Largest family without a pair of posets on consecutive levels of the Boolean lattice

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Abstract

Suppose $k \geq 2$ is an integer. Let Y_k be the poset with elements $x_1, x_2, y_1, y_2, \dots, y_{k-1}$ such that $y_1 < y_2 < \dots < y_{k-1} < x_1, x_2$ and let Y'_k be the same poset but all relations reversed. We say that a family of subsets of $[n]$ contains a copy of Y_k on consecutive levels if it contains $k+1$ subsets $F_1, F_2, G_1, G_2, \dots, G_{k-1}$ such that $G_1 \subset G_2 \subset \dots \subset G_{k-1} \subset F_1, F_2$ and $|F_1| = |F_2| = |G_{k-1}| + 1 = |G_{k-2}| + 2 = \dots = |G_1| + k - 1$. If both Y_k and Y'_k on consecutive levels are forbidden, the size of the largest such family is denoted by $\mathrm{La_c}(n, Y_k, Y'_k)$. In this paper, we will determine the exact value of $\mathrm{La_c}(n, Y_k, Y'_k)$.

Keywords: forbidden subposets, extremal set theory, double counting

MSC: 05D05

1 Introduction

Given two partially ordered sets (posets) P and Q, we say that P is a *subposet* of Q if there exists an injection $\phi : P \to Q$ such that $x \leq_P y$ implies $\phi(x) \leq_Q \phi(y)$. Viewing collections of sets as posets under the inclusion relation, we have the following extremal functions, first introduced by Katona and Tarján [9]. For any collection of finite posets $\mathcal{P}$, let $\mathrm{La}(n, \mathcal{P})$ be the maximum size of a family of subsets of $[n] = \{1, 2, \dots, n\}$ which does not contain any $P \in \mathcal{P}$ as a subposet.

This type of problems was first studied by Sperner [14].

Theorem 1.1 (Sperner [14]). *Let $\mathcal{F}$ be a family of subsets of $[n]$ without inclusion relation between any two of the subsets. Then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

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A chain of length k is a poset with elements $x_1, x_2, \dots, x_k$ such that $x_1 < x_2 < \cdots < x_k$. Let $\sum(n,k)$ be the sum of the k largest binomial coefficients of the form $\binom{n}{i}$. Then Sperner's theorem can be extended as follows:

Theorem 1.2 (Erdős [3]). *Let $\mathcal{F}$ be a family of subsets of $[n]$ without a chain of length k. Then*

$$|\mathcal{F}| \le \sum(n, k-1).$$

Before stating the next result, we need the following notation. Let $2 \le k \le n$ and $0 \le r \le n$ be two integers. The following lacunary sum of binomial coefficients was first introduced by Ramus [13] in 1834.

$$S(n,k,r) = \sum_{\substack{i=0 \\ i \equiv r \bmod k}}^{n} \binom{n}{i}.$$

Clearly, if $r_1 \equiv r_2 \pmod{k}$, then $S(n,k,r_1) = S(n,k,r_2)$. So there are k distinct such sums $S(n,k,0), S(n,k,1), \dots, S(n,k,k-1)$.

The first author [8] published the next two theorems which are analogs of the two theorems above.

Theorem 1.3 (Katona [8]). *Let $\mathcal{F}$ be a family of subsets of $[n]$ such that no two of the subsets F_i, F_j satisfy $F_i \subset F_j$ and $|F_j| - |F_i| < k$. Then*

$$|\mathcal{F}| \le \max\{S(n,k,r) \mid r \in \{0, \dots, k-1\}\}.$$

Theorem 1.4 (Katona [8]). *Let $h \le k$ be an integer, and let $\mathcal{F}$ be a family of subsets of $[n]$ such that no $h+1$ of the subsets $F_{i_1}, \dots, F_{i_{h+1}}$ satisfy $F_{i_1} \subset \cdots \subset F_{i_{h+1}}$ and $|F_{i_{h+1}}| - |F_{i_1}| < k$. Then*

$$|\mathcal{F}| \le \max\{S(n,k,r_1) + S(n,k,r_2) + \cdots + S(n,k,r_h) \mid r_1 \ne r_2 \ne \cdots \ne r_h \in \{0, \dots, k-1\}\}.$$

Another type of generalizations of Sperner Theorem is to determine the largest size of a family of subsets of $[n]$ without a copy of poset Y_k and Y_k' defined below. Let Y_k be the poset with elements $x_1, x_2, y_1, y_2, \dots, y_{k-1}$ such that $y_1 < y_2 < \cdots < y_{k-1} < x_1, x_2$, and let Y_k' be the same poset but all relations reversed. Katona and Tarján [9] gave the following result for $k = 2$. (In their paper, posets Y_2 and Y_2' are denoted by V (the cherry poset) and Λ (the fork poset) respectively.)

Theorem 1.5 (Katona and Tarján [9]).

$$\mathrm{La}(n, Y_2, Y_2') = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

De Bonis, Katona and Swanepoel [2] studied the case $k = 3$. We remark that they actually proved a result for the so-called butterfly poset, but their proof implies the following theorem.

Theorem 1.6 (De Bonis, Katona and Swanepoel [2])**.**

$$\mathrm{La}(n, Y_3, Y_3') = \sum(n, 2).$$

For the case $k \geq 4$, Methuku and Tompkins [12] got the next theorem.

