The Turán number of the square of a path

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Abstract The Turán number of a graph H, $\mathrm{ex}(n, H)$, is the maximum number of edges in a graph on n vertices which does not have H as a subgraph. Let P_k be the path with k vertices, the square P_k^2 of P_k is obtained by joining the pairs of vertices with distance one or two in P_k. The powerful theorem of Erdős, Stone and Simonovits determines the asymptotic behavior of $\mathrm{ex}(n, P_k^2)$. In the present paper, we determine the exact value of $\mathrm{ex}(n, P_5^2)$ and $\mathrm{ex}(n, P_6^2)$ and pose a conjecture for the exact value of $\mathrm{ex}(n, P_k^2)$.

Keywords Turán number, Extremal graphs, Square of a path.

1 Introduction

In this paper, all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph, the vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. We denote the degree of a vertex v by $d(v)$, the minimum degree in graph G by $\delta(G)$, the neighborhood of v by $N(v)$ and the chromatic number of graph G by $\chi(G)$.

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The following graphs will be studied in the present paper. Let P_k be the path with k vertices, the square P_k^2 of P_k is obtained by joining the pairs of vertices with distance one or two in P_k, see Figure 1. The Turán number of a graph H, $\mathrm{ex}(n, H)$, is the maximum number of edges in a graph on n vertices which does not have H as a subgraph. Our goal in this paper is to study $\mathrm{ex}(n, P_k^2)$ and the extremal graphs for P_k^2. The Erdős-Stone-Simonovits Theorem [3, 4] asymptotically determines $\mathrm{ex}(n, H)$ for all non-bipartite graphs H:

$$\mathrm{ex}(n, H) = (1 - \tfrac{1}{\chi(H)-1})\binom{n}{2} + o(n^2).$$

Since $\chi(P_k^2) = 3$, $k \geq 3$, we have $\mathrm{ex}(n, P_k^2) = \frac{n^2}{4} + o(n^2)$. Yet, it still remains interesting to determine the exact value of $\mathrm{ex}(n, P_k^2)$.

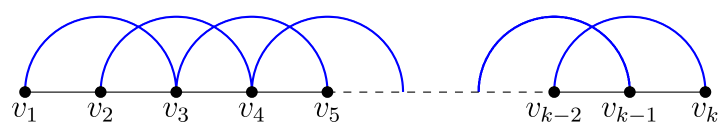

Figure 1: Graph P_k^2

The very first result of extremal graph theory gave the value of $\mathrm{ex}(n, P_3^2)$.

Theorem 1 (**Mantel** [8])**.** *The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$, that is $\mathrm{ex}(n, P_3^2) = \lfloor \frac{n^2}{4} \rfloor$. Furthermore, the only triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

The case $k = 4$ was solved by Dirac in a more general context.

Theorem 2 (**Dirac** [1])**.** *The maximum number of edges in an n-vertex P_4^2-free graph is $\lfloor \frac{n^2}{4} \rfloor$, that is $\mathrm{ex}(n, P_4^2) = \lfloor \frac{n^2}{4} \rfloor$, $(n \geq 4)$. Furthermore, when $n \geq 5$, the only extremal graph is the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

For $k = 5$, our results are given in the next two theorems, where we separate the result for the Turán number and the extremal graphs for P_5^2.

Theorem 3. *The maximum number of edges in an n-vertex P_5^2-free graph is $\lfloor \frac{n^2+n}{4} \rfloor$, that is $\mathrm{ex}(n, P_5^2) = \lfloor \frac{n^2+n}{4} \rfloor$, $(n \geq 5)$.*

Definition 1. *Let E_n^i denote a graph obtained from a complete bipartite graph $K_{i,n-i}$ plus a maximum matching in the class which has i vertices, see Figure 2.*

Theorem 4. *Let n be a natural number, when $n = 5$, the extremal graphs for P_5^2 are E_5^2, E_5^3 and G_0, where G_0 is obtained from a K_4 plus a pendent edge. When $n \geq 6$, if $n \equiv 1, 2 \pmod 4$, the extremal graphs for P_5^2 are $E_n^{\lceil \frac{n}{2} \rceil}$ and $E_n^{\lfloor \frac{n}{2} \rfloor}$, otherwise, the extremal graph for P_5^2 is $E_n^{\lceil \frac{n}{2} \rceil}$.*

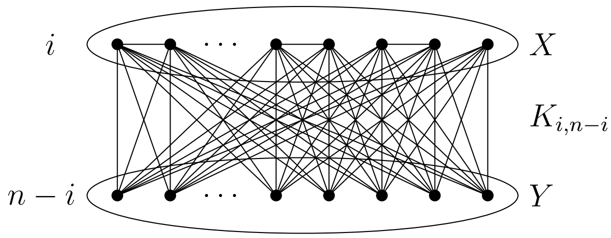

Figure 2: Graph E_n^i

Definition 2. *Let T denote the flattened tetrahedron, see T in Figure 3.*

Although the determination of $\mathrm{ex}(n, T)$ is not within the main lines of our paper, we need the exact value of $\mathrm{ex}(n, T)$ in order to determine $\mathrm{ex}(n, P_6^2)$.

Theorem 5. *The maximum number of edges in an n-vertex T-free graph ($n \neq 5$) is,*

$$\mathrm{ex}(n, T) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n}{2} \right\rfloor, & n \not\equiv 2 \pmod 4, \\ \dfrac{n^2}{4} + \dfrac{n}{2} - 1, & n \equiv 2 \pmod 4. \end{cases}$$

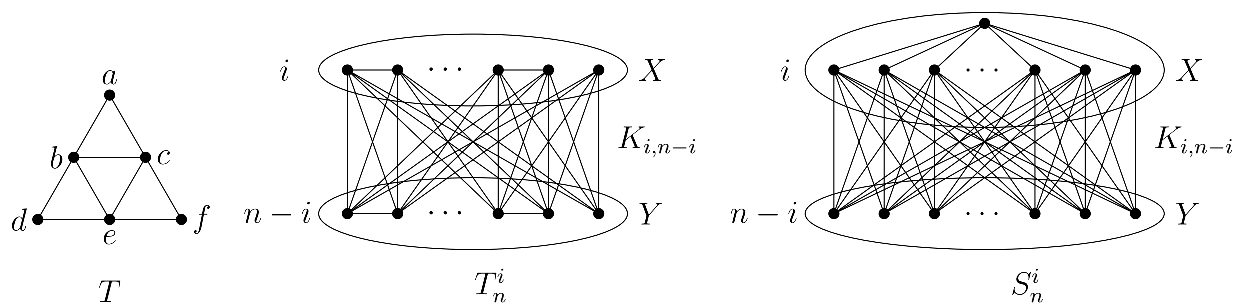

Figure 3: Graphs T, T_n^i and S_n^i

Definition 3. *Let T_n^i denote a graph obtained from a complete bipartite graph $K_{i,n-i}$ plus a maximum matching in the class X which has i vertices and a maximum matching in the class Y which has $n - i$ vertices, see T_n^i in Figure 3. Let S_n^i denote a graph obtained from $K_{i,n-i}$ plus an i-vertex star in the class X, see S_n^i in Figure 3.*

Theorem 6. *Let n $(n \neq 5, 6)$ be a natural number,*

when $n \equiv 0 \pmod 4$, the extremal graph for T is $T_n^{\frac{n}{2}}$,

when $n \equiv 1 \pmod 4$, the extremal graphs for T are $T_n^{\lceil \frac{n}{2} \rceil}$ and $S_n^{\lceil \frac{n}{2} \rceil}$,

when $n \equiv 2 \pmod 4$, the extremal graphs for T are $T_n^{\frac{n}{2}}$, $T_n^{\frac{n}{2}+1}$ and $S_n^{\frac{n}{2}}$,

when $n \equiv 3 \pmod 4$, the extremal graphs for T are $T_n^{\lceil \frac{n}{2} \rceil}$ and $S_n^{\lceil \frac{n}{2} \rceil}$.

These two results are known for sufficiently large $n's$ [7], here we are able to determine the value for small $n's$.

Using Theorems 5 and 6, we are able to prove the next two results for P_6^2.

