Results on intersecting families of subsets, a survey

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1 Introduction

The underlying set will be $\{1, 2, \ldots, n\}$. The family of all k-element subsets of $[n]$ is denoted by $\binom{[n]}{k}$. Its subfamilies are called *uniform*. A family $\mathcal{F}$ of some subsets of $[n]$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every pair $F, G \in \mathcal{F}$. The whole story has started with the seminal paper of Erdős, Ko and Rado [7]. Their first observation was that an intersecting family in $2^{[n]}$ can contain at most one of the complementing pairs, therefore the size of an intersecting family cannot exceed the half of the number of all subsets of $[n]$.

Observation 1 (Erdős, Ko, Rado [7]) *If $\mathcal{F} \subset 2^{[n]}$ is intersecting then $|\mathcal{F}| \leq 2^{n-1} = 2^n/2$.*

The following trivial construction shows that the bound is sharp.

Construction 1 *Take all subsets of $[n]$ containing the element 1.*

However there are many other construction giving equality in Observation 1. The following one will be interesting for our further investigations.

Construction 2 *If n is odd take all sets of size at least $\frac{n-1}{2}$. If n is even then choose all the sets of size at least $\frac{n}{2} - 1$ and the sets of size $\frac{n}{2}$ not containing the element n.*

*This research was supported by the National Research, Development and Innovation Office – NKFIH Fund No's SSN117879, NK104183 and K116769.

The analogous problem when the intersecting subsets have size exactly k, that is the case of uniform families, is not so trivial when $k \leq \frac{n}{2}$. (Otherwise all k-element subsets can be chosen.)

Theorem 1 (Erdős, Ko, Rado [7]) *If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting where $k \leq \frac{n}{2}$ then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

The original proof uses the so called *shifting method*. There is a shorter proof based on the *cycle method* in [20]. It can also be found in the books [2] and [5]. We will give an algebraic (using eigenvalues) proof in Section 4. In the case of Theorem 1 there is only one extremal construction, mimicking Construction 1.

Construction 3 *Take all subsets of $[n]$ having size k and containing the element 1.*

It is worth mentioning that the results contained in [7] were obtained in the late 1930's when all three authors worked in England, but they did not publish them because they did not think that the mathematical community would find them interesting. They sent the paper for publication only in 1960 when they realized that the "mathematical climate" has changed: Combinatorics became a science. It was a very good idea, this paper is the second most cited paper of Erdős according to MathSciNet, although the competition is tough.

We say that a family $\mathcal{F}$ is *trivially intersecting* if there is an element $a \in [n]$ such that all members of $\mathcal{F}$ contain a. Constructions 1 and 3 are trivially intersecting. Construction 2 shows that in the non-uniform case non-trivially intersecting families can be as large as the trivially intersecting one. But this seemed not to be true for the uniform families. [7] posed the problem of finding the largest k-uniform non-trivially intersecting family. It was found by Hilton and Milner.

Theorem 2 [15] *If $\mathcal{F}$ is an intersecting but not a trivially intersecting family, $\mathcal{F} \subset \binom{[n]}{k} (2k \leq n)$ then*

$$|\mathcal{F}| \leq 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

The construction giving equality is the following.

Construction 4 *Let $K = \{2, 3, \ldots, k+1\}$. The extremal family will consist of all k-element sets containing 1 and intersecting K.*

The goal of the present paper is to survey only some directions of this theory. A comprehensive survey would be a book. (Let us call the reader's attention to the forthcoming book of Gerbner and Patkós [14].) The author, of course, selected the directions according his own interest, covering his own results. There is an overlapping with the paper [21].

2 t-intersecting families

Already Erdős, Ko and Rado ([7]) considered a more general problem. A family $\mathcal{F} \subset 2^{[n]}$ is *t-intersecting* if $|F \cap G| \geq t$ holds for every pair $F, G \in \mathcal{F}$. They posed a conjecture for the maximal size of non-uniform a t-intersecting family. This conjecture was justified in the following theorem.

Theorem 3 (Katona [18]) *If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+t}{2}}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \sum_{i=\frac{n+t+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+t-1}{2}} & \text{if } n+t \text{ is odd .} \end{cases}$$

Here the generalization of Construction 1 gives only 2^{n-t}, less than the upper bound in Theorem 3 (if $t > 1$). In order to obtain a sharp construction we have to mimic Construction 2.

Construction 5 *If $n+t$ is even, take all sets of size at least $\frac{n+t}{2}$. If $n+t$ is odd then choose all the sets of size at least $\frac{n+t+1}{2}$ and the sets of size $\frac{n+t-1}{2}$ not containing the element n.*

The uniform case is harder, again. A natural trial to obtain the best construction is the obvious generalization of Construction 3.

Construction 6 *Take all subsets of $[n]$ having size k and containing $[t]$ as a subset.*

This is really t-intersecting, but [7] contains an example when this construction is not the best. Let $n = 8, k = 4, t = 2$. Divide the underlying set into two parts $[8] = X_1 \cup X_2$ where $X_1 = [4], X_2 = \{5,6,7,8\}$. Let $\mathcal{F}$ consist of all 4-element subsets F satisfying $|X_1 \cap F| \geq 3$. This family is 2-intersecting and has 17 members, while Construction 6 has only $\binom{6}{2} = 15$ in this case.