Theorem 1.7 (Methuku and Tompkins [12])**.** *For $n \geq k \geq 4$,*

$$\mathrm{La}(n, Y_k, Y_k') = \sum(n, k-1).$$

For other results related to the La function, see [1, 5, 6, 7, 11, 15]. Now, we consider the following problem, which is a combination of the two types of problems above. We say that a family of subsets of $[n]$ contains a copy of Y_k on consecutive levels if it contains $k+1$ subsets $F_1, F_2, G_1, G_2, \ldots, G_{k-1}$ such that $G_1 \subset G_2 \subset \cdots \subset G_{k-1} \subset F_1, F_2$ and $|F_1| = |F_2| = |G_{k-1}|+1 = |G_{k-2}|+2 = \cdots = |G_1|+k-1$. We denote by $\mathrm{La_c}(n, Y_k, Y_k')$ the largest size of a family of subsets of $[n]$ containing neither Y_k nor Y_k' on consecutive levels. We note that this problem is mentioned in [4] in the language of the oriented hypercube. The authors in [4] gave an asymptotic formula of the size of largest family without tree posets in consecutive levels. For exact result, they proved

$$\mathrm{La_c}(n, Y_2, Y_2') = 2^{n-1}.$$

Let $m = \lceil (n-k)/2 \rceil$ in the rest of this paper. For $k \geq 3$, we have the following theorem.

Theorem 1.8. *Let $n \geq k \geq 3$, then*

$$\mathrm{La_c}(n, Y_k, Y_k') = 2^n - S(n, k, m).$$

Remark 1.9. (I) *Loehr and Michael [10] showed that*

$$S(n, k, m) = \min_{r:\ 0 \leq r \leq k-1} S(n, k, r).$$

So our result implies that the trivial construction is the best. Here, a trivial construction consists of all subsets except the ones with size $s \equiv m \pmod k$.
(II) *If $n = k \geq 3$, then $m = 0$ and $S(n, k, m) = \binom{n}{0} + \binom{n}{n} = 2$. In this case, Theorem 1.8 is trivial, since every family $\mathcal{F}$ with size $2^n - 1$ contains a copy of Y_k or Y_k' on consecutive levels.*

The rest of the paper is organized as follows. In the next section, we present some preliminary results. In Section 3, we will prove our main theorem (Theorem 2.1), which implies Theorem 1.8.

2 Preliminary results

A cyclic permutation σ of $[n]$ is a cyclic ordering $a_1, a_2, \ldots, a_n, a_1$, where $a_i \in [n]$ for $i = 1, 2, \ldots, n$. Let $\mathcal{F}$ be a family of subsets of $[n]$ containing neither Y_k nor Y_k' on consecutive levels. We say a set $F \in \sigma$ if F is an interval along the cyclic permutation σ.

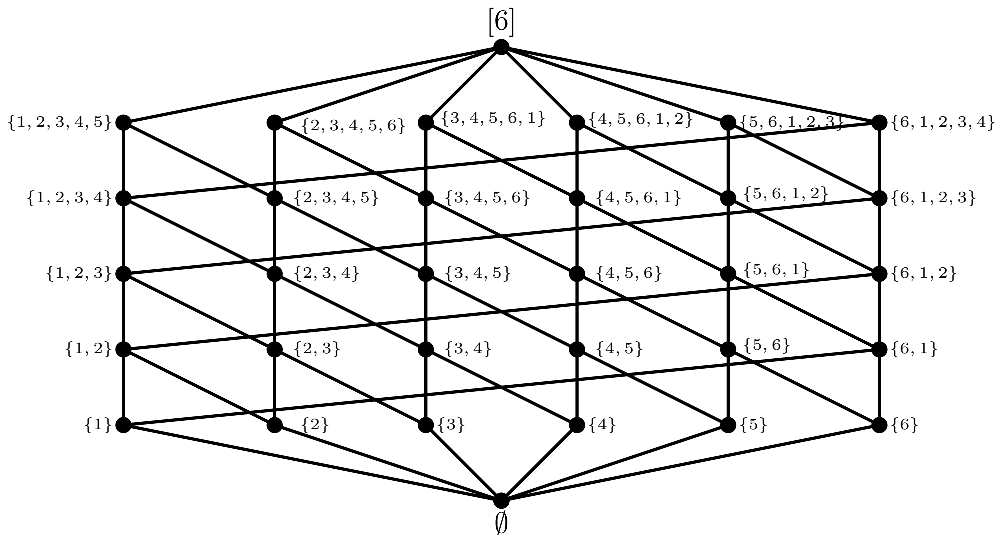

Figure 1: $\mathcal{I}(n)^\sigma$ for $n = 6$ and $\sigma = 1, 2, 3, 4, 5, 6, 1$ is shown by bold vertices, and two vertices from consecutive levels are connected by an edge if they have inclusion relation.

Now, we double count the sum S of ϕ_F over all cyclic permutations σ and $F \in \mathcal{F}$ such that $F \in \sigma$, where

$$\phi_F = \begin{cases} \binom{n}{|F|}, & \text{if } F \neq \emptyset \text{ and } F \neq [n]; \\ n, & \text{if } F = \emptyset \text{ or } F = [n]. \end{cases}$$

For any $F \in (\mathcal{F} \setminus \{\emptyset, [n]\})$, we have $|F|!(n - |F|)!$ cyclic permutations σ satisfying $F \in \sigma$. If $\emptyset \in \mathcal{F}$ or $[n] \in \mathcal{F}$, all cyclic permutations satisfy the above condition, and the number of cyclic permutations is $(n-1)!$. So

$$\begin{aligned} S =& |\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n \cdot (n-1)! + \sum_{F \in (\mathcal{F} \setminus \{\emptyset, [n]\})} |F|!(n - |F|)! \binom{n}{|F|} \\ =& \left(|\{\emptyset, [n]\} \cap \mathcal{F}| + \sum_{F \in (\mathcal{F} \setminus \{\emptyset, [n]\})} 1 \right) \cdot n! = |\mathcal{F}| \cdot n!. \end{aligned}$$