Theorem 7. *The maximum number of edges in an n-vertex P_6^2-free graph $(n \neq 5)$ is:*

$$\mathrm{ex}(n, P_6^2) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n-1}{2} \right\rfloor, & n \equiv 1, 2, 3 \pmod 6, \\ \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil, & \text{otherwise}. \end{cases}$$

Definition 4. *Suppose $3 \nmid n$, and $1 \leq j \leq i$. Let $F_n^{i,j}$ be the graph obtained by adding vertex disjoint triangles (possibly 0) and one star with j vertices in the class X of size i of $K_{i,n-i}$, see Figure 4 (Of course $3 \mid (i-j)$ is supposed). On the other hand if $3 \mid i$ then add $\frac{i}{3}$ vertex disjoint triangles in the class X of size i. The so obtained graph is denoted by H_n^i, see Figure 4.*

Theorem 8. *Let $n \geq 6$ be a natural number. The extremal graphs for P_6^2 are the following ones.*

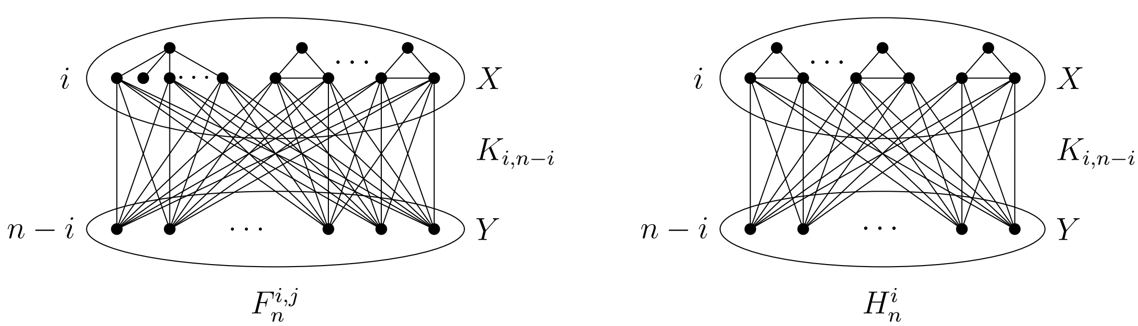

Figure 4: Graphs $F_n^{i,j}$ and H_n^i

When $n \equiv 1 \pmod 6$ *then* $F_n^{\lceil \frac{n}{2} \rceil, j}$ *and* $H_n^{\lfloor \frac{n}{2} \rfloor}$,

when $n \equiv 2 \pmod 6$ *then* $F_n^{\frac{n}{2}, j}$ *and* $F_n^{\frac{n}{2}+1, j}$,

when $n \equiv 3 \pmod 6$ *then* $F_n^{\lceil \frac{n}{2} \rceil, j}$ *and* $H_n^{\lceil \frac{n}{2} \rceil + 1}$,

when $n \equiv 0, 4, 5 \pmod 6$ *then* $H_n^{\frac{n}{2}}$, $H_n^{\frac{n}{2}+1}$ *and* $H_n^{\lceil \frac{n}{2} \rceil}$, *respectively.* ($j$ *can have all the values satisfying the conditions* $j \leq i$ *and* $3 \mid (i - j)$).

On the basis of these results let us pose a conjecture for the general case.

Conjecture 1.

$$\mathrm{ex}(n, P_k^2) \leq \max \left\{ \frac{i \left(\left\lfloor \frac{2k}{3} \right\rfloor - 2 \right)}{2} + i(n - i) \right\}.$$

If $\left\lfloor \frac{2k}{3} \right\rfloor - 1$ *divides* i *then the following graph gives equality here. Take a complete bipartite graph with parts of size* i *and* $n - i$, *add vertex disjoint complete graphs on* $\left\lfloor \frac{2k}{3} \right\rfloor - 1$ *vertices to the part with* i *elements.*

Observe that Theorems 1, 2, 3 and 7 justify our conjecture for the cases when $k = 3, 4, 5, 6$. We will give some hints in Section 3 how we arrived to this conjecture. A weaker form of this conjecture is the following one.

Conjecture 2.

$$\mathrm{ex}(n, P_k^2) = \frac{n^2}{4} + \frac{\left(\left\lfloor \frac{k}{3} \right\rfloor - 1 \right) n}{2} + O_k(1)$$

where $O_k(1)$ *depends only on* k.

2 Proofs of the main results

2.1 The Turán number and the extremal graphs for P_5^2

Proof of Theorem 3. The fact that $\mathrm{ex}(n, P_5^2) \geq \left\lfloor \frac{n^2+n}{4} \right\rfloor$ follows from the construction $E_n^{\lceil \frac{n}{2} \rceil}$. We prove the inequality

$$\mathrm{ex}(n, P_5^2) \leq \left\lfloor \frac{n^2 + n}{4} \right\rfloor \ (n \geq 5) \tag{1}$$

by induction on n.

We check the base cases first. Since our induction step will go from $n-4$ to n, we have to find a base case in each residue class mod 4.

Let G be an n-vertex P_5^2-free graph. When $n \leq 3$, K_n is the graph with the most number of edges and does not contain P_5^2, $e(K_n) \leq \left\lfloor \frac{n^2+n}{4} \right\rfloor$. This settles the cases $n = 1, 2, 3$. However, when $n = 4$, $e(K_4) = 6 > \lfloor \frac{4^2+4}{4} \rfloor$, the statement is not true. Then we show that the statement is true for $n = 8$. If $P_4^2 \nsubseteq G$, $e(G) \leq \lfloor \frac{8^2}{4} \rfloor$. If $P_4^2 \subseteq G$ and $K_4 \nsubseteq G$, each vertex $v \in V(G - P_4^2)$ can be adjacent to at most 2 vertices of the copy of P_4^2, since $e(G - P_4^2) \leq 5$, we have $e(G) \leq 5 + 8 + 5 \leq 18 = \lfloor \frac{8^2+8}{4} \rfloor$. If $K_4 \subseteq G$, then each vertex $v \in V(G - K_4)$ can be adjacent to at most one vertex of the K_4, since $e(G - P_4^2) \leq 6$, we have $e(G) \leq 16$.

Suppose (1) holds for all $k \leq n-1$, the proof is divided into 3 parts,

Case 1. If $P_4^2 \nsubseteq G$, then by Theorem 2, $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$.

Case 2. If $P_4^2 \subseteq G$ and $K_4 \nsubseteq G$, then each vertex $v \in V(G - P_4^2)$ can be adjacent to at most 2 vertices of the copy of P_4^2, otherwise, $P_5^2 \subseteq G$. Since $G - P_4^2$ is an $(n-4)$-vertex P_5^2-free graph, we have

$$e(G) \leq 5 + 2(n-4) + e(G - P_4^2) \leq 2n - 3 + \mathrm{ex}(n-4, P_5^2).$$

By the induction hypothesis, $\mathrm{ex}(n-4, P_5^2) \leq \left\lfloor \frac{(n-4)^2+n-4}{4} \right\rfloor$ then

$$e(G) \leq 2n - 3 + \mathrm{ex}(n-4, P_5^2) \leq 2n - 3 + \left\lfloor \frac{(n-4)^2 + n - 4}{4} \right\rfloor = \left\lfloor \frac{n^2+n}{4} \right\rfloor \quad (n \geq 5). \tag{2}$$

Case 3. If $K_4 \subseteq G$, then each vertex $v \in V(G - K_4)$ can be adjacent to at most one vertex of the K_4, otherwise, $P_5^2 \subseteq G$. Since $G - K_4$ is an $(n-4)$-vertex P_5^2-free graph, we have

$$e(G) \leq 6 + (n-4) + e(G - K_4) \leq n + 2 + \mathrm{ex}(n-4, P_5^2).$$

By the induction hypothesis, $\mathrm{ex}(n-4, P_5^2) \leq \left\lfloor \frac{(n-4)^2+n-4}{4} \right\rfloor$, thus

$$e(G) \leq n + 2 + \left\lfloor \frac{(n-4)^2 + n - 4}{4} \right\rfloor = 5 + \left\lfloor \frac{n^2 - 3n}{4} \right\rfloor \leq \left\lfloor \frac{n^2+n}{4} \right\rfloor \quad (n \geq 5). \tag{3}$$

□

Proof of Theorem 4. We determine the extremal graphs for P_5^2 by induction on n. Let G be an n-vertex P_5^2-free graph satisfying (1) with equality. It is easy to check, when $n = 5$, the extremal graphs for P_5^2 are G_0, E_5^2 and E_5^3. When $n = 6, 7, 8$, the extremal graphs for P_5^2 are E_6^3 and E_6^4, E_7^4, E_8^4, respectively.

Suppose Theorem 4 is true for $k \leq n-1$, when $n \geq 9$, the proof is divided into 3 parts.

Case 1. If $P_4^2 \nsubseteq G$, the equality in (1) cannot hold, then we cannot find any extremal graph for P_5^2 in this case.

Case 2. If $P_4^2 \subseteq G$ and $K_4 \nsubseteq G$, the equality holds in inequality (2) if and only if each vertex $v \in V(G-P_4^2)$ is adjacent to 2 vertices of the P_4^2 and $G-P_4^2$ is an extremal graph on $n-4$ vertices for P_5^2. Let $a,\ b,\ c$ and d be four vertices of a copy of P_4^2, $d_{P_4^2}(b) = d_{P_4^2}(c) = 3$. By the induction hypothesis, $G - P_4^2$ is obtained from a complete bipartite graph $K_{i,n-4-i}$ plus a maximum matching in $X^{'}$, where $X^{'}$ is the class of $G - P_4^2$ with size i. It is easy to check that every vertex $v \in V(G - P_4^2)$ can be adjacent to either a and d or b and c.

