n-1}{2} + 1 + t$, we have $d_2 = \frac{n-1}{2} + t > \frac{n-1+2\sqrt{t}}{2}$, which contradicts $d_2 \leq |Y'| = n - 1 - i' \leq \frac{n-1+2\sqrt{t}}{2}$.

(iii) $t \geq 1$ and $d_1 = 2$. By $d_1 + d_2 = \frac{n-1}{2} + 1 + t$, we have $d_2 = \frac{n-1}{2} + t - 1$. However, $T(G) \geq d_1 d_2 - t = 2(\frac{n-1}{2} + t - 1) - t \geq n - 2$, which contradicts $T(G) \leq n - 3$.

In conclusion, when n is even, G is $K^{-}_{\frac{n}{2},\frac{n}{2}}$. When n is odd, G is either $K^{-}_{\frac{n-1}{2},\frac{n+1}{2}}$ or $K^{-}_{\frac{n+1}{2},\frac{n-1}{2}}$ or $K^{\mathrm{T}}_{\frac{n+1}{2},\frac{n-1}{2}}$. □

Using Lemma 3, we are able to give the proof of Theorem 4.

Proof of Theorem 4. We prove our result by induction on n. The induction step will go from $n-2$ to n, so we check the bases when $n = 3$ and $n = 4$, obviously, our statement is true for these two cases. Suppose Theorem 4 holds for $k = n - 2$ ($n \geq 5$), we separate the rest of the proof into 2 cases.

Case 1. Every edge in G is contained in at least one triangle. Then $T(G) \geq \left\lceil \frac{\left\lfloor \frac{n^2}{4} \right\rfloor + 1}{3} \right\rceil \geq n - 2$.

Case 2. There exists at least one edge uv which is not contained in any triangle. Then u and v cannot have common neighbor in $G \setminus \{u, v\}$, which implies that $e(\{u, v\}, G \setminus \{u, v\}) \leq n - 2$. Therefore, $e(G \setminus \{u, v\}) \geq \left\lfloor \frac{n^2}{4} \right\rfloor - (n-2) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$. In this point, we split the rest of the proof into 3 subcases.

Case 2.1 $e(G \setminus \{u, v\}) \geq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 3$. By Theorem 2, we get $T(G \setminus \{u, v\}) \geq 3\left\lfloor \frac{n-2}{2} \right\rfloor$, which implies that $T(G) \geq 3\left\lfloor \frac{n-2}{2} \right\rfloor \geq n - 2$.

Case 2.2. $e(G \setminus \{u, v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 2$. When n is even, by Theorem 2, we get

$T(G \setminus \{u,v\}) \geq n-2$, since $T(G) \geq T(G \setminus \{u,v\})$, we are done. When n is odd, we have $e(\{u,v\}, G \setminus \{u,v\}) = n-3$, then there exists $w \in V(G \setminus \{u,v\})$ such that edges $vw, uw \notin E(G)$. If $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 1$, then the number of triangles which contains u or v is at least 1. By Theorem 2, $T(G \setminus \{u,v\}) \geq n-3$ holds, thus, $T(G) \geq n-2$. Otherwise, $G \setminus \{u,v,w\}$ is bipartite and all triangles in $G \setminus \{u,v\}$ are adjacent to w. since $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 0$, no triangle contains u or v. Therefore, $\tau_\triangle(G) = \tau_\triangle(G \setminus \{u,v\}) = 1$ and all triangles in G are adjacent to w.

Case 2.3. $e(G \setminus \{u,v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$, then $e(\{u,v\}, G \setminus \{u,v\}) = n-2$. When $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 0$, $G \setminus \{u,v\}$ is bipartite, it has at most $\left\lfloor \frac{(n-2)^2}{4} \right\rfloor$ edges, contradicting the assumption of the case.

