New results on intersecting families of subsets

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Abstract

A family $\mathcal{F} \subset \binom{[n]}{k}$ of k-element subsets of an n-element set is called intersecting if $F, G \in \mathcal{F}$ implies $F \cap G \neq \emptyset$. The celebrated Erdős-Ko-Rado theorem says that it has at most $\binom{n-1}{k-1}$ members. Equality can be obtained for the family of sets containing one fixed element.

Some of the results from the history of the area are surveyed and some new developments are introduced. One such direction is the problem of "two-part intersecting" families. The underlying set $[n]$ is partitioned into X_1 and X_2. It is still true that the largest intersecting family, for large n, is the one consisting of members containing one fixed element. Even in the following very general form when those sets are considered which satisfy $|F \cap X_1| = k_i, |F \cap X_2| = \ell_i$ for some $k_i, \ell_i (1 \leq i \leq m)$. The statement was known for the case $m = 2$ as a result of Frankl.

The shadow $\sigma(\mathcal{F})$ of a family $\mathcal{F} \subset \binom{[n]}{k}$ is the family of all $k-1$-element subsets of members of $\mathcal{F}$. The shadow theorem determines the minimum size of the shadow family for fixed n, k and $|\mathcal{F}|$. To find the smallest shadow of an intersecting family is very different from the traditional problem. Some old and new results of this kind are exhibited.

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1 Introduction

The underlying set will be $\{1, 2, \ldots, n\}$. The family of all k-element subsets of $[n]$ is denoted by $\binom{[n]}{k}$. Its subfamilies are called *uniform.* A family $\mathcal{F}$ of some subsets of $[n]$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every pair $F, G \in \mathcal{F}$. The whole story has started with the seminal paper of Erdős, Ko and Rado [4]. The main result of the paper determines the largest intersecting family consisting of subsets of size exactly k, that is the case of uniform families. The problem is trivial when $k > \frac{n}{2}$: all k-element subsets can be chosen. It is not so trivial at all when $k \leq \frac{n}{2}$.

Theorem 1 (Erdős, Ko, Rado [4]) *If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting where $k \leq \frac{n}{2}$ then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

The original proof uses the so called *shifting method.* There is a shorter proof based on the *cycle method* in [11]. It can also be found in the books [1] and [2]. If $k < \frac{n}{2}$ there is only one extremal family.

Construction 1 *Take all subsets of $[n]$ having size k and containing the element 1.*

Construction 2 *If $k = \frac{n}{2}$ one can choose one from each complementing pair, freely.*

We say that a family $\mathcal{F}$ is *trivially intersecting* if there is an element $a \in [n]$ such that all members of $\mathcal{F}$ contain a. Construction 1 is trivially intersecting, Construction 2 not necessarily. [4] posed the problem of finding the largest k-uniform non-trivially intersecting family. It was found by Hilton and Milner.

Theorem 2 [8] *If $\mathcal{F}$ is an intersecting but not a trivially intersecting family, $\mathcal{F} \subset \binom{[n]}{k} (2k \leq n)$ then*

$$|\mathcal{F}| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

The construction giving equality is the following.

Construction 3 *Let $K = \{2, 3, \ldots, k+1\}$. The extremal family will consist of all k-element sets containing 1 and intersecting K.*

Let me call the reader's attention to the forthcoming book of Gerbner and Patkós [7], containg many related results.

2 Two-part intersecting families

Now we will consider the problem when the underlying set is partitioned into two parts X_1, X_2 and the sets $F \in \mathcal{F}$ have fixed sizes in both parts. For some motivation see [12] (Section 4). More precisely let X_1 and X_2 be disjoint sets of n_1, respectively n_2 elements. [6] considered such subsets of $X = X_1 \cup X_2$ which had k elements in X_1 and ℓ elements in X_2. The family of all such sets is denoted by

$$\binom{X_1, X_2}{k, \ell} = \binom{X_1}{k} \biguplus \binom{X_2}{\ell} = \{F \subset X_1 \cup X_2 : \ |F \cap X_1| = k, |F \cap X_2| = \ell\}.$$

The construction above, taking all possible sets containing a fixed element also works here. If the fixed element is in X_1 then the number of these sets is

$$\binom{n_1 - 1}{k - 1}\binom{n_2}{\ell},$$

otherwise it is

$$\binom{n_1}{k}\binom{n_2 - 1}{\ell - 1}.$$

The following theorem of Frankl [6] claims that the larger one of these is the best.