On the other hand, for any cyclic permutation σ, let $\mathcal{I}(n)^\sigma$ be the family of intervals along σ. The i^{th} level of $\mathcal{I}(n)^\sigma$ is the collection of its elements of size i. (See Figure 1 for an example of $\mathcal{I}(6)^\sigma$, where $\sigma = 1, 2, 3, 4, 5, 6, 1$.) For $0 \leq i \leq n$, let x_i be the number of subsets in the i^{th} level of $\mathcal{F} \cap \mathcal{I}(n)^\sigma$. Then

$$S = \sum_\sigma \left(|\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n + \sum_{i=1}^{n-1} \binom{n}{i} x_i \right).$$

Now, we need the following theorem to give an upper bound on $|\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n + \sum_{i=1}^{n-1} \binom{n}{i} x_i$ for every cyclic permutation σ.

Theorem 2.1. *Let $n > k \geq 3$ and $\mathcal{F}$ be a family of subsets of $[n]$ containing neither Y_k nor Y'_k on consecutive levels. Then for every cyclic permutation σ, we have*

$$|\{\emptyset, [n]\} \cap \mathcal{F}| \cdot n + \sum_{i=1}^{n-1} \binom{n}{i} x_i \leq n \cdot \big(2^n - S(n, k, m)\big) + n - 1.$$

Supposing that we know Theorem 2.1, then

$$S = |\mathcal{F}| \cdot n! \leq \sum_{\sigma} \Big(n \cdot \big(2^n - S(n,k,m)\big) + n - 1\Big) = (n-1)! \cdot \Big(n \cdot \big(2^n - S(n,k,m)\big) + n - 1\Big).$$

Since $|\mathcal{F}|$ is an integer, we have

$$|\mathcal{F}| \leq \left\lfloor \frac{n \cdot \big(2^n - S(n,k,m)\big) + n - 1}{n} \right\rfloor = 2^n - S(n,k,m),$$

as desired. So by (II) of Remark 1.9, it is sufficient to prove Theorem 2.1. The following two lemmas are needed to give constraints of $x_0, x_1, \ldots, x_n$.

Lemma 2.2. *If $0 \leq i \leq n-k+1$, then for all σ,*

$$x_i + x_{i+1} + \cdots + x_{i+k-1} \leq (k-1)n.$$

Lemma 2.3. *For all σ, if $x_0 = 1$ and $x_1 + x_2 + \cdots + x_k = (k-1)n$, then*

$$x_0 + x_1 + \cdots + x_{k-1} \leq (k-1)n - \lfloor n/2 \rfloor.$$

For all σ, if $x_n = 1$ and $x_{n-k} + x_{n-k+1} + \cdots + x_{n-1} = (k-1)n$, then

$$x_{n-k+1} + x_{n-k+2} + \cdots + x_n \leq (k-1)n - \lfloor n/2 \rfloor.$$

Before starting the proofs of the two lemmas above, we introduce some notations. Let $\sigma = a_1, a_2, \ldots, a_n, a_1$, where $a_j \in [n]$ for $1 \leq j \leq n$. In the rest of this paper, we view $\mathcal{I}(n)^\sigma$ (a subfamily of $\mathcal{I}(n)^\sigma$) as a graph. The vertex set of this graph is $\mathcal{I}(n)^\sigma$ (the subfamily of $\mathcal{I}(n)^\sigma$), and two vertices are connected by an edge if they are from consecutive levels and they have inclusion relation (see Figure 2). We denote by I_t^s the vertex $\{a_{t+1}, a_{t+2}, \ldots, a_{t+s}\}$ (with addition taken modulo n) of $\mathcal{I}(n)^\sigma$. Clearly, $|I_t^s| = s$, I_t^{s-1}, $I_{t+1}^{s-1} \subset I_t^s$ for $2 \leq s \leq n-1$, and $I_t^s \subset I_{t-1}^{s+1}$, I_t^{s+1} for $1 \leq s \leq n-2$ (see Figure 2).

In order to prove the two kinds of constraints of k consecutive x_i's above, we consider some typical structures in k consecutive levels of $\mathcal{I}(n)^\sigma$. If the k levels are the levels from 0^{th} to $(k-1)^{st}$, then every Y_k on these levels must contain $\emptyset$. In this case, we consider a special kind of Y_k, which we denote by

$$Y_k(j) = \{\emptyset, I_j^1, I_j^2, \ldots, I_j^{k-2}, I_j^{k-1}, I_{j-1}^{k-1}\}$$

where $1 \leq j \leq n$. (See Figure 3 for an example of $Y_k(1)$.) If the k levels are the levels from $(i+1)^{st}$ to $(i+k)^{th}$ (the middle part of Figure 2), then we introduce a new kind of structure on k consecutive levels. A family of $k+2$ subsets is called X_k if it is

$$\{I_{t+1}^{i+1}, I_t^{i+1}, I_t^{i+2}, I_t^{i+3}, \ldots, I_t^{i+k}, I_{t-1}^{i+k}\} \text{ or } \{I_{t-1}^{i+1}, I_t^{i+1}, I_{t-1}^{i+2}, I_{t-2}^{i+3}, \ldots, I_{t-k+1}^{i+k}, I_{t-k+2}^{i+k}\},$$

where $1 \leq t \leq n$. (See Figure 3 for two types of examples of $t = 2$ and $t = k$ respectively.) In each X_k, we call the 3 elements in the $(i+k)^{th}$ level and the $(i+k-1)^{st}$ level a cherry, and the 3 elements in the $(i+1)^{st}$ level and the $(i+2)^{nd}$ level a fork.

