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Matching problems

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We have two types of lattice-figures: a "big" and a "small" one. We should like to cover the "big" figure with disjoint replicas of the "small" one. An old problem of this type is the well-known chessboard problem: is it possible to cover with 31 dominoes a chessboard deprived of two diagonal fields?

The following interesting matching problem of N.G. de Bruijn was published in 1962 in Matematikai Lapok [1]:

An n-dimensional rectangular parallelotop is to be decomposed into such congruent rectangular parallelotops, the edge-lengths of which are the given natural numbers $a_1, a_2, \dots, a_n$. Under which conditions can we say that such a decomposition exists if and only if there exists a decomposition with parallel parallelotops (i.e. the parallel edges of the parallelotops involved in the decomposition are equal).

(The solution of the problem was sent in by G. Hajós and the authors [2].)

In a not too general sense, matching problems deal with the coverability and the number od different coverings of a lattice-figure - say B - with the replicas of another lattice-figure - say A. Rotation and symmetry are or are not allowed. Sometimes more types of A's are also allowed. The notion of coverability can be also generalized (see Definition B). The principal results of the paper give necessary and sufficient conditions for the coverability of a lattice parallelotop with

α) lattice-parallelotops of one or more given types;

β) lattice-figures consisting of two cubes (this is a generalization of the domino).

Here we summarize only the main definitions and results of our paper being in the press at the Journal of Combinatorial Theory.

The proofs can be found in that paper, and the numbering of definitions and theorems is also taken from there.

DEFINITIONS

Let us consider the set of n-dimensional lattice points (i.e. the points with integer coordinates).

An n-dimensional lattice-figure is an arbitrary subset of lattice-points.

There exists a natural correspondence between the lattice-points and lattice-fields (unit cubes). Thus, sometimes we shall use the more illustrative expression "lattice field" instead of "lattice-point".

We accept the usual concept of congruency, that is we allow of shift, rotation and symmetry.

For the sake of simplicity we suppose (unless we emphasize the contrary) that the parallelotop which we want to decompose will be situated in the non-negative octant, and that one of its vertices is the origin. (I.e. a parallelotop B with edge-lengths $b_1, b_2, \dots, b_n$ consists of the lattice-points $(x_1, x_2, \dots, x_n)$ satisfying the conditions $0 \le x_i < b_i \quad (1 \le i \le n)$.)

DEFINITION A.

We say that a parallelotop B can be filled up (covered) by the given lattice-figures $A_1, A_2, \dots, A_m$ if we can decompose B into disjoint subsets, each of which is congruent to one of A_i's; and in this case we write

$$(A_1, A_2, \dots, A_m) \mid B .$$

(If $m=1$, we write simply $A_1 \mid B$.)

If we use the above natural definition of coverability, then the necessary and sufficient conditions are valid only if all the edges of the parallelotop B are large enough. However, in the case of the next definition we can omit this.

DEFINITION B.

We say that a parallelotop B can be filled up (covered) in the weak sense by the given lattice-figures $A_1, A_2, \dots, A_m$ if there exist the parallelotops $A_1(1), \dots, A_1(r_1), A_2(1), \dots, A_2(r_2), \dots, A_m(1), \dots, A_m(r_m)$ and the integers $\nu_{11}, \dots, \nu_{1r_1}, \nu_{21}, \dots, \nu_{2r_2}, \dots, \nu_{m1}, \dots, \nu_{mr_m}$ such that

$$\sum_{\substack{1 \le i \le m \\ 1 \le j \le r_i \\ x \in A_i(j)}} \nu_{ij} = \begin{cases} 1 & \text{if} \quad x \in B \\ 0 & \text{if} \quad x \notin B , \end{cases}$$

where $A_i(j) \quad (1 \le j \le r_i)$ is congruent to $A_i \quad (1 \le i \le m)$

In this case we write

$$(A_1, A_2, \dots, A_m) \mid^* B .$$

The number ν_{ij} is called the multiplicity of $A_i(j)$.