Since $|V(G-P_4^2)| \geq 5$, we have $|V(X')| \geq 2$. The endpoints of an edge in $G-P_4^2$ cannot be both adjacent to b and c, otherwise, they form a K_4. Also, the endpoints of an edge in $G-P_4^2$ which have one end vertex as a matched vertex in X' and one end vertex in $Y^{'}$ can be both adjacent to none of $\{a, b, c\}$ and d, otherwise, these would create a P_5^2. If there exists a matched vertex $v \in X'$ which is adjacent to b and c, then all vertices $w \in N(v)$ should be adjacent to a and d, these form a P_5^2. Hence, it is only possible that all matched vertices in $X^{'}$ are adjacent to both a and d, all vertices in $Y^{'}$ are adjacent to b and c. When there exists an unmatched vertex $v_0 \in X^{'}$, since $N(v_0) = Y^{'}$, if v_0 is adjacent to b and c, we have $P_5^2 \subseteq G$. Thus G is obtained from a complete bipartite graph $K_{i+2,n-i-2}$ plus a maximum matching in X, where $X = X^{'} \cup \{b, c\}$ and $Y = Y^{'} \cup a \cup d$. Therefore, if $G - P_4^2$ is $E_{n-4}^{\lceil \frac{n-4}{2} \rceil}$ then G is $E_n^{\lceil \frac{n}{2} \rceil}$, if $E_{n-4}^{\lfloor \frac{n-4}{2} \rfloor}$ then G is $E_n^{\lfloor \frac{n}{2} \rfloor}$.

Case 3. If $K_4 \subseteq G$, the inequality in (3) can be equality only when $n = 5$ and the vertex $v \in\ V(G - K_4)$ is adjacent to one vertex of the K_4, that is G_0. $\square$

2.2 The Turán number and the extremal graphs for T

To prove Theorem 5, we need the following lemmas.

Lemma 9. *Let G be an n-vertex T-free nonempty graph such that for each edge $\{x, y\} \in E(G)$, $d(x) + d(y) \geq n + 2$ holds, then we have $K_4 \subseteq G$.*

Proof. From the condition we know that each edge belongs to at least two triangles. Let abc and bcd be two triangles, if a is adjacent to d then a, b, c and d induce a K_4, if not, since edge $\{b, d\}$ is contained in at least two triangles, there exists at least one vertex e such that bde is a triangle. Similarly, edge $\{c, d\}$ is also contained in at least two triangles, then, either there exists a vertex f which is adjacent to c and d, this implies that vertices a, b, c, d, e and f induce a T, or c is adjacent to e, this implies that vertices b, c, d and e induce a K_4. □

Lemma 10. *Let G be an n-vertex $(n \geq 7)$ T-free graph and $K_4 \subseteq G$, then $e(G) \leq 2n - 2 + ex(n-4, T)$. For $n \geq 8$, the equality might hold only if each vertex $v \in V(G - K_4)$ is adjacent to 2 vertices of the K_4.*

Proof. If there exists vertex $v \in V(G - K_4)$, such that v is adjacent to at least 3 vertices of the K_4, it is simple to check that every other vertex $u \in V(G - K_4)$ can be adjacent to at most one vertex of the K_4, otherwise $T \subseteq G$, then $e(G) \leq 6 + 4 + (n-5) + e(G - K_4) \leq n + 5 + ex(n-4, T)$. If not, each vertex in $G - K_4$ is adjacent to at most 2 vertices of the K_4, then $e(G) \leq 6 + 2(n-4) + e(G - K_4) \leq 2n - 2 + ex(n-4, T)$. When $n \geq 8$, $e(G) \leq 2n - 2 + ex(n-4, T)$, the equality holds only if each vertex $v \in V(G - K_4)$ is adjacent to 2 vertices of the K_4. □

Proof of Theorem 5. Let

$$f_T(n) = \begin{cases} \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor, \ n \not\equiv 2 \pmod 4, \\ \frac{n^2}{4} + \frac{n}{2} - 1, \ n \equiv 2 \pmod 4. \end{cases}$$

The fact that $\mathrm{ex}(n, T) \geq f_T(n)$ follows from the construction $T_n^{\lceil \frac{n}{2} \rceil}$. Next, we show the inequality

$$\mathrm{ex}(n, T) \leq f_T(n) \tag{4}$$

by induction on n.

Let G be an n-vertex T-free graph. first, we show the induction steps, in the end we will show the base cases which are needed to complete the induction.

Suppose (4) holds for all $l \leq n-1$, in the following cases, we will assume that $k \geq 2$, the proof is divided into 4 cases.

Case 1. When $n = 4k$, we divide the proof of $\mathrm{ex}(4k, T) \leq f_T(4k) = 4k^2 + 2k$ into 2 subcases. Let G be a $4k$-vertex T-free graph.
(i) If $\delta(G) \leq 2k+1$, after removing a vertex of minimum degree and by the induction hypothesis $\mathrm{ex}(4k-1, T) = 4k^2 - 1$, we get

$$e(G) \leq \mathrm{ex}(4k-1, T) + 2k + 1 \leq 4k^2 - 1 + 2k + 1 = f_T(4k). \tag{5}$$

(ii) If $\delta(G) \geq 2k+2$, then for each edge $\{u, v\} \in E(G)$, $d(u) + d(v) \geq 4k+4$. By Lemmas 9 and 10 and the induction hypothesis $\mathrm{ex}(4k-4, T) = 4(k-1)^2 + 2(k-1)$, we get

$$e(G) \leq 2n - 2 + \mathrm{ex}(4k-4, T) = 8k - 2 + 4(k-1)^2 + 2(k-1) = f_T(4k). \tag{6}$$

Therefore, $\mathrm{ex}(4k, T) \leq f_T(4k)$.

Case 2. When $n = 4k+1$, we divide the proof of $\mathrm{ex}(4k+1, T) \leq f_T(4k+1) = 4k^2 + 4k$ into 3 subcases. Let G be a $(4k+1)$-vertex T-free graph.
(i) If $\delta(G) \leq 2k$, after removing a vertex of minimum degree and by the induction hypothesis $\mathrm{ex}(4k, T) = 4k^2 + 2k$, we have

$$e(G) \leq \mathrm{ex}(4k, T) + 2k \leq f_T(4k+1). \tag{7}$$

Now, we assume that in the following two cases $\delta(G) \geq 2k+1$. Then for any pair of vertices $\{u, v\} \in E(G)$, $d(u) + d(v) \geq 4k+2$ holds.
(ii) Suppose that there exists an edge $\{u, v\} \in E(G)$, such that $d(u) + d(v) = 4k+2$. This implies that u and v have at least one common neighbor. Deleting $\{u, v\}$ we can use the induction hypothesis $\mathrm{ex}(4k-1, T) = 4k^2 - 1$. Then

$$e(G) \leq 4k + 1 + \mathrm{ex}(4k-1, T) = f_T(4k+1). \tag{8}$$

(iii) For each edge $\{u,v\} \in E(G)$, $d(u)+d(v) \geq 4k+3$ holds. By Lemmas 9 and 10 and the induction hypothesis $\mathrm{ex}(4k-3,T) = 4(k-1)^2 + 4(k-1)$ we get

$$e(G) \leq 2n-2+\mathrm{ex}(4k-3,T) = 8k+4(k-1)^2+4(k-1) = f_T(4k+1). \tag{9}$$

Therefore, $\mathrm{ex}(4k+1,T) \leq f_T(4k+1)$.

Case 3. When $n = 4k+2$, we divide the proof of $\mathrm{ex}(4k+2,T) \leq f_T(4k+2) = 4k^2+6k+1$ into 2 subcases. Let G be a $(4k+2)$-vertex T-free graph.

(i) If $\delta(G) \leq 2k+1$, after removing a vertex of minimum degree and by the induction hypothesis $\mathrm{ex}(4k+1,T) = 4k^2+4k$, we get

$$e(G) \leq \mathrm{ex}(4k+1,T)+2k+1 \leq 4k^2+6k+1 = f_T(4k+2). \tag{10}$$

(ii) If $\delta(G) \geq 2k+2$, then for each edge $\{u,v\} \in E(G)$, $d(u)+d(v) \geq 4k+4$. By Lemmas 9 and 10 and the induction hypothesis $\mathrm{ex}(4k-2,T) = 4(k-1)^2+6(k-1)+1$, we get

$$e(G) \leq 2n-2+\mathrm{ex}(4k-2,T) = 8k+2+4(k-1)^2+6(k-1)+1 = f_T(4k+2). \tag{11}$$

Therefore, $\mathrm{ex}(4k+2,T) \leq f_T(4k+2)$.

Case 4. When $n = 4k+3$, we divide the proof of $\mathrm{ex}(4k+3,T) \leq f_T(4k+3) = 4k^2+8k+3$ into 2 subcases. Let G be a $(4k+3)$-vertex T-free graph.

(i) If $\delta(G) \leq 2k+2$, after removing a vertex of minimum degree and by the induction hypothesis $\mathrm{ex}(4k+2,T) = 4k^2+6k+1$, we get

$$e(G) \leq \mathrm{ex}(4k+2,T)+2k+2 \leq 4k^2+8k+3 = f_T(4k+3). \tag{12}$$

(ii). If $\delta(G) \geq 2k+3$, then for each edge $\{u,v\} \in E(G)$, $d(u)+d(v) \geq 4k+6$. By Lemmas 9 and 10 and the induction hypothesis $\mathrm{ex}(4k-1,T) = 4(k-1)^2+8(k-1)+3$, we get

$$e(G) \leq 2n-2+\mathrm{ex}(4k-1,T) = 8k+4+4(k-1)^2+8(k-1)+3 = f_T(4k+3). \tag{13}$$

Therefore, $\mathrm{ex}(4k+3,T) \leq f_T(4k+3)$.