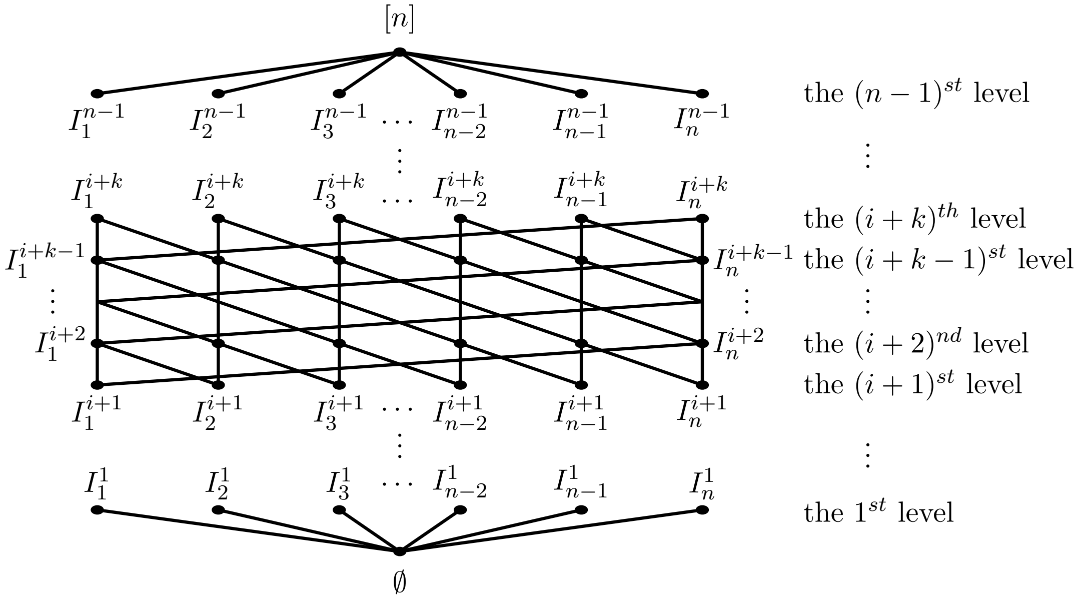

Figure 2: Vertices I_t^s of $\mathcal{I}(n)^\sigma$.

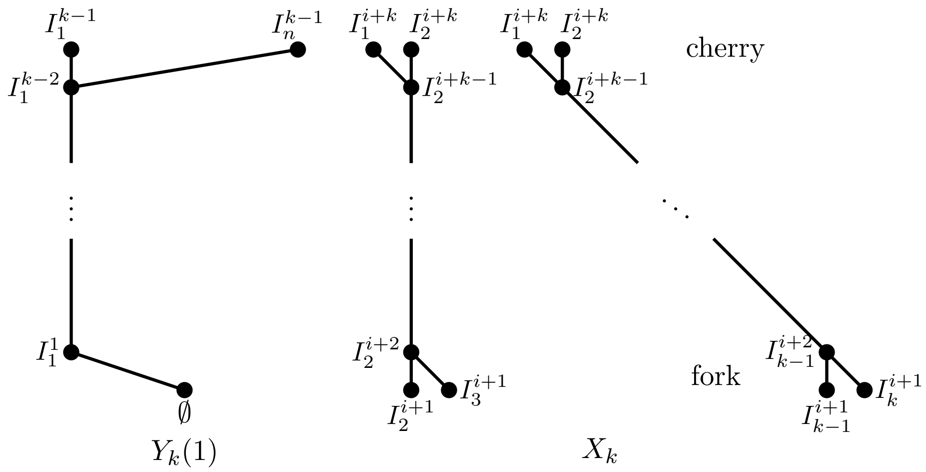

Figure 3: Examples of structures on k consecutive levels.

Remark 2.4. (I) *The vertices of the* n Y_k*'s* $(Y_k(1), Y_k(2), \dots, Y_k(n))$ *cover the vertices of the levels of* $\mathcal{I}(n)^\sigma$ *from* 1^{st} *to* $(k-2)^{nd}$ *once.*
(II) *For* $k \geq 3$ *and* $0 \leq i \leq n-k-1$*, the edges of the* $2n$ X_k*'s cover the edges within the* $(i+1)^{st}$ *level and the* $(i+2)^{nd}$ *level of* $\mathcal{I}(n)^\sigma$ *twice, the edges within the levels of* $\mathcal{I}(n)^\sigma$ *from* $(i+2)^{nd}$ *to* $(i+k-1)^{st}$ *once, and the edges within the* $(i+k-1)^{st}$ *level and the* $(i+k)^{th}$ *level of* $\mathcal{I}(n)^\sigma$ *twice.*

Now, we are ready to start the proofs of Lemmas 2.2 and 2.3. We note that a vertex I of $\mathcal{I}(n)^\sigma$ is a subset of $[n]$. So in the two proofs, we write $I \in \mathcal{F}$ if the subset I is an element of the family $\mathcal{F}$.

Proof of Lemma 2.2: First, we show $x_0 + x_1 + \cdots + x_{k-1} \leq (k-1)n$. Since $x_0 \leq 1$ and $x_i \leq n$ for $1 \leq i \leq k-1$, we have $x_0+x_1+\cdots+x_{k-1} \leq (k-1)n+1$. If $x_0+x_1+\cdots+x_{k-1} = (k-1)n+1$, every subset of the levels from 0^{th} to $(k-1)^{st}$ is in $\mathcal{F}$, then one can easily find a copy of Y_k on consecutive levels. (See $Y_k(1)$ for an example in Figure 3.) Similarly, we have $x_{n-k+1} + x_{n-k+2} + \cdots + x_n \leq (k-1)n$.