It is easy to see that $(A_1, A_2, \dots, A_m) | B$ implies $(A_1, A_2, \dots, A_m) |^* B$ and we can choose the integers ν_{ij} so that $\nu_{ij} = 1$ $(1 \le i \le m,\ 1 \le j \le r_i)$.

DEFINITION C.

We say that a parallelotop B can be filled up (covered) in a parallel manner by a given parallelotop A if we can decompose B into disjoint subsets, each of which is congruent to A and the parallel edges are equal. In this case we write $A |^P B$.

DEFINITION D.

If $a_1, a_2, \dots, a_n$ are given nonnegative integers $(\sum_{i=1}^{n} a_i > 0)$ then the lattice points $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n)$ form a knight-figure of type $a_1 \times a_2 \times \dots \times a_n$ if $|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|$ is a permutation of the integers $a_1, a_2, \dots, a_n$. The knight-figure of type $a_1 \times a_2 \times \dots \times a_n$ will be denoted by $K(a_1, \dots, a_n)$.

In Part 1 we give a necessary and sufficient condition for the validity of $(A_1, A_2, \dots, A_m) |^* B$ (Theorem 2), and for the validity of $(A_1, A_2, \dots, A_m) | B$ and $A | B$ if B is large enough (Theorem 3 and Theorem 4, resp.). Some special cases are also explained because of the simpler form of conditions (Theorem 1, 5, 6, 7). In the course of the proofs we need a generalization (Lemma 7) of the well-known marriage principle which may be interesting in itself.

In Part 2 we give a necessary and sufficient condition for the validity of $K(a,b) | B$ if B is large enough. (Theorem 12). For the case $K(a,1)$ a covering is constructed. Two simple n-dimensional generalizations (Theorem 13, 14) are also given.

1. COVERINGS WITH PARALLELOTOPS

The simplest but very interesting case is the case of parallelotop with edge-lengths $1,1,\dots,1,a$. The following theorem concerning this type is obviously a special case of general Theorem 2.

Theorem 1. [6], [7] Let A and B be n-dimensional parallelotops and the edge-lengths of A be $1,1,\dots,1,a$, then $A|B$ if and only if at least one edge of B is divisible by a.

DEFINITION 2.

If $e_1,\dots,e_n$ are natural numbers and B is an n-dimensional parallelotop, then $M(B,e_1,\dots,e_n)$ denotes the divisibility matrix: the j-th element of the i-th row is 1 if $e_i \mid b_j$ (where b_j is the j-th side of B) and 0 if $e_i \nmid b_j$.

DEFINITION 3.

We say that an $n\times n$ matrix M has not independent 0's (or 1's) if there are no n 0's (or 1's) in different rows and columns.

Theorem 2. $(A_1,A_2,\dots,A_m)|^*B$ holds if and only if, choosing in an arbitrary manner $k_i(\ge 1)$ edges of A_i, denoting by d_i their greatest common divisor and making n sets of the numbers d_i in an arbitrary manner,
(F_1) but using every d_i exactly in $n-k_i+1$ sets, finally, denoting by $e_1,\dots,e_n$ the greatest common divisors of the numbers in one set ($e_j=\infty$ if the j-th set is void), the matrix $M(B,e_1,\dots,e_n)$ has no n independent 0's.

Theorem 3. In the case of

$$b_i \ge 3^{nm\cdot 2^{nm}}\cdot a^{2^{nm}+2} \tag{5}$$

(where a is the maximum of edges of A_i's) $(A_1,\dots,A_m)|B$ holds if and only if (F_1) holds.

DEFINITION 4.

Let ε_1 and ε_2 be equal to 0 or 1. We call the logical sum of ε_1 and ε_2 the number

$$\varepsilon_1 \vee \varepsilon_2 = \begin{cases} 0 & \text{if} \quad \varepsilon_1 = \varepsilon_2 = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, if t_1 and t_2 are row vectors with 0,1 coordinates then the coordinates of the logical sum of t_1 and t_2 are the logical sums of the corresponding coordinates.