Theorem 3 *Let X_1, X_2 be two disjoint sets of n_1 and n_2 elements, respectively. The positive integers k, ℓ satisfy the inequalities $2k \le n_1, 2\ell \le n_2$. If $\mathcal{F}$ is an intersecting subfamily of $\binom{X_1, X_2}{k, \ell}$ then*

$$|\mathcal{F}| \le \max\left\{\binom{n_1 - 1}{k - 1}\binom{n_2}{\ell}, \binom{n_1}{k}\binom{n_2 - 1}{\ell - 1}\right\}.$$

Actually his theorem is formulated for an arbitrary number of parts.

Theorem 3 could be formulated in such a way that the largest subfamily of $\binom{X_1, X_2}{k, \ell}$ is one of the trivially intersecting families. It is natural to ask what is the largest non-trivially intersecting subfamily.

Take a Hilton-Milner family (Construction 3) in X_1, denote it by $\mathrm{HM}(X_1, k)$. Extend its members in all possible ways by ℓ-element subsets chosen from X_2:

$$\mathrm{HM}_1(X_1, k; X_2, \ell) = \{F \cup G : F \in \mathrm{HM}(X_1, k), G \subset X_2, |G| = \ell\}.$$

Define, similarly,

$$\mathrm{HM}_2(X_1,k;X_2,\ell)=\{F\cup G: F\subset X_1, |F|=k, G\in \mathrm{HM}(X_2,\ell)\}.$$

It was conjectured in [12] that either $\mathrm{HM}_1(X_1,k;X_2,\ell)$ or $\mathrm{HM}_2(X_1,k;X_2,\ell)$ is the largest nontrivially intersecting subfamily of $\binom{X_1,X_2}{k,\ell}$. Kwan, Sudakov and Vieira [15] showed that this is not true: there are other, "mixed" Hilton-Milner families which are better in some cases.

Fix an element $a\in X_1$, a set $A\subset X_1$ such that $a\notin A$, $|A|=k$ and a set $B\subset X_2$ such that $|B|=\ell$ and define

$$\mathrm{HM}_1^{\mathrm{mix}}(X_1,k;X_2,\ell)=\{F:|F\cap X_1|=k, |F\cap X_2|=\ell, a\in F, F\cap(A\cup B)\neq\emptyset\}.$$

$\mathrm{HM}_2^{\mathrm{mix}}(X_1,k;X_2,\ell)$ is the symmetric construction.

Theorem 4 (Kwan, Sudakov, Vieira [15]) *If both $|X_1|$ and $|X_2|$ are large enough then the largest non-trivially intersecting subfamily of $\binom{X_1,X_2}{k,\ell}$ is one of* $\mathrm{HM}_1(X_1,k;X_2,\ell), \mathrm{HM}_2(X_1,k;X_2,\ell), \mathrm{HM}_1^{\mathrm{mix}}(X_1,k;X_2,\ell)$ *and* $\mathrm{HM}_2^{\mathrm{mix}}(X_1,k;X_2,\ell)$.

Their result actually claims the analogous statement for more parts. The proof uses the shifting method.

Consider now the case when two sizes are also allowed in both parts (but not independently!) that is the family consists of sets satisfying either $|F\cap X_1|=k, |F\cap X_2|=\ell$ or $|F\cap X_1|=r, |F\cap X_2|=s$. Using the notation above, we will consider intersecting subfamilies of

$$\binom{X_1,X_2}{k,\ell}\bigcup\binom{X_1,X_2}{r,s}.$$

In Theorem 3

$$\binom{n_1-1}{k-1}\binom{n_2}{\ell}\geq\binom{n_1}{k}\binom{n_2-1}{\ell-1}$$

holds if and only if

$$\frac{k}{n_1}\binom{n_1}{k}\binom{n_2}{\ell}\geq\frac{\ell}{n_2}\binom{n_1}{k}\binom{n_2}{\ell}$$

that is when

$$\frac{k}{\ell}\geq\frac{n_1}{n_2}.$$

In this case the best is a trivially intersecting family with fixing one point on the left hand side. Otherwise the point should be fixed on the right hand side. Of course the same holds for the pair r, s. Therefore if

$$\frac{k}{\ell}, \frac{r}{s} \geq \frac{n_1}{n_2}$$

then the best, for both kinds of sets, is to fix one point on the left hand side.

But what happens if

$$\frac{k}{\ell} > \frac{n_1}{n_2} > \frac{r}{s}?$$

For the family of sets having k and ℓ elements in the two sizes, respectively, the best construction chooses the fixed element on the left hand side, for the other family on the right hand side. These two families together are not intersecting. The answer to our question is that one of them wins! That is if both n_1 and n_2 are large then the largest intersecting family is trivially intersecting, either on the left or on the right hand side.