Now, we prove that

$$x_{i+1} + x_{i+2} + \cdots + x_{i+k} \leq (k-1)n$$

for $0 \leq i \leq n-k-1$. Recall that there is an edge between two vertices I, I' if and only if $I' \subset I$ and $|I| - |I'| = 1$. Then, we double count the sum T of the weight function $\psi(e)$ over all edges $e = \{I, I'\}$ in the $2n$ X_k's, where

$$\psi(e) = \begin{cases} 0, & \text{if } I, I' \in \mathcal{F}; \\ 0, & \text{if } I, I' \notin \mathcal{F}; \\ 0, & I \in \mathcal{F},\ I' \notin \mathcal{F} \text{ and } |I| = i+k; \\ 0, & I \notin \mathcal{F},\ I' \in \mathcal{F} \text{ and } |I| = i+2; \\ 1, & \text{otherwise.} \end{cases}$$

On the one hand, recall that every vertex in the s^{th} level of $\mathcal{I}(n)^\sigma$ have two neighbors in the $(s+1)^{st}$ level and two neighbors in the $(s-1)^{st}$ level, and note that only the case when one of the two vertices in the edge is not in $\mathcal{F}$ counts nonzero. Then by (II) of Remark 2.4,

$$\begin{aligned} T =& \Big(2 \cdot \sum_{I \notin \mathcal{F}, |I| = i+k} \psi(e) + \sum_{I \notin \mathcal{F}, |I| \neq i+k} \psi(e)\Big) + \Big(2 \cdot \sum_{I' \notin \mathcal{F}, |I'| = i+1} \psi(e) + \sum_{I' \notin \mathcal{F}, |I'| \neq i+1} \psi(e)\Big) \\ \leq& \Big(2 \cdot 2(n - x_{i+k}) + 1 \cdot \sum_{j=i+3}^{i+k-1} 2(n - x_j)\Big) + \Big(2 \cdot 2(n - x_{i+1}) + 1 \cdot \sum_{l=i+2}^{i+k-2} 2(n - x_l)\Big) \\ =& 4\Big(\sum_{h=i+1}^{i+k} (n - x_h)\Big) - 2(n - x_{i+2}) - 2(n - x_{i+k-1}). \end{aligned}$$

On the other hand, we will prove below the claim that in each X_k, the sum of the weight function over its edges is at least $|\{I \in \mathcal{F} \cap X_k \mid |I| = i+2\}| + |\{I \in \mathcal{F} \cap X_k \mid |I| = i+k-1\}|$. Then if we sum over all $2n$ X_k's, T is at least $2(x_{i+2} + x_{i+k-1})$.

Now, we divide the cherries of all $2n$ X_k's into 4 types (see Figure 4), according to whether its 3 elements are in $\mathcal{F}$ or not. We call a cherry Type 1 (2 or 3) if the middle element and both (one or none) of its neighbors are in $\mathcal{F}$. The rest of the cases are Type 4, namely that the middle element is not in $\mathcal{F}$. Note that in Type 4, we do not distinguish the cases if two neighbors of the middle element are in $\mathcal{F}$ or not. Similarly, we divide all forks into Types $5, 6, 7$ and 8 (see Figure 4).

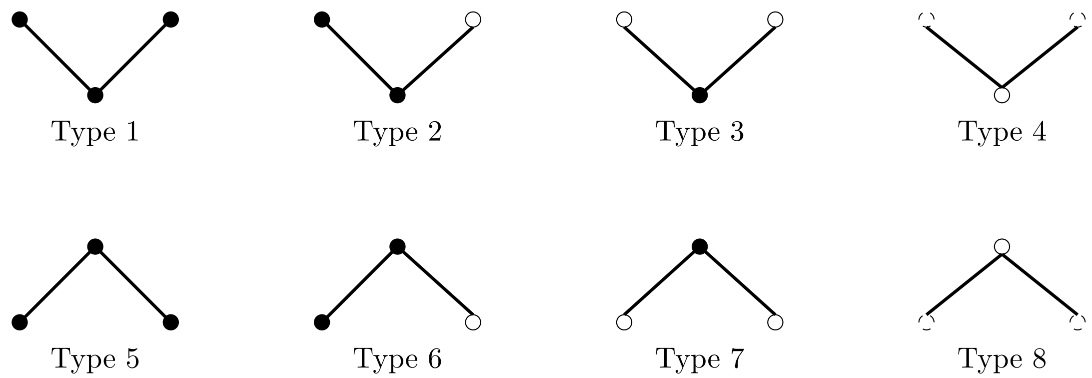

Figure 4: Types $1-8$: the vertices I in $\mathcal{F}$ (not in $\mathcal{F}$ or not clear) are denoted by solid (hollow or dashed) vertices, respectively.