Lemma 7. Let $M_1, M_2, \ldots, M_m$ be $n \times n$ matrices with elements 0 and 1. If they have the property that choosing in an arbitrary manner $k_i \geq 1$ rows from M_i $(1 \leq i \leq m)$, denoting by w_i the logical sum of these rows and making n sets of the row vectors w_i in an arbitrary manner, but using every w_i exactly in $n-k_i+1$ sets, finally denoting by z_j $(1 \leq j \leq n)$ the logical sum of the w_i's lying in the j'th set (if the j'th set is void, then $w_i = (0,0,\ldots,0)$) the matrix formed from $z_1, z_2, \ldots, z_n$, as rows has not n independent 0's, then there is an index p $(1 \leq p \leq m)$ such that M_p has n independent 1's. (F_2)

REMARK. This lemma is a generalization of the well-known marriage principle [3], which says:

MARRIAGE PRINCIPLE. Let M be an $n \times n$ matrix with elements 0 and 1. If choosing in an arbitrary manner k rows, the number of columns containing a 1 in of these rows (or the numbers of 1's of the logical sum of these rows) $\geq k$, then M has n independent 1's.

Now we consider some interesting special cases.

Theorem 4. In the case of

$$b_i \geq 3^{n \cdot 2^n} \cdot a^{2^n + 1}$$

(where a is the maximum of edges of A)

$$A|B$$

holds if and only if choosing k $(1 \le k \le n)$ edges of A in an arbitrary manner

(F_3) their greatest common divisor d is a divisor of at least k edges of B.

The problem of de Bruijn says that

Theorem 5. A has the property "$A|B$ if and only if $A|^p B$ " if and only if from any two edges of A one of them is the divisor of the other one.

In Theorem 5 we have shown that it is true only in a special case that we can fill up something only if we can fill it up in a "regular" wav. However, we may define the term "regular" in a wider sense.

DEFINITION 5.

A filling up $A|B$ is regular if we can reach to this filling up by cuts, where cut is the operation when we divide the whole parallelotop by an $n-1$-dimensional hyperplane.

Theorem 6. $(A_1, A_2, \dots, A_m)|B$ if and only if it is possible regularly, too.

Another interesting special case of Theorem 3 is the case when we have n-dimensional cubes with relative prime edges.

Theorem 7. Let $C_1, C_2, \dots, C_m$ be n-dimensional cubes with edges $c_1, c_2, \dots, c_m$ satisfying $(c_i, c_j) = 1$ $(i \neq j)$ and let

$$b_j > 3^{nm\,2^{nm}} \cdot c^{2^{nm}+1} \qquad (1 \le j \le n)$$

where $c = \max(c_1, \dots, c_m)$. Then

$$(C_1, \dots, C_m)|B$$

holds if and only if the $m \times n$ matrix $M(B, c_1, \dots, c_m)$ has no m independent 0's.

We have obtained the following modified form of Theorem 7.

Theorem 7a. Under the condition of Theorem 7 $(C_1, \dots, C_m) | B$ if and only if

a) $m > n$

b) $m \leq n$ and there are $n-m+1$ edges of B divisible by all the numbers $c_1, \dots, c_m$, or we can fill up B by less than m of C_i's.

2. COVERINGS WITH KNIGHT FIGURES

Theorem 12. Let $(a,b) = d$.

1. If $\frac{a}{d} \equiv \frac{b}{d} \pmod 2$, then a rectangular with large enough sizes $(m \geq m_1(a,b), n \geq n_1(a,b))$ is coverable with knight-figures of type $a \times b$, if and only if m and n are divisible by $2d$.

2. If $\frac{a}{d} \not\equiv \frac{b}{d} \pmod 2$, then a rectangular with large enough sizes $(m \geq m_1(a,b), n \geq n_1(a,b))$ is coverable with knight-figures of type $a \times b$, if and only if either m or n is divisible by $2d$.

Theorem 13. An n-dimensional parallelotop R can be covered with knight-figures of type $a \times 0 \times \dots \times 0$ if and only if one of its sizes is divisible by $2a$.

Theorem 14. An n-dimensional parallelotop R can be covered with knight-figures of type $a \times \dots \times a$ if and only if its sizes are divisible by $2a$.

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