Let us consider now the more general case when other sizes are also allowed that is the family consists of sets satisfying $|F \cap X_1| = k_i, |F \cap X_2| = \ell_i$ for certain pairs (k_i, ℓ_i) of positive integers. Using the notation above, we will consider subfamilies of

$$\bigcup_{i=1}^{m} \binom{X_1, X_2}{k_i, \ell_i}.$$

The generalization is however a little weaker at one point. In Theorem 10 the thresholds $2k \leq n_1, 2\ell \leq n_2$ for validity are natural. If either n_1 or n_2 is smaller then the problem becomes trivial, all such sets can be selected in $\mathcal{F}$. In the generalization below there is no such natural threshold. There will be another difference in the formulation. We give the construction of the extremal family rather than the maximum number of sets.

Theorem 5 [12] *Let X_1, X_2 be two disjoint sets of n_1 and n_2 elements, respectively. Some positive integers $k_i, \ell_i (1 \leq i \leq m)$ are given. Define $b = \max_i\{k_i, \ell_i\}$. Suppose that $9b^2 \leq n_1, n_2$. If $\mathcal{F}$ is an intersecting subfamily of*

$$\bigcup_{i=1}^{m} \binom{X_1, X_2}{k_i, \ell_i}$$

then $|\mathcal{F}|$ cannot exceed the size of the largest trivially intersecting family satisfying the conditions.

Sketch of the proof. The proof uses the so called *cycle method* used in a simple proof of Theorem 1 (see [11]). Its basic idea is to find the largest family of intersecting intervals of length k along a cycle of length n and then a simple double counting leads to the statement of the theorem. It is convenient to consider the cycle as $\mathbb{Z}_n$ and an interval as a set $\{i, i+1, \ldots, i+k-1\}$ mod k. It is easy to prove that the largest intersecting family of such intervals is trivially intersecting.

In the present proof cyclic permutation will be replaced by a product of two cyclic permutations. In notation: $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Of course intervals will be replaced by direct products of intervals of length k_i and ℓ_i, that is by $k_i \times \ell_i$ *rectangles.*The "intersecting condition" is that any two rectangles must meet in one of the coordinates. More precisely, if the two rectangles are $\{i_1, i_1+1, \ldots, i_1+k_u-1\} \times \{i_2, i_2+1, \ldots, i_2+\ell_u-1\}$ and $\{j_1, j_1+1, \ldots, j_1+k_v-1\} \times \{j_2, j_2+1, \ldots, j_2+\ell_v-1\}$ then either $\{i_1, i_1+1, \ldots, i_1+k_u-1\} \cap \{j_1, j_1+1, \ldots, j_1+k_v-1\}$ or $\{i_2, i_2+1, \ldots, i_2+\ell_u-1\} \cap \{j_2, j_2+1, \ldots, j_2+\ell_v-1\}$ is non-empty. We call a pair of rectangles having this property *proj-intersecting.*

Let $\mathcal{R}_i$ be a family of $k_i \times \ell_i$ rectangles in $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} (1 \le i \le m)$. We say that $\mathcal{R} = \bigcup_{i=1}^m \mathcal{R}_i$ is a *proj-intersecting family* if, any two members are proj-intersecting.

One can prove the statement analogous to the theorem for the rectangles, that is, the largest $\mathcal{R}$ is trivially intersecting (if n_1 and n_2 are large) either in the projections in $\mathbb{Z}_{n_1}$ or in the projections in $\mathbb{Z}_{n_2}$.

In other words

$$\sum_{i=1}^{m} |\mathcal{R}_i| \le \max \left\{ n_1 \sum_{i=1}^{m} \ell_i, n_2 \sum_{i=1}^{m} k_i \right\}$$

holds. However this is not sufficient for the proof of the theorem. A weighted version is needed.

Lemma 1 *Suppose that the positive integers k_i, ℓ_i, b, n_1, n_2 satisfy the inequalities $k_i, \ell_i \le b (1 \le i \le m), 9b^2 < n_1, n_2$. Let $\mathcal{R}_i$ be a family of $k_i \times \ell_i$ rectangles in $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} (1 \le i \le m)$. Suppose that $\mathcal{R} = \bigcup_{i=1}^m \mathcal{R}_i$ is a proj-intersecting family. Let $\lambda_i > 0 (1 \le i \le m)$ be real numbers. Then*

$$\sum_{i=1}^{m} \lambda_i |\mathcal{R}_i| \le \max \left\{ n_1 \sum_{i=1}^{m} \lambda_i \ell_i, n_2 \sum_{i=1}^{m} \lambda_i k_i \right\}$$

holds.