Suppose $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) = 1$. Since $|N(u) \setminus v \ \cup \ N(v) \setminus u| = n-2$, we have $e([N(u) \setminus v], [N(v) \setminus u]) \leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor$. Thus, $e(G \setminus \{u,v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$ implies that $G \setminus \{u,v\}$ is obtained from $K_{\lfloor \frac{n-2}{2} \rfloor, \lceil \frac{n-2}{2} \rceil}$ plus an edge, say $\{j,k\}$, in one class. Therefore, all triangles in G contain $\{j,k\}$ and hence $\tau_\triangle(G) = 1$ follows.

Now we assume that $e(G[N(u) \setminus v]) + e(G[N(v) \setminus u]) \geq 2$, then the number of the triangles containing u or v is at least 2. It is easy to check that if $v(G) = 5$ then $G \setminus \{u,v\}$ is a triangle and either $\tau_\triangle(G) = 1$ or $T(G) = 4$. Therefore, we may assume $n \geq 6$. Since $e(G \setminus \{u,v\}) = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 1$, by the induction hypothesis, either $\tau_\triangle(G \setminus \{u,v\}) = 1$ or $T(G \setminus \{u,v\}) \geq n-4$. When $T(G \setminus \{u,v\}) \geq n-4$, we have $T(G) \geq T(G \setminus \{u,v\}) + 2 \geq n-2$. Otherwise, $\tau_\triangle(G \setminus \{u,v\}) = 1$ and $T(G \setminus \{u,v\}) \leq n-5$ hold. By Lemma 3, we see that when n is even, $G \setminus \{u,v\}$ is $K^-_{\frac{n}{2}-1, \frac{n}{2}-1}$, when n is odd, $G \setminus \{u,v\}$ is either $K^-_{\frac{n-3}{2}, \frac{n-1}{2}}$ or $K^-_{\frac{n-1}{2}, \frac{n-3}{2}}$ or $K^{\mathrm{T}}_{\frac{n-1}{2}, \frac{n-3}{2}}$. Let us check what will happen in these cases.

We first give the following technical lemma:

Lemma 5. *Let $f(a,b) = ab + (A-a)(B-b)$, where A and B are integers, $1 \leq a \leq A$, $1 \leq b \leq B$, then $f(a,b) \geq min\{A,B\}$.*

Proof of Lemma 5. Obviously, when $AB = max\{A,B\}$, $f(a,b) \geq 1 = min\{A,B\}$. Otherwise, we have $A, B \geq 2$. Without loss of generality, fix b, then $f(a,b)$ is a linear function

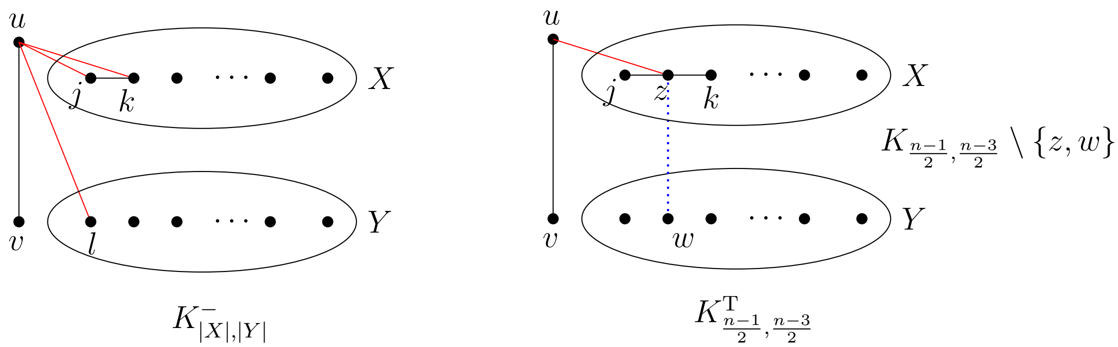

Figure 2:

of variable a. Since $\frac{\partial f}{\partial a} = b - (B - b)$, thus, $f(a, b)$ is decreasing when $b < \frac{B}{2}$ and $f(a, b)$ is increasing when $b > \frac{B}{2}$. Therefore,

$$f(a,b) \geq \begin{cases} f(A,b) = Ab, & b \leq \frac{B}{2}, \\ f(1,b) = b + (A-1)(B-b), & b > \frac{B}{2}. \end{cases}$$