To prove the claim, we distinguish several cases by the types defined above. Note that in each X_k, in order to avoid copies of Y_k and Y_k' in $\mathcal{F}$, either at least 2 of the 4 vertices in the $(i+1)^{st}$ level and the $(i+k)^{th}$ level are not in $\mathcal{F}$, or at least one of the $k-2$ vertices in the levels from $(i+2)^{nd}$ to $(i+k-1)^{st}$ is not in $\mathcal{F}$. If the cherry and the fork in some X_k are Types 1 and 5 respectively, then we must have some vertex I not in $\mathcal{F}$ in the chain connecting the cherry and the fork in this X_k. Then along this chain in both directions from I, take the first vertices in $\mathcal{F}$ respectively. Thus, we can find at least two edges with weight 1. Note that such vertices exist since in Types 1 and 5, the vertices in the $(i+2)^{nd}$ level and the $(i+k-1)^{st}$ level are in $\mathcal{F}$. The same argument works for the pairs Types 1 and 6, and Types 2 and 5. If they are Types 1 and 7, then we already have two desired edges in the fork. If they are Types 1 and 8, then we find the first vertex in $\mathcal{F}$ along the chain from the fork to the cherry, this vertex and its neighbor below in the chain will give the edge we want. The same argument works for Types 4 and 5. In the rest cases, it is easy to find the desired edges in the cherry and the fork. (If $k=3$, Types 1 or 2 or 3 and 8 (5 or 6 or 7 and 4) cannot appear in any X_k, since $i+1=i+k-2$. If $k=3$ or 4, Types 1 and 5 (1 and 6, or 2 and 5) cannot appear in any X_k, since they will form a Y_k or a Y_k' in $\mathcal{F}$.)

Therefore, we have

$$2(x_{i+2}+x_{i+k-1}) \leq 4\Big(\sum_{h=i+1}^{i+k}(n-x_h)\Big) - 2(n-x_{i+2}) - 2(n-x_{i+k-1}).$$

That is,

$$x_{i+1}+x_{i+2}+\cdots+x_{i+k} \leq (k-1)n,$$

as required. □

Proof of Lemma 2.3: By symmetry, it is enough to prove the first part of the lemma. That is, we need to prove

$$x_0 + x_1 + \cdots + x_{k-1} \le (k-1)n - \lfloor n/2 \rfloor = (k-2)n + \lceil n/2 \rceil.$$

We distinguish three cases to prove it.

Case 1. $x_{k-1} \le \lfloor n/2 \rfloor$ *when* n *is odd or* $x_{k-1} \le (n/2) - 1$ *when* n *is even.*

Note that $x_i \le n$ for $i = 1, 2, \ldots, k-2$, then by the assumption $x_0 = 1$, we have

$$x_0 + x_1 + \cdots + x_{k-1} \le 1 + (k-2)n + \lfloor n/2 \rfloor \le (k-2)n + \lceil n/2 \rceil,$$

if n is odd; and

$$x_0 + x_1 + \cdots + x_{k-1} \le 1 + (k-2)n + (n/2) - 1 \le (k-2)n + (n/2),$$

if n is even.

Case 2. $x_{k-1} = n/2$ *when* n *is even.*

Suppose that the result is not true. That is,

$$x_0 + x_1 + \cdots + x_{k-1} \ge (k-2)n + (n/2) + 1.$$

Note that $x_{k-1} = n/2$ in this case and $x_i \le n$ for $i = 1, 2, \ldots, k-2$. Then by the assumption $x_0 = 1$, we have $x_1 = x_2 = \cdots = x_{k-2} = n$. Therefore, by the assumption $x_1 + x_2 + \cdots + x_k = (k-1)n$, we have $x_k = n/2$. Since $x_{k-1} = n/2$ and $x_k = n/2$, we can find 2 vertices such that they are in the k^{th} level and the $(k-1)^{st}$ level respectively, they are in $\mathcal{F}$, and they have inclusion relation. Then a copy of Y'_k on the levels from 1^{st} to k^{th} can be found, a contradiction.

Case 3. $x_{k-1} \ge \lfloor n/2 \rfloor + 1$.

The number of pairs of vertices $(I^{k-1}_{j-1}, I^{k-1}_j)$ in $\mathcal{F}$ is at least $x_{k-1} - (n - x_{k-1}) = 2x_{k-1} - n$ (see Figure 3). To avoid a copy of $Y_k(j)$ in $\mathcal{F}$, at least one vertex of $Y_k(j)$ in the levels from 1^{st} to $(k-2)^{nd}$ is not in $\mathcal{F}$. So by (I) of Remark 2.4, we have

$$\sum_{i=0}^{k-1} x_i \le 1 + (k-2)n - (2x_{k-1} - n) + x_{k-1} = 1 + (k-1)n - x_{k-1} \le (k-1)n - \lfloor n/2 \rfloor,$$

since $x_{k-1} \ge \lfloor n/2 \rfloor + 1$. This completes the proof of this case and the lemma. □

Before starting the proof of Theorem 2.1, we need the following notations and lemmas for helping us to use the constraints above. Recall the definition of $S(n,k,r)$ and $m = \lceil (n-k)/2 \rceil$. Now let z be an integer such that $z \equiv r \pmod{k}$. Then we denote

$$S(n,k,r \mid z) = \sum_{\substack{i=0 \\ i \equiv r \bmod k \\ i \le z}}^{n} \binom{n}{i};$$

$$w_i = \begin{cases} S(n,k,i \mid i) - S(n,k,i-1 \mid i-1), & \text{if } i \le m; \\ S(n,k,n-i-k+1 \mid n-i-k+1) - S(n,k,n-i-k \mid n-i-k), & \text{if } m+1 \le i. \end{cases}$$

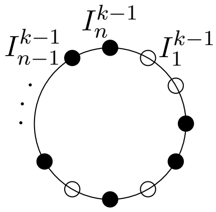

Figure 5: The cycle of the vertices in the $(k-1)^{st}$ level of $\mathcal{I}(n)^\sigma$.