But Erdős, Ko and Rado [7] were able to prove that Construction 6 gives the largest family when n is large with respect to k. The dependence of the threshold on t is not interesting here since $1 \leq t < k$ can be supposed.

Theorem 4 [7] *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and $n \geq n(k)$ then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}. \tag{1}$$

The next step towards the better understanding of the situation was when Frankl [9] and Wilson [29] determined the exact value of the threshold $n(k)$ in Theorem 3.

Let us now consider the following generalization of the counter-example above.

Construction 7 *Choose a non-negative integer parameter i and define the family*

$$\mathcal{A}(n,k,t,i) = \{A : \ |A| = k, |A \cap [t+2i]| \geq t+i\}. \tag{2}$$

It is easy to see that $\mathcal{A}(n,k,t,i)$ is t-intersecting for each i.

Introduce the following notation:

$$\max_{0 \leq i} |\mathcal{A}(n,k,t,i)| = \mathrm{AK}(n,k,t).$$

This is the size of the best of the constructions (2). Frankl [9] conjectured that this construction gives the largest k-uniform t-intersecting family. Frankl and Füredi [11] proved the conjecture for a very large class of parameters but the full conjecture remained open until 1996 when it became a theorem.

Theorem 5 (Ahlswede and Khachatrian, [1]) *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a t-intersecting family. Then*

$$|\mathcal{F}| \leq AK(n,k,t)$$

holds.

Of course Theorem 5 has many consequences. We will exhibit only one result of us, in Section 3, where this theorem is used and plays a role even in the formulation of the statement.

3 Largest union-intersecting families and related problems

The following problem was asked by János Körner.

Let $\mathcal{F} \subset 2^{[n]}$ and suppose that if $F_1, F_2, G_1, G_2 \in \mathcal{F}, F_1 \neq F_2, G_1 \neq G_2$ holds then

$$(F_1 \cup F_2) \cap (G_1 \cup G_2) \neq \emptyset.$$

What is the maximum size of such a family?

He conjectured that the following construction gives the largest one.

Construction 8 *If n is odd then take all sets of size at least $\frac{n-1}{2}$. If n is even then choose all the sets of size at least $\frac{n}{2}$ and the sets of size $\frac{n}{2}-1$ containing the element 1.*

We solved the problem in a more general setting. A family $\mathcal{F} \subset 2^{[n]}$ is called a *union-t-intersecting* if

$$|(F_1 \cup F_2) \cap (G_1 \cup G_2)| \geq t$$

holds for any four members such that $F_1 \neq F_2, G_1 \neq G_2$.

Theorem 6 (Katona-D.T. Nagy [24]) *If $\mathcal{F} \subset 2^{[n]}$ is a union-t-intersecting family then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+t}{2}-1}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \sum_{i=\frac{n+t-1}{2}}^{n} \binom{n}{i} + AK\left(n, \frac{n+t-3}{2}, t\right) & \text{if } n+t \text{ is odd}. \end{cases}$$

The following construction shows that the estimate is sharp.

Construction 9 *If $n+t$ is even, take all the sets with size at least $\frac{n+t}{2}-1$. Otherwise choose all the sets of size at least $\frac{n+t-1}{2}$ and the sets of size $\frac{n+t-3}{2}$ following Construction 7 where $k=\frac{n+t-3}{2}$ and i chosen to maximize (2).*

Since the result contains the AK-function, it is obvious that Theorem 5 must be used in the proof of this theorem.

As before, the uniform case is more difficult. Yet, we will treat it in an even more general form. A family $\mathcal{F} \subset 2^{[n]}$ is called a (u,v)-*union-intersecting* if for different members $F_1, \ldots, F_u, G_1, \ldots, G_v$ the following holds:

$$\left(\cup_{i=1}^{u} F_i\right) \cap \left(\cup_{j=1}^{v} G_j\right) \neq \emptyset.$$

Theorem 7 (Katona-D.T. Nagy [24]) *Let* $1 \leq u \leq v$ *and suppose that the family* $\mathcal{F} \subset \binom{[n]}{k}$ *is a* (u, v)*-union–intersecting family then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + u - 1$$

holds if $n > n(k, v)$.

The following construction shows that the estimate is sharp.

Construction 10 *Take all* k*-element subsets containing the element 1, and choose* $u - 1$ *distinct sets non containing 1.*

The theorem does not give a solution for small values.

Open Problem 1 [21] *Is there an Ahlswede-Khachatrian type theorem here, too?*

A new result of Alishahi and Taherkhani [3] gave Theorem 7 a wider perspective. The *Kneser graph* $\mathrm{K}(n, k)(1 \leq k \leq \frac{n}{2})$ is the graph whose vertices are all k-element subsets of an n-element set, where two vertices are adjacent iff the corresponding sets are disjoint. Using this terminology Erdős-Ko-Rado theorem claims that the largest independent set of vertices in this graph has size $\binom{n-1}{k-1}$. In other words, if a set S of vertices of the Kneser graph $\mathrm{K}(n, k)$ induces the empty graph then $|S| \leq \binom{n-1}{k-1}$.