Now we show the base cases which are needed to complete the induction steps. Since our induction steps will go from $n-1$ to n, $n-2$ to n and $n-4$ to n, we will require to show the statement is true for cases when $n = 3,4,6$ and 9.

When $n \leq 4$, K_n is the graph with the most number of edges, and $e(K_n) = f_T(n)$.

When $n = 5$, $e(K_5) = 10 > f_T(5)$, the statement is not true, but we will see that the statement is true for $n = 9$.

When $n = 6$, let v be a vertex with minimum degree. If $\delta(G) = 1$, since $e(G - v) \leq 10$, we get $e(G) \leq 11$. If $\delta(G) = 2$ and $e(G) = 12$, then the only possibility is that $G - v$ is K_5, but then $T \subseteq G$, and we have $e(G) \leq 11$. Suppose now $\delta(G) \geq 3$. If $K_4 \subseteq G$ and there exists a vertex $u \in V(G - K_4)$ which is adjacent to at least 3 vertices of the copy of K_4, then $w \in V(G - K_4 - u)$ can be adjacent to at most one vertex of the K_4, otherwise, $T \subseteq G$. This contradicts $\delta(G) \geq 3$. Then in this case it is only possible that $\{u, w\} \in E(G)$ and both u and w are adjacent to 2 vertices of the K_4 which implies that $e(G) \leq 11$. If $K_4 \nsubseteq G$, then by Turán's Theorem, we have $e(G) \leq 12$ and the Turán graph $T(6, 3)$ is the unique K_4-free graph which has 12 edges, however, $T \subseteq T(6, 3)$, then $e(G) \leq 11 = f_T(6)$. Summarizing: $e(G) \leq 11 \leq f_T(6)$.

When $n = 9$, suppose first that there exists a pair of vertices $\{u, v\} \in E(G)$, such that $d(u)+d(v) \leq 10$. Deleting $\{u, v\}$ and using $\mathrm{ex}(7, T) = 15$, we get $e(G) \leq 9+15 = 24 = f_T(9)$. If for each pair of vertices $\{u, v\} \in E(G)$, $d(u) + d(v) \geq 11$ holds, by Lemma 9, we obtain $K_4 \subseteq G$. Let G' denote the graph $G - K_4$. If $e(G') \leq 8$, since the number of edges between K_4 and G' is at most 10, we have $e(G) \leq 6 + 10 + 8 = 24$. If $e(G') \geq 9$, then $K_4 \subseteq G'$ and the vertex $w \in G' - K_4$ is adjacent to at least 3 vertices of the copy of K_4 in G'. This implies that each vertex from $G - G'$ can be adjacent to at most 1 vertex of $G' - w$, then the number of edges between $G - G'$ and G' is at most 8, we can conclude that, $e(G) \leq 6 + 8 + 10 = 24$, $e(G) \leq 24 = f_T(9)$.

It is easy to see that the case $n = 7$ can be proved using $n = 3$ and $n = 6$ (Case 4). Similarly, the case $n = 8$ follows by $n = 7$ and $n = 4$ (Case 1). Hence the cases $n = 6, 7, 8, 9$ are settled forming a good bases for the induction. □

Now, we determine the extremal graphs for T.

Proof of Theorem 6. Similarly to the proof of Theorem 5, first, we show the induction steps, in the end we will show the base cases which are needed to complete the induction.

Suppose that the extremal graphs for T are as shown in Theorem 5 for $l \leq n-1$. In the following cases, we will assume that $k \geq 2$.

Let G be an n-vertex T-free graph with $e(G) = f_T(n)$. The proof is divided into 4 cases following the steps of the proof of Theorem 5.

Case 1. When $n = 4k$, $f_T(n) = 4k^2 + 2k$.

(i) If $\delta(G) \leq 2k+1$, the equality in (5) holds only when there exists a $v \in V(G)$, such that $d(v) = \delta(G) = 2k+1$ and $G - v$ is an extremal graph for T on $4k-1$ vertices. By the induction hypothesis, $G - v$ can be either T_{4k-1}^{2k} or S_{4k-1}^{2k}. Let X' and Y' be the classes in $G - v$ with size $2k$ and $2k-1$, respectively.

When $G - v$ is T_{4k-1}^{2k}, it can be easily checked that v cannot be adjacent to the two endpoints of an edge which have two matched vertices located in different classes, otherwise, $T \subseteq G$, see Figure 5. Let w be the unmatched vertex in Y'. Since $d(v) = 2k+1$, $N(v)$ must contain the unmatched vertex $w \in Y'$, then the only way to avoid $T \subseteq G$ is choosing $N(v) = w \cup X'$. Consequently, $G = T_{4k}^{2k}$ holds.

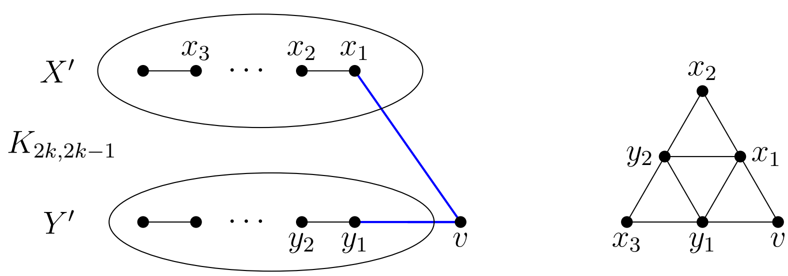

Figure 5:

When $G - v$ is S_{4k-1}^{2k}, let x_1 denote the center of the star in X'. If v is adjacent to the two endpoints of the edge $\{x_i, y_j\}$ ($x_i \in X', y_i \in Y', 2 \leq i \leq 2k,\ 1 \leq j \leq 2k-1$), then $T \subseteq G$ (see Figure 6). We obtained a contradiction. But $d(v) = 2k+1$ implies that this is always the case.

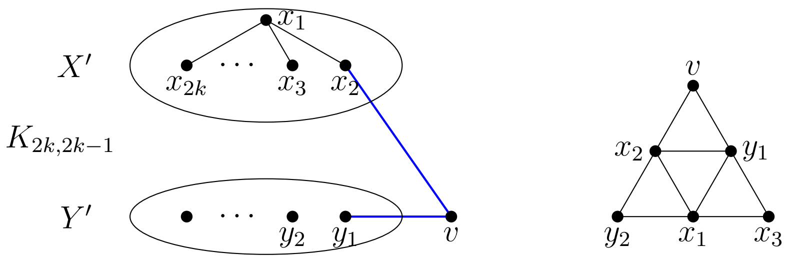

Figure 6:

(ii) If $\delta(G) \geq 2k+2$, this implies that $e(G) \geq 2k(2k+2) = 4k^2+4k$, which contradicts the fact that $\text{ex}(4k, T) = 4k^2 + 2k$.

That is, G can only be $T_n^{\frac{n}{2}}$.

Case 2. When $n = 4k+1$, $f_T(n) = 4k^2 + 4k$.

(i) If $\delta(G) \leq 2k$, the equality in (7) holds only if there exists $v \in V(G)$, such that $d(v) = \delta(G) = 2k$ and $G - v$ is an extremal graph for T on $4k$ vertices. By the induction hypothesis, $G - v$ is T_{4k}^{2k}. All neighbors of v should be located in the same class, otherwise, $T \subseteq G$, we get that G is T_{4k+1}^{2k+1}, that is $T_n^{\lceil \frac{n}{2} \rceil}$.

If $\delta(G) \geq 2k+1$, then for any pair of vertices $\{u, v\} \in V(G)$, $d(u)+d(v) \geq 4k+2$. Here we distinguishing two subcases.

(ii) Suppose that there exists an edge $\{u, v\} \in E(G)$ such that $d(u) + d(v) = 4k + 2$. The equality in (8) holds only if when $d(u) = d(v) = 2k + 1$ and $G - u - v$ is an extremal graph for T on $4k - 1$ vertices. By the induction hypothesis, $G - u - v$ can be either T_{4k-1}^{2k} or S_{4k-1}^{2k}. Let X' and Y' be the classes in $G - u - v$ with size $2k$ and $2k - 1$, respectively.

When $G - u - v$ is T_{4k-1}^{2k}, as in the previous case, neither u nor v can be adjacent to the two endpoints of an edge which have two matched vertices located in different classes, see Figure 5. If $N(u) - v \neq X'$, then u is adjacent to the unmatched vertex w in Y' and the other $2k - 1$ neighbors of u are all located in X', say, $N(u) - v - w = \{x_1, \ldots x_{2k-1}\}$ and $\{x_{2k-1}, x_{2k}\} \in E(X')$, otherwise, $T \subseteq G$. Since $|X'| \geq 4$, in this case, v cannot be adjacent to x_i $(1 \leq i \leq 2k-2)$, otherwise, $T \subseteq G$, see Figure 7. Now v should choose $2k$ neighbors among the rest $2k+1$ vertices in $V(G-u-v-\bigcup_{i=1}^{2k-2} x_i)$, which implies that v is adjacent to the two endpoints of an edge which have two matched vertices locate in different classes as endpoints, then $T \subseteq G$. Hence, $N(u)-v = X'$, similarly, $N(v)-u = X'$. Thus, G is $T_{4k+1}^{2k+1} = T_{4k+1}^{2k}$, that is $T_n^{\lceil \frac{n}{2} \rceil}$.