Lemma 2.5. *Let $t \le n$ be an integer. We have the following equalities.*
(I) *If $t < 0$, $S(n,k,t \mid t) = 0$, and if $0 \le t \le k-1$, $S(n,k,t \mid t) = \binom{n}{t}$.*
(II) *If $0 \le t \le n$, $S(n,k,t \mid t) = S(n,k,t-k \mid t-k) + \binom{n}{t}$.*
(III) *If $0 \le t \le n$, $S(n,k,t \mid t) + S(n,k,n-k-t \mid n-k-t) = S(n,k,t)$.*
(IV) *If $0 \le t \le n$, $S(n,k,t-k \mid t-k) + S(n,k,n-k-t \mid n-k-t) = S(n,k,t) - \binom{n}{t}$.*

Proof: (I) follows from the definition of $S(n,k,t \mid t)$. (II) follows from (I) if $0 \le t \le k-1$, and the definition of $S(n,k,t \mid t)$ if $k \le t \le n$. (IV) follows from (II) and (III). Now, we distinguish two cases to prove (III).

If $t > n-k$, then we have $t+k > n$ and $n-k-t < 0$. Hence, $S(n,k,t|t) = S(n,k,t)$ and $S(n,k,n-k-t|n-k-t) = 0$ by (I), and this completes the proof of this case.

If $0 \le t \le n-k$, let $0 \le r \le k-1$ be the remainder of $n-k-t$ divided by k. We have

$$\begin{aligned}
&S(n,k,t|t) + S(n,k,n-k-t|n-k-t)\\
=&S(n,k,t|t) + \binom{n}{r} + \binom{n}{r+k} + \cdots + \binom{n}{n-k-t}\\
=&S(n,k,t|t) + \binom{n}{n-r} + \binom{n}{n-r-k} + \cdots + \binom{n}{k+t}\\
=&S(n,k,t|t) + \binom{n}{t+k} + \cdots + \binom{n}{n-r-k} + \binom{n}{n-r} = S(n,k,t),
\end{aligned}$$

since $r < k$. □

Lemma 2.6. *Let s be an integer.*
(I) *$w_s = 0$ if $s < 0$ or $s > n-k+1$, and $w_s > 0$ if $0 \le s \le n-k+1$.*
(II) *If $0 \le s \le m$ or $m+k \le s \le n$, then*

$$\sum_{i=s-k+1}^{s} w_i = \binom{n}{s}.$$

(III) *If $m < s < m+k$,*

$$\sum_{i=s-k+1}^{s} w_i = S(n,k,m) - S(n,k,s) + \binom{n}{s}.$$

(IV) *$\sum_{i=0}^{n-k+1} w_i = S(n,k,m)$.*

Proof: (I) It follows from the definition of w_s and (I) of Lemma 2.5.
(II) By (II) of Lemma 2.5, if $0 \le s \le m$, it follows from (I) that

$$\sum_{i=s-k+1}^{s} w_i = \sum_{i=s-k+1}^{s} \big(S(n,k,i|i) - S(n,k,i-1|i-1)\big) = S(n,k,s|s) - S(n,k,s-k|s-k) = \binom{n}{s},$$

and if $m+k \le s \le n$, we have $s-k+1 \ge m+1$, and so

$$\begin{aligned}\sum_{i=s-k+1}^{s} w_i &= \sum_{i=s-k+1}^{s} \big(S(n,k,n-k-i+1|n-k-i+1) - S(n,k,n-k-i|n-k-i)\big)\\ &= S(n,k,n-s|n-s) - S(n,k,n-s-k|n-s-k) = \binom{n}{n-s} = \binom{n}{s}.\end{aligned}$$

(III) If $m < s < m+k$, then $s-k+1 \le m$ and $s \ge m+1$. So by (III) and (IV) of Lemma 2.5, we have

$$\begin{aligned}\sum_{i=s-k+1}^{s} w_i = &\sum_{i=s-k+1}^{m} \big(S(n,k,i|i) - S(n,k,i-1|i-1)\big)\\ &+ \sum_{j=m+1}^{s} \big(S(n,k,n-k-j+1|n-k-j+1) - S(n,k,n-k-j|n-k-j)\big)\\ =&S(n,k,m|m) - S(n,k,s-k|s-k)\\ &+ S(n,k,n-k-m|n-k-m) - S(n,k,n-k-s|n-k-s)\\ =&\big(S(n,k,m|m) + S(n,k,n-k-m|n-k-m)\big)\\ &- \big(S(n,k,s-k|s-k) + S(n,k,n-k-s|n-k-s)\big)\\ =&S(n,k,m) - \left(S(n,k,s) - \binom{n}{s}\right) = S(n,k,m) - S(n,k,s) + \binom{n}{s}.\end{aligned}$$

(IV)

$$\sum_{i=0}^{n-k+1} w_i = \sum_{\substack{i=0\\ i\equiv m \bmod k}}^{n} \left(\sum_{j=i-k+1}^{i} w_j\right) = S(n,k,m),$$

by (I) and (II). □

3 Proof of Theorem 2.1

First, we consider a linear programming problem: maximize $\sum_{i=0}^{n}\binom{n}{i}x_i$ subject to the constraints $x_i + x_{i+1} + \cdots + x_{i+k-1} \le (k-1)n$ for $i = 0, 1 \ldots, n-k+1$ (Lemma 2.2), $0 \le x_0, x_n \le 1$, and $0 \le x_i \le n$ for all $i = 1, \ldots, n-1$.