It is easy to check that $Ab \geq A$, when $b \leq \frac{B}{2}$, and $b+(A-1)(B-b) = B(A-1)+b(2-A) \geq B$ when $b > \frac{B}{2}$. Hence, we get $f(a, b) \geq min\{A, B\}$. Obviously, if $min\{A, B\} = A$, the equality holds only when $a = A$ and $b = 1$, if $min\{A, B\} = B$, the equality holds only when $a = 1$ and $b = B$. $\square$

Case 2.3.1. $G \setminus \{u, v\}$ is $K^{-}_{\lfloor\frac{n}{2}\rfloor-1,\lceil\frac{n}{2}\rceil-1}$, which implies that when n is even, $G \setminus \{u, v\}$ is $K^{-}_{\frac{n}{2}-1,\frac{n}{2}-1}$ and when n is odd, $G \setminus \{u, v\}$ is $K^{-}_{\frac{n-3}{2},\frac{n-1}{2}}$. Let X and Y be the two classes of $K^{-}_{\lfloor\frac{n}{2}\rfloor-1,\lceil\frac{n}{2}\rceil-1}$ and $\{j, k\}$ be the extra edge in X, where $|X| = \lfloor\frac{n}{2}\rfloor - 1$, see Figure 2. Since $e(G[N(u)\setminus v])+e(G[N(v)\setminus u]) \geq 2$, $| N(u)\setminus v \cup N(v)\setminus u |= n-2$ and $N(u)\setminus v \cap N(v)\setminus u = \emptyset$, we see that either $N(u) \setminus v$ or $N(v) \setminus u$ contains at least one vertex in both classes X and Y. Without loss of generality, say at least $N(u) \setminus v$ has this property.

Let $|N(u)\setminus v \cap X| = a$ and $|N(u)\setminus v \cap Y| = b$, where $1 \leq a \leq \lfloor\frac{n}{2}\rfloor - 1$ and $1 \leq b \leq \lceil\frac{n}{2}\rceil - 1$. Then the number of triangles which are adjacent to u, containing one vertex in X and one in Y is ab while the number of triangles which are adjacent to v, containing one vertex in X and one in Y is $(A-a)(B-b)$. Hence, we get $T(G) \geq ab + \left(\lfloor\frac{n}{2}\rfloor - 1 - a\right)\left(\lceil\frac{n}{2}\rceil - 1 - b\right) + \lceil\frac{n}{2}\rceil - 1$. By Lemma 5, we see $T(G) \geq \lfloor\frac{n}{2}\rfloor - 1 + \lceil\frac{n}{2}\rceil - 1 = n - 2$.

Case 2.3.2. n is odd and $G \setminus \{u, v\}$ is $K^{-}_{\frac{n-1}{2},\frac{n-3}{2}}$. Let X and Y be the two classes of $K^{-}_{\frac{n-1}{2},\frac{n-3}{2}}$ and $\{j, k\}$ be the extra edge in X, where $|X| = \frac{n-1}{2}$. Similarly as in the previous case, either $N(u) \setminus v$ or $N(v) \setminus u$ contains at least one vertex in both classes X and Y. Without loss of generality, say at least $N(u) \setminus v$ has this property.

Let $|N(u)\setminus v \cap X| = a$ and $|N(u)\setminus v \cap Y| = b$, where $1 \leq a \leq \frac{n-1}{2}$ and $1 \leq b \leq \frac{n-3}{2}$, then $T(G) \geq ab + \left(\frac{n-1}{2} - a\right)\left(\frac{n-3}{2} - b\right) + \frac{n-3}{2}$. By Lemma 5, we get $T(G) \geq \frac{n-3}{2} + \frac{n-3}{2} \geq n - 3$, the equality holds only if $a = 1$ and $b = \frac{n-3}{2}$. Let $s \in X$ and $\{u, s\} \in E(G)$, $a = 1$ and $b = \frac{n-3}{2}$ implies that either $s \in \{j, k\}$ then $\tau_{\triangle}(G) = 1$, or $s \notin \{j, k\}$ then there exists one more triangle $\{v, j, k\}$, thus $T(G) \geq n - 3 + 1 = n - 2$.