Define the families

$$\mathcal{F}_i = \{F \in \mathcal{F} : \ |F \cap X_1| = k_i, |F \cap X_2| = \ell_i\}.$$

We use double counting for the sum

$$\sum_{F, \mathcal{C}_1, \mathcal{C}_2} s(F)$$

where $\mathcal{C}_j$ is a cyclic permutation of $\mathbb{Z}_{n_j} (j = 1, 2)$, $F \in \mathcal{F}$ and it forms a rectangle for the product of these two cyclic permutations and the weight $s(F)$ is defined in the following way:

$$s(F) = s_i(F) = \frac{1}{n_1!} \cdot \frac{1}{n_2!} \binom{n_1}{k_i} \binom{n_2}{\ell_i} \text{ if } F \in \mathcal{F}_i.$$

Some tedious calculations and the usage of the lemma leads to the proof of the theorem.

3 A small detour: shadows

Let $\mathcal{F} \subset \binom{[n]}{k}$ be a family of k-element subsets of $[n]$. Its *shadow* is defined as

$$\sigma(\mathcal{F}) = \{G : \ |G| = k - 1, G \subset F \text{ for some } F \in \mathcal{F}\}.$$

The shadow problem is the following: given n, k and $|\mathcal{F}|$, minimize $|\sigma(\mathcal{F})|$. It is obvious to believe that if we are lucky and $|\mathcal{F}| = \binom{a}{k}$ holds for an integer a then the best construction is "to push all these k-element subsets into the corner" that is to take all k-element subsets of an a-element set A. Then the shadow will be $\min |\sigma(\mathcal{F})| = \binom{a}{k-1}$.

This is really true and this pattern can be continued using the following lemma.

Lemma 2 *If $0 < k, m$ are integers then one can find integers $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$ such that*

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

and they are unique.

This is called the *canonical form* of m. Now we can formulate the solution to the shadow problem.

Theorem 6 (Shadow Theorem, [14], [10]) . *If n, k and $|\mathcal{F}|$ are given, the canonical form of $|\mathcal{F}|$ is*

$$|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

then

$$\min |\sigma(\mathcal{F})| = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_t}{t-1}.$$

We might also want to minimize the "deeper" shadow, the so called *s-shadow*: $\sigma_s(\mathcal{F}) = \{G : \ |G| = k - s, G \subset F$ for some $F \in \mathcal{F}\}$. Theorem 6 can be formulated in this general form.

Theorem 7 (Shadow Theorem, [14], [10]) . *If n, k and $|\mathcal{F}|$ are given, the canonical form of $|\mathcal{F}|$ is*

$$|\mathcal{F}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

then

$$\min |\sigma_s(\mathcal{F})| = \binom{a_k}{k-s} + \binom{a_{k-s}}{k-1-s} + \ldots + \binom{a_t}{t-s}.$$

Lovász [16] found an estimate which is not sharp in most cases but is easier to handle. We need to generalize the binomial coefficients for real numbers. If x is a real number, $\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}$.

Theorem 8 (Lovász' version of the Shadow theorem, [16]). *If $\mathcal{A}$ is a family of k-element sets,*

$$|\mathcal{A}| = \binom{x}{k}$$

then

$$|\sigma_s(\mathcal{A})| \geq \binom{x}{k-s}.$$

This estimate is sharp only when x is an integer.

Daykin [3] noticed that the shadow theorem implies the Erdős-Ko-Rado theorem.

Proof. Let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting $(2k \leq n)$. Define the complementing family $\mathcal{F}^- = \{[n] - F : \ F \in \mathcal{F}\} \subset \binom{[n]}{n-k}$ where $k \leq n-k$. If $A \in \mathcal{F}$ then $\overline{A} \in \mathcal{F}^-$ has $n-k$ elements. Deleting $s = n-2k$ elements from the $n-k$-element set $\overline{A}$ we obtain a k-element shadow set. Hence $\sigma_{n-2k}(\mathcal{F}^-) \subset \binom{[n]}{k}$. The members of $\sigma_{n-2k}(\mathcal{F}^-)$ are all disjoint to A therefore they cannot be in the intersecting $\mathcal{F}$. We obtained

$$\mathcal{F} \cap \sigma_{n-2k}(\mathcal{F}^-) = \emptyset. \tag{1}$$

Suppose $|\mathcal{F}^-| = |\mathcal{F}| > \binom{n-1}{k-1} = \binom{n-1}{n-k}$. Then by Theorem 7 $|\sigma_{n-2k}(\mathcal{F}^-)| \geq \binom{n-1}{k}$ and by (1), the number of k-element subsets is at least $|\mathcal{F}|+|\sigma_{n-2k}(\mathcal{F}^-)| > \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. This contradiction proves the statement. $\square$

4 Shadows of intersecting families

Suppose $\mathcal{F}$ is intersecting and $|\mathcal{F}| = \binom{a}{k}$ where $2k < a < n$. If we want to find the minimum of $\sigma(\mathcal{F})$ it is easy to see that the old construction does not work here since one cannot choose all k-element sets of the a-element set, since there are disjoint ones among them.