We assign the weight w_i (see the definition and properties of w_i in Section 2) to the constraint $x_i + x_{i+1} + \cdots + x_{i+k-1} \le (k-1)n$ for every $0 \le i \le n-k+1$. By (I) of Lemma 2.6, $w_i > 0$ for $0 \le i \le n-k+1$, so we have

$$\sum_{i=0}^{n-k+1} w_i(x_i + x_{i+1} + \cdots + x_{i+k-1}) \le (k-1)n \cdot \sum_{i=0}^{n-k+1} w_i. \tag{1}$$

Then by (I), (II) and (III) of Lemma 2.6, the LHS of (1) is equal to

$$\sum_{i=0}^{n}\left(\left(\sum_{j=i-k+1}^{i} w_j\right) x_i\right) = \sum_{i=0}^{n}\binom{n}{i}x_i + \sum_{j=m+1}^{m+k-1}\left(S(n,k,m) - S(n,k,j)\right) x_j.$$

On the other hand, by (IV) of Lemma 2.6, the RHS of (1) is equal to $(k-1)n \cdot S(n,k,m)$. Hence, by (1), we have

$$\sum_{i=0}^{n}\binom{n}{i}x_i + \sum_{j=m+1}^{m+k-1}\left(S(n,k,m) - S(n,k,j)\right) x_j \leq (k-1)n \cdot S(n,k,m). \tag{2}$$

Note that $S(n,k,m) = \min_{r:\ 0\leq r\leq k-1} S(n,k,r)$ by Remark 1.9, and $0 \leq x_j \leq n$ for $m+1 \leq j \leq m+k-1$. So we have

$$\sum_{j=m+1}^{m+k-1}\left(S(n,k,j) - S(n,k,m)\right) x_j \leq \sum_{j=m+1}^{m+k-1}\left(S(n,k,j) - S(n,k,m)\right) n. \tag{3}$$

Therefore, combining (2) and (3), we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}x_i &\leq (k-1)n \cdot S(n,k,m) + \sum_{j=m+1}^{m+k-1}\left(S(n,k,j) - S(n,k,m)\right) n \\ &= \sum_{j=m+1}^{m+k-1}\left(S(n,k,j) - S(n,k,m) + S(n,k,m)\right) n \\ &= n \cdot \sum_{j=m+1}^{m+k-1} S(n,k,j) = n \cdot \left(2^n - S(n,k,m)\right). \end{aligned} \tag{4}$$

Now, we use (4) to prove Theorem 2.1. If $|\mathcal{F} \cap \{\emptyset, [n]\}| = 0$, then the statement of the theorem follows from (4). Then if $|\mathcal{F} \cap \{\emptyset, [n]\}| = 1$, we may suppose that $\emptyset \in \mathcal{F}$ and $[n] \notin \mathcal{F}$, and so $x_0 = 1$ and $x_n = 0$. Thus, it follows that

$$n \cdot x_0 + \sum_{i=1}^{n-1}\binom{n}{i}x_i + n \cdot x_n = \sum_{i=0}^{n}\binom{n}{i}x_i + n - 1 \leq n \cdot \big(2^n - S(n,k,m)\big) + n - 1,$$

as required.

If $\emptyset \in \mathcal{F}$ and $[n] \in \mathcal{F}$, we have $x_0 = 1$ and $x_n = 1$. By Lemma 2.2, $x_1 + x_2 + \cdots + x_k \leq (k-1)n$ and $x_{n-k} + x_{n-k+1} + \cdots + x_{n-1} \leq (k-1)n$. According to these two constraints, we distinguish two subcases.

If $x_1 + x_2 + \cdots + x_k \leq (k-1)n - 1$ or $x_{n-k} + x_{n-k+1} + \cdots + x_{n-1} \leq (k-1)n - 1$, we may assume $x_1 + x_2 + \cdots + x_k \leq (k-1)n - 1$. Then in this case, (1) should be

$$\sum_{i=0}^{n-k+1} w_i(x_i + x_{i+1} + \cdots + x_{i+k-1}) \leq \left((k-1)n \cdot \sum_{i=0}^{n-k+1} w_i\right) - w_1.$$

By the assumption of Theorem 2.1, $n > k$, so $m \geq 1$ and $w_1 = n - 1$. Then (4) should be

$\sum_{i=0}^{n}\binom{n}{i}x_i \leq n\cdot(2^n - S(n,k,m)) - (n-1)$, and so

$$\begin{aligned}n(x_0+x_n)+\sum_{i=1}^{n-1}\binom{n}{i}x_i &= (n-1)(x_0+x_n)+\sum_{i=0}^{n}\binom{n}{i}x_i\\ &\leq 2(n-1)+n\cdot\big(2^n-S(n,k,m)\big)-(n-1)\\ &= n\cdot\big(2^n-S(n,k,m)\big)+(n-1).\end{aligned}$$

If $x_1+x_2+\cdots+x_k=(k-1)n$ and $x_{n-k}+x_{n-k+1}+\cdots+x_{n-1}=(k-1)n$, then by Lemma 2.3, $x_0+x_1+\cdots+x_{k-1}\leq(k-1)n-\lfloor\frac{n}{2}\rfloor$ and $x_{n-k+1}+x_{n-k+2}+\cdots+x_n\leq(k-1)n-\lfloor\frac{n}{2}\rfloor$. Similarly, one can modify (1) and (4) to show that

$$\begin{aligned}n\cdot(x_0+x_n)+\sum_{i=1}^{n-1}\binom{n}{i}x_i &\leq 2(n-1)+n\cdot\big(2^n-S(n,k,m)\big)-2\lfloor\frac{n}{2}\rfloor\\ &\leq (n-1)+n\cdot\big(2^n-S(n,k,m)\big)\end{aligned}$$

by $w_0=w_{n-k+1}=1$. This completes the proofs of Theorems 2.1 and 1.8. □

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