Case 2.3.3. n is odd and $G \setminus \{u, v\}$ is $K^{\mathrm{T}}_{\frac{n-1}{2},\frac{n-3}{2}}$. Since $\frac{n-1}{2} \geq 3$, we get $n \geq 7$. Let X and Y be the classes of $K^{\mathrm{T}}_{\frac{n-1}{2},\frac{n-3}{2}}$, $\{j, z\}$ and $\{z, k\}$ be the two extra edges in X and $\{z, w\}$ be the missing edge in $K_{\frac{n-1}{2},\frac{n-3}{2}}$, see Figure 2.

Let $|N(u) \setminus v \cap X| = a$ and $|N(u) \setminus v \cap Y| = b$. Since $\mid N(u) \setminus v \cup N(v) \setminus u \mid = n-2$ and $N(u) \setminus v \cap N(v) \setminus u = \emptyset$, when $a = 0$, we have $X \subseteq N(v) \setminus u$. If $N(v) \setminus u = X$, clearly, all triangles in G contain z and hence $\tau_{\triangle}(G) = 1$. Otherwise, $|(N(v) \setminus u) \cap Y| \geq 1$. It is easy to check that $T(K^{\mathrm{T}}_{\frac{n-1}{2},\frac{n-3}{2}}) = n-5$, therefore, in this case we get $T(G) \geq n-5+2+\frac{n-1}{2}-1 \geq n-1$ ($n \geq 7$). When $b = 0$, then $Y \subseteq N(v) \setminus u$. If $N(v) \setminus u = Y$ then $N(u) \setminus v = X$, we see that all triangles in G contain z and hence $\tau_{\triangle}(G) = 1$. Otherwise, $|(N(v) \setminus u) \cap X| \geq 1$. When $|(N(v) \setminus u) \cap X| = 1$, if $(N(v) \setminus u) \cap X = \{z\}$, obviously, all triangles in G contain z, hence $\tau_{\triangle}(G) = 1$. If not, then clearly $T(G) \geq n - 5 + 1 + \frac{n-3}{2} \geq n - 2$ ($n \geq 7$). It is easy to check that $T(G)$ reaches the lower bound when $|(N(v) \setminus u) \cap X| = 1$ for $n \geq 9$ and when $n = 7$, $T(G) \geq 5$ holds in all cases. Therefore, we get either $\tau_{\triangle}(G) = 1$ or $T(G) \geq n - 2$.

Now suppose that, $1 \leq a \leq \frac{n-1}{2}$ and $1 \leq b \leq \frac{n-3}{2}$. Then $T(G) \geq ab + (\frac{n-1}{2} - a)(\frac{n-3}{2} - b) + n - 5$, by Lemma 5, we get $T(G) \geq \frac{n-3}{2} + n - 5 \geq n - 2$ ($n \geq 9$). Since $T(G) \geq 5$ when $n = 7$, we see that $T(G) \geq n - 2$ holds in this case.

This completes the proof. $\square$

3 Open problems

Let $V_1, V_2, \ldots, V_r$ be pairwise disjoint sets where $\left\lceil \frac{n}{2} \right\rceil \geq |V_1| \geq |V_2| \geq \ldots \geq |V_r| \geq \left\lfloor \frac{n}{2} \right\rfloor$ and $\sum |V_i| = n$ hold. Define the graph $T_r(n)$ with vertex set $\cup V_i$ where $\{u, v\}$ is an edge if $u \in V_i$, $v \in V_j (i \neq j)$, but there is no edge within a V_i. The number of edges of the graph $T_r(n)$ is denoted by $t_r(n)$. The following fundamental theorem of Turán is a generalization of Mantel's theorem.