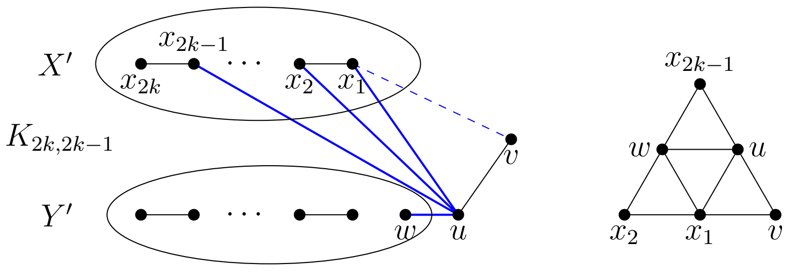

Figure 7:

Let us now consider the case when $G-u-v$ is S_{4k-1}^{2k}. Let x_1 denote the center of the star in X'. If u is adjacent to the two endpoints of the edge $\{x_i, y_j\}$ $(2 \le i \le 2k,\ 1 \le j \le 2k-1)$, then $T \subseteq G$. Thus, there are only two possibilities for $T \nsubseteq G$: $N(u)-v = X'$ or $N(u)-v = Y' \cup x_1$. The same holds for v and it is easy to check that if $N(u)-v = N(v)-u$, then $T \subseteq G$. From the above, the only possibility for $T \nsubseteq G$ is that when $N(u)-v = X'$ and $N(v)-u = Y' \cup x_1$ or in the another way around, which implies that G is S_{4k+1}^{2k+1}, that is $S_n^{\lceil \frac{n}{2} \rceil}$.

(iii) Suppose that for each edge $\{u,v\} \in E(G)$, $d(u)+d(v) \ge 4k+3$ holds. Let $d(v) = \delta(G)$, then either $d(v) = 2k+1$ or $d(v) \ge 2k+2$, but in both cases, each neighbor of v has degree at least $2k+2$. Then all $4k+1$ vertices have degree at least $2k+1$, but $2k+1$ of them, which are the neighbors of v, have degree at least one larger. This implies that $e(G) \ge \frac{(4k+1)(2k+1)+2k+1}{2} = 4k^2+4k+1$, which contradicts the fact that $\mathrm{ex}(4k+1, T) = 4k^2+4k$.

That is, G can be either $T_n^{\lceil \frac{n}{2} \rceil}$ or $S_n^{\lceil \frac{n}{2} \rceil}$.

Case 3. When $n = 4k+2$ we have $f_T(n) = 4k^2+6k+1$.

(i) If $\delta(G) \le 2k+1$, the equality holds in (10) only if there exists $v \in V(G)$, such that $d(v) = \delta(G) = 2k+1$ and $G-v$ is an extremal graph for T on $4k+1$ vertices. By the induction hypothesis, $G-v$ can be either T_{4k+1}^{2k+1} or S_{4k+1}^{2k+1}.

Suppose first that $G-v$ is T_{4k+1}^{2k+1}. Let X' any Y' be the classes in $G-v$ with size $2k+1$ and $2k$, w be the unmatched vertex in X'. The vertex v cannot be adjacent to the two endpoints of an edge which have two matched vertices located in different classes. Since $d(v)=2k+1$, there are two possibilities to avoid T: $N(v)=X'$ or $N(v)=Y'\cup w$, which implies that G is either T_{4k+2}^{2k+1} or T_{4k+2}^{2k+2}, that is $T_n^{\frac{n}{2}}$ or $T_n^{\frac{n}{2}+1}$.

When $G-v$ is S_{4k+1}^{2k+1}. Let X' be the class in $G-v$ which contains a star and Y' be the other class of the $G-v$. Also, let x_1 denote the center of the star in X'. Since, $d(v)=2k+1$ and v cannot be adjacent to the two endpoints of an edge which is not incident with x_1, we get either $N(v)=Y'\cup x_1$ or $N(v)=X'$. If $N(v)=X'$, G is S_{4k+2}^{2k+1}, that is $S_n^{\frac{n}{2}}$. If $N(v)=Y'\cup x_1$, G is S_{4k+2}^{2k+2}, that is $S_n^{\frac{n}{2}+1}$. It is easy to see that $S_n^{\frac{n}{2}+1}$ is isomorphic to $S_n^{\frac{n}{2}}$.

(ii) If $\delta(G)\geq 2k+2$, then $e(G)\geq (k+1)(4k+2)=4k^2+6k+2$, which contradicts the fact that $\mathrm{ex}(4k+2,T)=4k^2+6k+1$.

Therefore, G can be $T_n^{\frac{n}{2}}$, $T_n^{\frac{n}{2}+1}$ or $S_n^{\frac{n}{2}}$.

Case 4. When $n=4k+3$ we have $f_T(n)=4k^2+8k+3$.

(i) If $\delta(G)\leq 2k+2$, the equality holds in (12) only if there exists $v\in V(G)$, such that $d(v)=\delta(G)=2k+2$ and $G-v$ is an extremal graph for T on $4k+2$ vertices. By the induction hypothesis, $G-v$ can be T_{4k+2}^{2k+1}, T_{4k+2}^{2k+2} or S_{4k+2}^{2k+1}.
When $G-v$ is T_{4k+2}^{2k+1} or T_{4k+2}^{2k+2}, similarly to Case 1 (i), G can only be T_{4k+3}^{2k+2}, that is $T_n^{\lceil\frac{n}{2}\rceil}$.
When $G-v$ is S_{4k+2}^{2k+1}, similarly to Case 2 (ii), G can only be S_{4k+3}^{2k+2}, that is $S_n^{\lceil\frac{n}{2}\rceil}$.

(ii) If $\delta(G)\geq 2k+3$, then $e(G)\geq \frac{(2k+3)(4k+3)}{2}>4k^2+9k+4>4k^2+8k+3$, which contradicts the fact that $\mathrm{ex}(4k+3,T)=4k^2+8k+3$.

Therefore, in this case, G is either $T_n^{\lceil\frac{n}{2}\rceil}$ or $S_n^{\lceil\frac{n}{2}\rceil}$.

Now we check the base cases which are needed to complete the induction.

When $n=4$, $\mathrm{ex}(4,T)=6$, K_4 is the extremal graph which has the maximum number of edges on 4 vertices that does not contain T as a subgraph.

Although the Theorem does not hold for $n=6$, we determine the extremal graphs in this case because it will help us to determine them for some other n's.

When $n = 6$, $\text{ex}(6, T) = 11$. It follows from the proof of Theorem 5, when $\delta(G) = 1$, the only extremal graph for T is as shown in Figure 8(a). When $\delta(G) = 2$, the only extremal graph for T is as shown in Figure 8(b). Since $\delta(G) \geq 4$ implies $e(G) \geq 12$, this is not possible. The only remaining case is $\delta(G) = 3$. When $\delta(G) = 3$ and $K_4 \subseteq G$, by case analysis we obtain that the extremal graphs for T can be Figure 8(c) and Figure 8(d), which are T_6^3 and T_6^4. Suppose now that $\delta(G) = 3$ and $K_4 \nsubseteq G$. Let $d(v) = \delta(G) = 3$, then $e(G - v) = 8$, the only possibility is that $G - v$ is $T(5, 3)$. It is easy to check that G can only be S_6^3, see Figure 8(e).

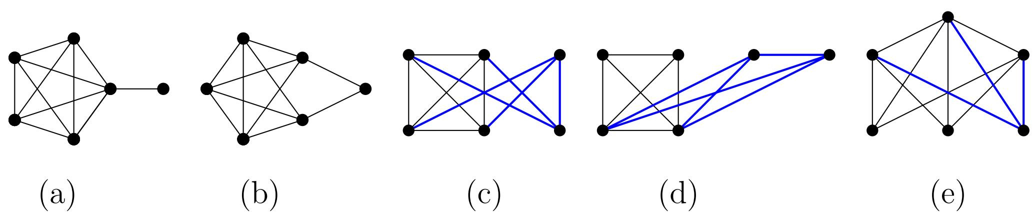

Figure 8: Extremal graphs for T when $n = 6$.

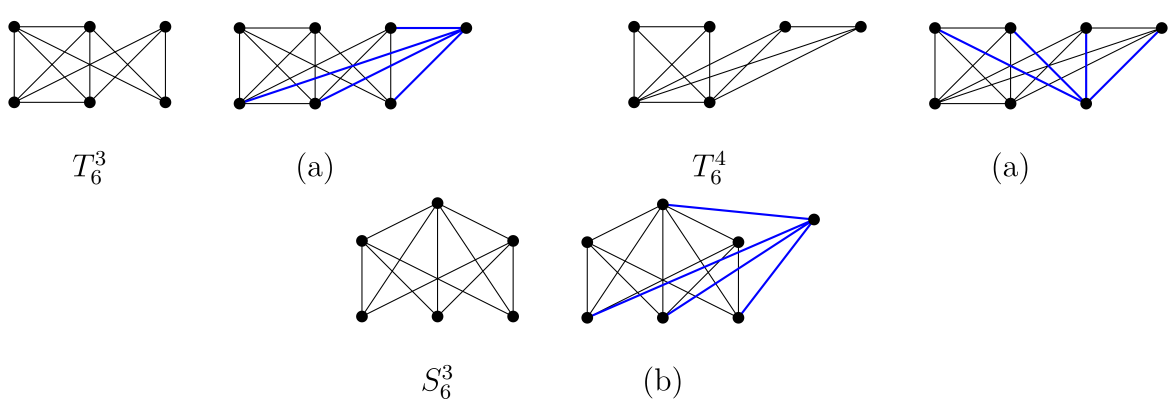

Figure 9: Extremal graphs for T when $n = 7$.