Let us consider the following more general case. $\mathcal{F}$ is *t-intersecting* if $F, G \in \mathcal{F}$ implies $|F \cap G| \geq t$. Our question is, again what is the minimum of $|\sigma_s(\mathcal{F})|$ under the condition that $\mathcal{F}$ is t-intersecting?

The disappointing answer is that we do not know! This is why we must ask a more modest question. What is the minimum of

$$\frac{|\sigma_s(\mathcal{F})|}{|\mathcal{F}|}$$

under the condition that $\mathcal{F}$ is t-intersecting?

Theorem 9 (Intersecting shadow theorem, [9]) *If $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting family, $s \leq t$ then*

$$\frac{|\sigma_s(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{\binom{2k-t}{k-s}}{\binom{2k-t}{k}}.$$

The family $\mathcal{F} = \binom{2k-t}{k}$ gives equality in the theorem.

Now we will show that the Intersecting shadow theorem implies EKR. This has an importance because it is a less difficult theorem than the Shadow theorem, yet it has the same implication at this place.

Proof ([9]). We will start in the same way as in the proof of Daykin. (Observe that [9] was published earlier than [3].) As before let $\mathcal{F} \subset \binom{[n]}{k}$ be intersecting ($2k \leq n$) and $\mathcal{F}^- = \{[n] - F : F \in \mathcal{F}\} \subset \binom{[n]}{n-k}$ where $k \leq n-k$. We saw that (1) holds.

$\mathcal{F}$ is intersecting therefore $\mathcal{F}^- = \{[n] - F : F \in \mathcal{F}\} \subset \binom{[n]}{n-k}$ is $n-2k+1$-intersecting. Here $2(n-k)-(n-2k+1) = n-1$ and by the intersecting shadow theorem we obtain

$$\frac{|\sigma_{n-2k}(\mathcal{F}^-)|}{|\mathcal{F}^-|} \geq \frac{\binom{n-1}{k}}{\binom{n-1}{n-k}} = \frac{n-k}{k}.$$

Hence by (1):

$$\binom{n}{k} \geq |\sigma_s(\mathcal{F}^-)| + |\mathcal{F}| \geq |\mathcal{F}| \left(\frac{n-k}{k} + 1\right) = |\mathcal{F}|\frac{n}{k},$$

which implies EKR. □

Return now to Theorem 9. The problem answered by it is not just for itself. The solution of the maximization of the non-uniform t-intersecting family was based on that (see [9]). Repeat the result of Theorem 9 for the case $s = 1$.

$$\frac{|\sigma(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{\binom{2k-t}{k-1}}{\binom{2k-t}{k}} = \frac{k-1}{k-t+1}.$$

It was mentioned above that this estimate is sharp. If $\mathcal{F}$ consists of all k-element subsets of a $2k-t$-element set then the size of the shadow is $\binom{2k-t}{k-1}$, the ratio is exactly the above one. In this construction however the size $|\mathcal{F}|$ of the family is "small", does not depend on n. What happens if we suppose that $|\mathcal{F}|$ is large? We have a slight improvement in this case.

Theorem 10 [13]. *If $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting family, $1 \leq t$ then*

$$|\sigma(\mathcal{F})| \geq |\mathcal{F}|\frac{k-1}{k-t} - c(k,t)$$

where $c(k,t)$ does not depend on n and $|\mathcal{F}|$.

This is an improvement only when $t > 1$ and $\mathcal{F}|$ is large. A better multiplicative constant cannot be expected as the following example shows.

Divide $[n]$ into two parts, X_1, X_2 where $|X_1| = 2k - t - 2, |X_2| = n - 2k + t + 2$ and define $\mathcal{F}$ as the family of all k-element sets F such that $|F \cap X_1| = k - 1, |F \cap X_2| = 1$. Here $|\mathcal{F}| = \binom{2k-t-2}{k-1}(n - 2k + t + 2), |\sigma(\mathcal{F})| = \binom{2k-t-2}{k-2}(n - 2k + t + 2) + \binom{2k-t-2}{k-1}$. Their ratio tends to $\frac{k-1}{k-t}$.

Let us mention that there is a similar result of Frankl [5]. See also a forthcoming papers of Frankl and the present author.

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