Theorem 6 (**Turán** [6])**.** *If a graph on n vertices has more than $t_{k-1}(n)$ edges then it contains a copy of the complete graph K_k as a subgraph.*

The most natural construction is to add one edge to $T_{k-1}(n)$ in the set V_1. This graph is denoted by $T_{k-1}^{-}(n)$. It contains not only one copy of K_k but $|V_2| \cdot |V_3| \cdots |V_{k-1}|$ of them. [3] proved that this is the least number. Observe that the intersection of all of these copies of K_k is a pair of vertices (in V_1). If this is excluded, the number of copies probably increases. This is expressed by the following conjecture. Take $T_{k-1}(n)$, add an edge $\{x, y\}$ in V_1, an edge $\{u, v\}$ in V_2 and delete the edge $\{u, x\}$. This graph is denoted by $T_{k-1}^{\sqsubset}$. It contains almost the double of the number of copies of K_k in $T_{k-1}^{-}(n)$.

Conjecture 1. *If a graph on n vertices has $t_{k-1}(n) + 1$ edges and the copies of K_k have an empty intersection then the number of copies of K_k is at least as many as in $T_{k-1}^{\sqsubset}$:* $(|V_2| - 1)|V_3| \cdot |V_4| \cdots |V_{k-1}| + (|V_1| - 1)|V_3| \cdot |V_4| \cdots |V_{k-1}| = (|V_1| + |V_2| - 2)|V_3| \cdot |V_4| \cdots |V_{k-1}|$.

Of course this would be a generalization of our Theorem 4. Now we try to generalize it in a different direction. What is the minimum number of triangles in an n-vertex graph G containing $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges if $\tau_\triangle(G) \geq s$ is also supposed. The problem is interesting only when $0 < t < s$. Otherwise, if $t \geq s$ then $\tau_\triangle(G) = t$ is allowed. By Lovász-Simonovits' theorem [4], we know that the number of triangles is at least $t \left\lfloor \frac{n}{2} \right\rfloor$ with equality for the following graph. Take $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ where the two parts are $V_1 (|V_1| = \left\lceil \frac{n}{2} \right\rceil)$ and $V_2 (|V_2| = \left\lfloor \frac{n}{2} \right\rfloor)$, respectively. Add t edges to V_1. Here all triangles contain one of the new added edges, therefore $\tau_\triangle(G) \leq t$ and the extra condition on $\tau_\triangle(G)$ is not a real restriction.

Hence we may suppose $0 < t < s$. Choose $2(s-1)$ distinct vertices in V_1 (of $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$):

$x_1, x_2, \ldots, x_{s-1}, y_1, y_2, \ldots, y_{s-1}$ and two distinct vertices in $V_2 : u_1, u_2$. Add the edges $\{x_1, y_1\}$, $\{x_2, y_2\}, \ldots, \{x_{s-1}, y_{s-1}\}, \{u_1, u_2\}$ to $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ and delete the edges $\{x_1, u_1\}, \ldots, \{x_{s-t}, u_1\}$. Let $K^{s,t}_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ denote this graph. It is easy to see that it contains $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges. On the other hand it contains s vertex disjoint triangles if $\left\lceil \frac{n}{2} \right\rceil \geq 2(s-1)+1$ and $\left\lfloor \frac{n}{2} \right\rfloor \geq s+1$. Therefore, $\tau_\triangle(K^{s,t}_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}) = s$ holds if n is large enough. We believe that this is the best possible construction.

Conjecture 2. *Suppose that the graph G has n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges, it satisfies $\tau_\triangle(G) \geq s$ and $n \geq n(t,s)$ is large. Then G contains at least as many triangles as $K^{s,t}_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ has, namely $(s-1)\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil - 2(s-t)$.*

In the case $t = 1, s = 2$ our Theorem 4 is obtained. There is an obvious common generalization of our two conjectures.

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