Suppose now that $n = 7$, $\text{ex}(7, T) = 15$. It is not possible that $\delta(G) \leq 3$, otherwise, $e(G) \leq 3 + ex(6, T) = 14$. Also, it is not possible that $\delta(G) \geq 5$, otherwise, $e(G) > 17$. Both are contradict with $e(G) = 15$. Let $d(v) = \delta(G)$, the only possibility is that $\delta(G) = 4$ and

$G-v$ is a 6-vertex T-free graph. Since $d(v)=4$, we have $\delta(G-v)\geq 3$, which implies that structures (a) and (b) in Figure 8 are not possible. If $G-v$ is T_6^3 or T_6^4, then G can only be (a) in Figure 9, that is T_7^4. If $G-v$ is S_6^3, then G can only be (b) in Figure 9, that is S_7^4.

Because case $n=8$ needs only the case $n=7$ (Case 1), case $n=9$ needs cases $n=7$ and $n=8$ (Case 2). These base cases complete the proof. $\square$

We will need the following statement later. It express that the "second best" graphs can be also well described if $4|n$.

Proposition 11. *Let n $(n\geq 8)$ be a natural number such that $4|n$ and G be an n-vertex T-free graph with $\frac{n^2}{4}+\frac{n}{2}-1$ edges, then G can only be $T_n^{\frac{n}{2}}$ minus an edge, $S_n^{\frac{n}{2}}$ or $S_n^{\frac{n}{2}+1}$.*

Proof. We can suppose that $\delta(G)\leq\frac{n}{2}$, otherwise, $e(G)\geq\frac{n^2}{4}+\frac{n}{2}$. Let $v\in V(G)$ and $d(v)=\delta(G)$, then $e(G)\leq d(v)+\mathrm{ex}(n-1,T)\leq\frac{n^2}{4}+\frac{n}{2}-1$, the equality holds only if $d(v)=\frac{n}{2}$ and $G-v$ is either $T_{n-1}^{\lceil\frac{n-1}{2}\rceil}$ or $S_{n-1}^{\lceil\frac{n-1}{2}\rceil}$. When $G-v$ is $T_{n-1}^{\lceil\frac{n-1}{2}\rceil}$, let w be the unmatched vertex in Y' and $X'=\{x_1,\ldots,x_{\lceil\frac{n-1}{2}\rceil}\}$, X' and Y' be the classes of $G-v$ with size $\left\lceil\frac{n-1}{2}\right\rceil$ and $\left\lfloor\frac{n-1}{2}\right\rfloor$, respectively. Since $d(v)=\frac{n}{2}$ and v cannot be adjacent to the two endpoints of an edge which have two matched vertices located in different classes, no matter $N(v)=X'$ or $N(v)=X'-x_i\cup w$ $(1\leq i\leq\left\lceil\frac{n-1}{2}\right\rceil)$, G is $T_n^{\frac{n}{2}}$ minus an edge in both cases. When $G-v$ is $S_{n-1}^{\lceil\frac{n-1}{2}\rceil}$, let x_1 be the center of the star in X', $X'=\{x_1,\ldots,x_{\lceil\frac{n-1}{2}\rceil}\}$ and $Y'=\{y_1,\ldots,y_{\lfloor\frac{n-1}{2}\rfloor}\}$ be the classes of $G-v$ with size $\left\lceil\frac{n-1}{2}\right\rceil$ and $\left\lfloor\frac{n-1}{2}\right\rfloor$, respectively. Since v cannot be adjacent to the two endpoints of the edge $\{x_i,y_i\}$ $(2\leq i\leq\left\lceil\frac{n-1}{2}\right\rceil, 1\leq j\leq\left\lfloor\frac{n-1}{2}\right\rfloor)$ and $d(v)=\frac{n}{2}$, which implies that $N(v)=x_1\cup Y'$ or $N(v)=X'$. Therefore, G can be either $S_n^{\frac{n}{2}}$ or $S_n^{\frac{n}{2}+1}$. $\square$

2.3 The Turán number and the extremal graphs for P_6^2

Proof of Theorem 7. Let

$$f_{P_6^2}(n)=\begin{cases}\left\lfloor\dfrac{n^2}{4}\right\rfloor+\left\lfloor\dfrac{n-1}{2}\right\rfloor, & n\equiv 1,2,3\ (\mathrm{mod}\ 6),\\ \left\lfloor\dfrac{n^2}{4}\right\rfloor+\left\lceil\dfrac{n}{2}\right\rceil, & \text{otherwise.}\end{cases}$$

The fact that $\mathrm{ex}(n, P_6^2) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil$, when $n \equiv 0, 4, 5 \pmod 6$, follows from the constructions $H_n^{\frac{n}{2}}$, $H_n^{\frac{n}{2}+1}$ and $H_n^{\lceil \frac{n}{2} \rceil}$, respectively. The fact that $\mathrm{ex}(n, P_6^2) \geq \left\lfloor \frac{n^2}{4} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor$, when $n \equiv 1, 2, 3 \pmod 6$, follows from the constructions $F_n^{\lceil \frac{n}{2} \rceil, j}$.
It remains to show the inequality

$$\mathrm{ex}(n, P_6^2) \leq f_{P_6^2}(n) \tag{14}$$

by induction on n.

Let G be an n-vertex P_6^2-free graph. Since our induction step will go from $n-6$ to n, we have to find a base case in each residue class mod 6.

When $n \leq 4$, K_n is the graph with the most number of edges and $e(K_n) = f_{P_6^2}(n)$.

When $n = 6$, if $P_5^2 \nsubseteq G$, by Theorem 3, $e(G) \leq \left\lfloor \frac{5^2+5}{4} \right\rfloor = 7 < f_{P_6^2}(6)$. If $P_5^2 \subseteq G$, $K_5 \nsubseteq G$ and $e(G) \geq 13$, it can be checked that the vertex $v \in V(G - P_5^2)$ can be adjacent to at most 3 vertices of the copy of P_5^2, otherwise $P_6^2 \subseteq G$, in this case, $d(v) \geq 13 - 9 = 4$ then $P_6^2 \subseteq G$. If $K_5 \subseteq G$, the vertex $v \in V(G - K_5)$ is adjacent to at most one vertex of the K_5, otherwise, $P_6^2 \subseteq G$. Therefore, $e(G) \leq 11 < f_{P_6^2}(6)$.

When $n = 5$, since $e(K_5) = 10 > f_{P_6^2}(5)$, the statement is not true, then we show that the statement is true for $n = 11$. If $P_5^2 \nsubseteq G$, by Theorem 3, $e(G) \leq \left\lfloor \frac{11^2+11}{4} \right\rfloor < f_{P_6^2}(11)$. If $P_5^2 \subseteq G$, first suppose that the graph spanned by the vertices of the copy of P_5^2 have at most 8 edges. It can be checked that every triangle can be adjacent to at most 7 edges of the P_5^2, otherwise, $P_6^2 \subseteq G$. When there exists a triangle as subgraph in $G - V(P_5^2)$, we get $e(G) \leq 8 + 7 + 9 + \mathrm{ex}(6, P_6^2) = 36 = f_{P_6^2}(6)$. If not, $e(G) \leq 8 + 18 + 9 = 35 < f_{P_6^2}(6)$. If $K_5^- \subseteq G$ (K_5 minus an edge) then each vertex $v \in V(G - K_5^-)$ is adjacent to at most 2 vertices of K_5^-. We get $e(G) \leq 9 + 12 + \mathrm{ex}(6, P_6^2) = 33 < f_{P_6^2}(6)$. If $K_5 \subseteq G$ then each vertex $v \in V(G - P_5^2)$ is adjacent to at most one vertex of K_5. Altogether we have at most $10 + 6 + \mathrm{ex}(6, P_6^2) = 28$ edges. From the above, $e(G) \leq 36 = f_{P_6^2}(11)$.

Suppose (14) holds for all $l \leq n-1$ ($l \neq 5$), the following proof is divided into 2 parts.
Case 1. If $T \subseteq G$, then each vertex $v \in V(G - T)$ is adjacent to at most 3 vertices of the copy of T, otherwise, $P_6^2 \subseteq G$. The graph spanned by the vertices of the copy of T cannot

have more than $ex(6, P_6^2) = 12$ edges. Since $G - T$ is an $(n-6)$-vertex P_6^2-free graph and $\mathrm{ex}(6, T) = 12$, we have

$$e(G) \leq 12 + 3(n-6) + e(G - T) \leq 3n - 6 + \mathrm{ex}(n-6, P_6^2). \tag{15}$$

By the induction hypothesis,

$$\mathrm{ex}(n-6, P_6^2) \leq f_{P_6^2}(n-6) = \begin{cases} \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lfloor \dfrac{n-7}{2} \right\rfloor, & n \equiv 1, 2, 3 \pmod 6, \\ \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lceil \dfrac{n-6}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

We get

$$\mathrm{ex}(n, P_6^2) \leq \begin{cases} 3n - 6 + \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lfloor \dfrac{n-7}{2} \right\rfloor = \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n-1}{2} \right\rfloor, & n \equiv 1, 2, 3 \pmod 6, \\ 3n - 6 + \left\lfloor \dfrac{(n-6)^2}{4} \right\rfloor + \left\lceil \dfrac{n-6}{2} \right\rceil = \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil, & \text{otherwise.} \end{cases}$$

Case 2. If $T \nsubseteq G$, by Theorem 5, $e(G) \leq \mathrm{ex}(n, T) \leq f_{P_6^2}(n)$ holds unless $n \equiv 8 \pmod{12}$. When $n \equiv 8 \pmod{12}$, then $e(G) \leq \mathrm{ex}(n, T) = f_{P_6^2}(n) + 1$, however, by Theorem 6, the equality holds only if G is $T_n^{\frac{n}{2}}$, but $P_6^2 \subseteq T_n^{\frac{n}{2}}$ $(n \geq 8)$, which implies that $e(G) \leq \mathrm{ex}(n, T) - 1 = f_{P_6^2}(n)$.
Summarizing, we obtain

$$\mathrm{ex}(n, P_6^2) = f_{P_6^2}(n) = \begin{cases} \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lfloor \dfrac{n-1}{2} \right\rfloor, & n \equiv 1, 2, 3 \pmod 6, \\ \left\lfloor \dfrac{n^2}{4} \right\rfloor + \left\lceil \dfrac{n}{2} \right\rceil, & \text{otherwise}. \end{cases}$$

□

Proof of Theorem 8. It is obvious that

$$\mathrm{ex}(n, T) \leq \mathrm{ex}(n, P_6^2), \text{ except when } n \equiv 8 \pmod{12}. \tag{16}$$

with strict inequality only when

$$n \equiv 5, 6, 7, \text{or } 11 \pmod{12}. \tag{17}$$

We want to determine the graphs G containing no copy of P_6^2 as a subgraph and satisfying $e(G) = \mathrm{ex}(n, P_6^2)$. Therefore suppose that G possesses these properties. We claim that G either contains a copy of T as a subgraph or it is either $F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$ or $F_n^{\frac{n}{2}+1, \frac{n}{2}+1}$. If n belongs to the set of integers given in (17) then this is obvious, since we have a strict inequality in (16). On the other hand for the other values of n (except $n \equiv 8 \pmod{12}$) we obtain $\mathrm{ex}(n, P_6^2) = \mathrm{ex}(n, T) = e(G)$. Theorem 6 describes these graphs. However G cannot be $T_n^{\lceil \frac{n}{2} \rceil}$ or $T_n^{\frac{n}{2}+1}$, because these graphs contain P_6^2 as a subgraph if $n \geq 7$. (In the case of $n = 6$ we had strict inequality in (16).) The other possibility by Theorem 6 is that $G = S_n^{\lceil \frac{n}{2} \rceil} = F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$. In the exceptional case we can use Proposition 11. According to this G could be $T_n^{\frac{n}{2}}, S_n^{\frac{n}{2}}$ or $S_n^{\frac{n}{2}+1}$. The first of them is excluded since $P_6^2 \subset T_n^{\frac{n}{2}}$ the second and third ones can be written in the form $F_n^{\frac{n}{2}, \frac{n}{2}}$ and $F_n^{\frac{n}{2}+1, \frac{n}{2}+1}$.

From now on we suppose that $e(G) = \mathrm{ex}(n, P_6^2)$, the graph G contains a copy of T and no copy of P_6^2 and prove by induction that G is a graph given in the theorem.

Let us list some graphs L (coming up in the forthcoming proofs) containing P_6^2 as a subgraph:

(α) L is obtained by adding any edge to T different from $\{a, e\}, \{d, c\}$ and $\{b, f\}$ on Figure 4.

(β) Add the edges $\{a, e\}, \{d, c\}, \{b, f\}$ to T resulting in T'. The graph L is obtained by adding a new vertex u to T' which is adjacent to three vertices of T' different from the sets $\{b, c, e\}$ and $\{a, d, f\}$.

(γ) L is obtained by adding two new adjacent vertices u and v to T', which are both adjacent to b, c and e. Then e.g. the square of the path $\{u, v, c, e, b, d\}$ is in L.

(δ) L is obtained by adding 4 new vertices u, v, w, x, forming a complete graph, to T', all of them adjacent to a, d and f. Then e.g. the square of the path $\{a, u, v, w, x, d\}$ is in L.

(ϵ) L consists of a complete graph on 5 vertices and a 6th vertex adjacent to two of them.

(ζ) The vertices of L are $p_i (1 \leq i \leq 4)$ and $q_j (1 \leq j \leq 2)$ where p_1, p_2, p_3, p_4 span a path and all pairs (p_i, q_j) are adjacent. Then the square of the path $\{p_1, q_1, p_2, p_3, q_2, p_4\}$ is L.

Let us start with the base cases. Let $n = 6$ and suppose $T \subset G$. By (α) only the edges $\{a, e\}, \{d, c\}$ and $\{b, f\}$ can be added to T. To obtain $\mathrm{ex}(6, P_6^2) = 12$ edges all three of them should be added. The so obtained graph T' is really H_6^3.

Consider now the case $n = 7$. It is clear that (15) holds with equality only when the subgraph spanned by T contains 12 edges and the vertex u not in T is adjacent with exactly 3 vertices of T. Hence the subgraph spanned by T is really T'. By (β) u can be adjacent to either b, c, e or a, d, f. In the first case $G = H_7^3$, in the second one $G = F_7^{4,1}$, as desired.

If $n = 8$, $e(G) = \mathrm{ex}(8, P_6^2) = 19$ and the equality in (15) implies, again, that T must span T' and the remaining two vertices u and v are adjacent to exactly 3 vertices of T': either to the set $\{b, c, e\}$ or to $\{a, d, f\}$ and $\{u, v\}$ is an edge. If both u and v are adjacent to $\{b, c, e\}$ then (γ) leads to a contradiction. If one of u and v is adjacent to $\{b, c, e\}$, the other one to $\{a, d, f\}$, then $G = F_8^{4,1}$. Finally if both of them are adjacent to $\{a, d, f\}$, then $G = F_8^{5,2}$.

Suppose now that $n = 9$, when $e(G) = \mathrm{ex}(9, P_6^2) = 24$ and (15) implies that the three vertices u, v, w not in T' form a triangle and all three possess the properties mentioned in the previous case. If two of them are adjacent to $\{b, c, e\}$ then (γ) gives the contradiction. If one of the them is adjacent to $\{b, c, e\}$, the two other ones are adjacent to $\{a, d, f\}$, then $G = F_9^{5,2}$. Finally if all three are adjacent to $\{a, d, f\}$, then $G = H_9^6$.

The case $n = 10$ and $e(G) = \mathrm{ex}(10, P_6^2) = 30$ is very similar to the previous ones. If one of the new vertices, u, v, w, x is adjacent to $\{b, c, e\}$ and the other 3 are adjacent to $\{a, d, f\}$, then $G = H_{10}^6$. Here it cannot happen, by (δ), that all 4 are adjacent to $\{a, d, f\}$.

Finally let $n = 11$ where $e(G) = \mathrm{ex}(11, P_6^2) = 36$. This case is different from the previous ones, since we cannot have all the potential edges (12 in the graph spanned by T, 10 among the other 5 vertices u, v, w, x, y, and 15 between the two parts) one is missing. We distinguish 3 cases according the place of the missing edge.

(i) $T' \subset G$, $\{u, v, w, x, y\}$ spans a copy of K_5, but there are only 14 edges between the two parts. Then T' has one vertex $z \in \{a, b, c, d, e, f\}$ incident to at least two of the 14 edges. Then (ϵ) leads to a contradiction.

(ii) $T' \subset G$, $\{u, v, w, x, y\}$ spans a copy of K_5 minus one edge, say $\{x, y\}$, and all 15 edges between the two parts are in G.

If two adjacent vertices from the set $\{u, v, w, x, y\}$ are both adjacent to $\{b, c, e\}$ then (γ) gives the contradiction. Therefore if x is adjacent to $\{b, c, e\}$ then u, v and w must be adjacent to $\{a, d, f\}$. If y is also adjacent to $\{a, d, f\}$ then we have 4 vertices spanning a K_4 and all adjacent to $\{a, d, f\}$. Then we obtain a contradiction by (δ). Otherwise y is adjacent to $\{b, c, e\}$ and $G = H_{11}^6$.

Suppose now that x is adjacent to $\{a, d, f\}$. If u, v, w are all adjacent to $\{a, d, f\}$ then (δ) leads to a contradiction. Hence at least one of them, say u is adjacent to $\{b, c, e\}$. But (γ) implies that two adjacent ones from from the set $\{u, v, w, x, y\}$ cannot be adjacent to $\{b, c, e\}$. Hence v, w, x, y are all adjacent to $\{a, d, f\}$ giving a contradiction again, by (δ).

(iii) T spans only 11 edges, $\{u, v, w, x, y\}$ determines a K_5 and all 15 edges are connecting the two parts. Then T must have a vertex incident to two edges connecting T with $\{u, v, w, x, y\}$. Here (ϵ) gives a contradiction.

Now we are ready to start the inductional step. Suppose that the statement is true for $n-6$ where $n \geq 12$. Prove it for n. Let $e(G) = \mathrm{ex}(n, P_6^2)$ and suppose that $T \subset G$. We have to prove that G is of the form described in the theorem. By (15) we know that the equality implies that T must span the the subgraph T' with 12 edges, every vertex of $G' = G - T'$ is adjacent either to the vertices b, c, e or the vertices a, d, f and G' is an extremal graph for $n-6$. That is G' is one the following graphs: $F_n^{\lceil \frac{n-6}{2} \rceil, j}, F_n^{\frac{n-6}{2}+1, j}, H_n^{\lfloor \frac{n-6}{2} \rfloor}, H_n^{\lceil \frac{n-6}{2} \rceil}, H_n^{\lceil \frac{n-6}{2} \rceil + 1}$. All these graphs have $n-6$ vertices, their vertex sets are divided into two parts, X' and Y' where $|X'|$ is either $\lfloor \frac{n-6}{2} \rfloor$ or $\lceil \frac{n-6}{2} \rceil$ or $\lceil \frac{n-6}{2} \rceil + 1$, there is a bipartite graph between X' and Y' and X' is covered by vertex-disjoint triangles and at most one star.

Color a vertex of G' by red if it is adjacent to the vertices b, c, e and blue otherwise. By (γ) two red vertices cannot be adjacent. On the other hand 4 blue vertices cannot span a path by (ζ). Suppose that there is a red vertex in X'. Then all vertices of Y' are colored blue. (It is easy to check that $n \geq 12$ implies $|Y'| \geq 2$.) If there are two blue vertices also in X' then they span a path of length 4 that is a contradiction. We can have one blue vertex in X' only when it contains no triangle and the center s of the star is blue, the other vertices are all red. This is called the first coloring. It is easy to see that the choice $X = \{b, c, e, s\} \cup Y', Y = \{a, d, f\} \cup (X' - \{s\})$ defines a graph possessing the properties of the expected extremal graphs: X and Y span a complete bipartite graph, there are no edges within Y, and X is covered by one triangle and one star which are vertex disjoint.

The other case is when all vertices of X' are blue. In this case no vertex of Y' can be blue, otherwise this vertex and the 3 vertices of a triangle or the center of the star with two other vertices would span a path of length 4. That is all vertices of Y' are red. This is the second coloring. Then the choice $X = \{b, c, e\} \cup X', Y = \{a, d, f\} \cup Y'$ defines a graph possessing the properties of the expected extremal graphs.

We have seen that G has the expected structure in both cases. We only have to check the parameters. If $n \equiv 0, 4, 5 \pmod 6$ then X' contains no star, the first coloring cannot occur, in the case of the second coloring 3-3 vertices are added to both parts, containing a triangle ($\{b, c, e\}$) in the X-part. The upper index increases by 3 in all cases when moving from $n - 6$ to n.

Consider now the case $n \equiv 1 \pmod 6$. If $G' = H_{n-6}^{\lfloor \frac{n-6}{2} \rfloor}$ then we can proceed like in the previous cases, and $G = H_n^{\lfloor \frac{n}{2} \rfloor}$ is obtained. Suppose that $G' = F_{n-6}^{\lceil \frac{n-6}{2} \rceil, j}$. If $j < \lceil \frac{n-6}{2} \rceil$ then, again, the second coloring applies and we obtain $G = F_n^{\lceil \frac{n}{2} \rceil, j}$. If, however, $j = \lceil \frac{n-6}{2} \rceil$ then both colorings result in $G = F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil - 3}$. Let us recall that $G = F_n^{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$ was obtained in the case when $T \not\subset G$.

The cases $n \equiv 2, 3 \pmod 6$ can be checked similarly. □

3 Open problems

The following paragraphs show why we think that Conjecture 1 is true.

Lemma 12. *If the graph G is obtained by adding a path of r vertices to one of the classes of the complete bipartite graph $K_{n,n}(n \geq r)$ then G contains the square of a path containing $\lfloor \frac{3r}{2} \rfloor + 1$ vertices.*

Proof. Suppose first that $r = 2s$ is even. Let X and Y be the two parts, where $|X| = |Y| = n$ all edges $\{x, y\}(x \in X, y \in Y)$ are in G. Moreover, X contains the path $\{x_1, x_2, \ldots, x_{2s}\}$. Then the square of the path $\{y_1, x_1, x_2, y_2, x_3, x_4, y_3, \ldots, x_{2s-1}, x_{2s}, y_{s+1}\}$ is in G for an arbitrary set of distinct vertices $y_1, y_2, \ldots, y_{s+1} \in Y$. The number of vertices of this path is really $3s + 1$.

If $k = 2s + 1$ is an odd number then the desired path is $\{y_1, x_1, x_2, y_2, x_3, x_4, y_3, \ldots, x_{2s-1}, x_{2s}, y_{s+1}, x_{2s+1}\}$. □

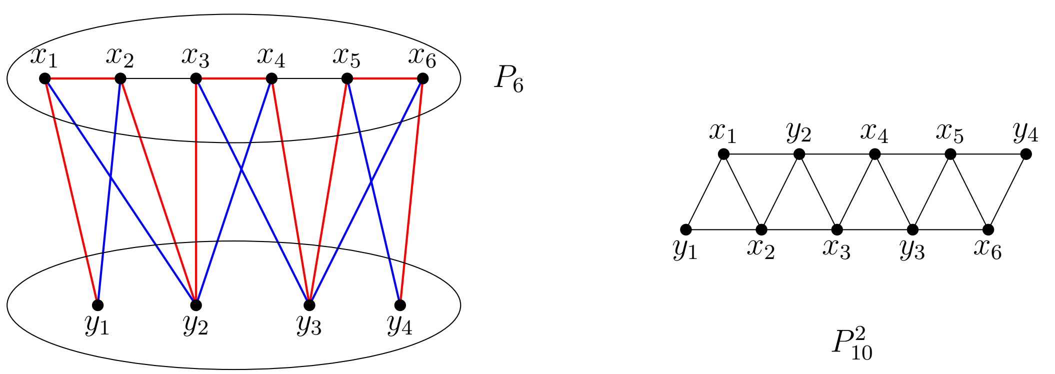

Figure 10:

It is easy to see, on the basis of Lemma 12 that if this graph does not contain P_k^2 then X cannot contain a path of length $\lfloor \frac{2k}{3} \rfloor$. Now the obvious question is that at most how many edges can be chosen in X without having a path of given length. As one of the earliest results in extremal Graph Theory Erdős and Gallai [2] proved the following result on the extremal number of paths.

Theorem 13 (**Erdős and Gallai** [2])**.** *The maximum number of edges in an n-vertex P_l-free graph is $\frac{n(l-2)}{2}$, that is $\text{ex}(n, P_l) \leq \frac{n(l-2)}{2}$ with equality if and only if $(l-1)|n$ and the graph is a vertex disjoint union of $\frac{n}{l-1}$ complete graphs on $l-1$ vertices.*

Faudree and Schelp [5] and independently Kopylov [6] improved this result determining $\mathrm{ex}(n, P_l)$ for every $n > l > 0$ as well as the corresponding extremal graphs.

Theorem 14 (**Faudree and Schelp** [5] **and independently Kopylov** [6])**.** *Let* $n \equiv r \pmod{l-1}$, $0 \leq r \leq l-1, l \geq 2$. *Then*

$$\mathrm{ex}(n, P_l) = \frac{1}{2}(l-2)n - \frac{1}{2}r(l-1-r).$$

Faudree and Schelp also described the extremal graphs which are either

(a) vertex disjoint union of m ($n = m(l-1) + r$) complete graphs K_{l-1} and a K_r or

(b) l is even and $r = \frac{l}{2}$ or $\frac{l}{2} - 1$ then another extremal graph can be obtained by taking a vertex disjoint union if t copies of K_{l-1} ($0 \leq t \leq m$) and a copy of $K_{\frac{l}{2}-1} + \overline{K}_{n-(t+\frac{1}{2})(l-1)+\frac{1}{2}}$. Where $\overline{G}$ denotes the edge complement of the graph G, and $G + H$ is defined as the graph obtained from the vertex disjoint union of G and H together with all edges between G and H.

We believe that the extremal graph for $\mathrm{ex}(n, P_k^2)$ is a complete bipartite graph plus one of the constructions above in the larger class. Check now the cases solved.

If $k = 4$, by Lemma 12 we cannot have a path of length 2 (that is an edge) in one side.

If $k = 5$ then $l = 3$, a path of length 3 is forbidden in one side. According to statements above we can have only vertex disjoint edges.

If $k = 6$ then $l = 4$ and a path of length 4 is forbidden in one side. Now the extremal constructions for P_l are either (a) triangles plus eventually one edge or (b) t triangles plus a star with $n - 3t$ vertices.

These are in accordance with our results. Note that in the case of $k = 7$, the value $l = 4$ obtained again. The expected maximum value is the same as in the case of $k = 6$, but the assumptions are weaker!

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