What is now the maximum of the size of S if it does not induce a star S_r (a graph with $r+1$ vertices, in which one vertex (the center) is adjacent to all other ones)? This was answered by Gerbner, Lemons, Palmer, Patkós, and Szécsi [13] in the following theorem. (Formulated by intersecting subsets, again.)

Theorem 8 [13] *Let* $\mathcal{F} \subset \binom{[n]}{k}$ *be a family in which no member is disjoint to* r *other members. If* $n \geq n(k, r)$ *then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Suppose now that S is such a set of vertices of the Kneser graph $\mathrm{K}(n, k)$ that it does not induce a complete bipartite graph $K_{u,v}(u \leq v)$. The maximum size of S under this condition is determined by Theorem 7 for large enough n.

But [3] solves the problem in full generality, for an arbitrary graph G instead of a complete bipartite graph. Let $\chi(G)$ be the chromatic number of G, furthermore let $\eta(G)$ be the size of the smallest color class for all proper colorings with $\chi(G)$ colors.

Theorem 9 [3] *Let S be a set of vertices of the Kneser graph $K(n,k)$ not inducing G as a subgraph. If n is large enough ($n \geq n(k,G)$) then*

$$|S| \leq \binom{n}{k} - \binom{n-\chi(G)+1}{k} + \eta(G) - 1.$$

4 Two- or more-part intersecting families

Before starting the real subject of the present section, we will give an algebraic proof of Theorem 1. This proof is spectral, based on the approach in [28] and in [4]. Before really starting the proof we have to remind the reader some known definitions and facts from the literature.

Let G be a simple graph on N vertices with adjacency matrix A. The number λ is called an *eigenvalue* of a matrix A if there is a non-zero vector x such that $Ax = \lambda x$. The vector x is the *associated eigenvector*. If I is the identity matrix of the same size then $\det(A - \lambda I)$ is a polynomial of λ. All the roots of the equation $\det(A - \lambda I) = 0$ are real. These roots are the eigenvalues and their number with multiplicities is N. Index them according to their natural ordering: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. It is known that if G is a regular graph then λ_1 is the common degree. (See e.g. [27].) If $\alpha(G)$ is the maximum number of independent vertices then (see [16], [28])

$$\alpha(G) \leq -N \frac{\lambda_N}{\lambda_1 - \lambda_N} \tag{3}$$

holds.

Proof of Theorem 1. Let $n \geq 2k$ be positive integers. The *Kneser graph* $\mathrm{K}(n,k)$ is the graph whose vertices are all k-element subsets of an n-element set, where two vertices are adjacent iff the corresponding sets are disjoint. (It was define in the previous section, we repeated here for the case, the reader did not read the whole paper.) The number of vertices of this graph is thus $N = \binom{n}{k}$, and it is known that its eigenvalues are all numbers of the form $(-1)^j \binom{n-k-j}{k-j}$, for $j \in \{0, 1, \ldots, k\}$ (See, e.g., [28] for a proof. They have different multiplicities.) In particular, the largest eigenvalue is the degree of regularity $d = \lambda_1 = \binom{n-k}{k}$ and the smallest (most negative) one is $\lambda_N = -\binom{n-k-1}{k-1} = -\frac{k}{n-k}d$. Substituting these values into (3)

$$-\binom{n}{k}\frac{-\frac{k}{n-k}d}{d+\frac{k}{n-k}d} = \binom{n-1}{k-1}$$

is obtained, finishing the elegant algebraic proof of Theorem 1. (See [28].) $\square$

The goal of this section is to consider the problem when the underlying set is partitioned into two (or more) parts X_1, X_2 and the sets $F \in \mathcal{F}$ have fixed sizes in both parts. For some motivation see [22] (Section 4). More precisely let X_1 and X_2 be disjoint sets of n_1, respectively n_2 elements. [10] considered such subsets of $X = X_1 \cup X_2$ which had k elements in X_1 and ℓ elements in X_2. The family of all such sets is denoted by

$$\binom{X_1, X_2}{k, \ell} = \binom{X_1}{k} \uplus \binom{X_2}{\ell} = \{F \subset X_1 \cup X_2 :\ |F \cap X_1| = k, |F \cap X_2| = \ell\}.$$

The construction above, taking all possible sets containing a fixed element also works here. If the fixed element is in X_1 then the number of these sets is

$$\binom{n_1-1}{k-1}\binom{n_2}{\ell},$$

otherwise it is

$$\binom{n_1}{k}\binom{n_2-1}{\ell-1}.$$

The following theorem of Frankl [10] claims that the larger one of these is the best.

Theorem 10 *Let X_1, X_2 be two disjoint sets of n_1 and n_2 elements, respectively. The positive integers k, ℓ satisfy the inequalities $2k \le n_1, 2\ell \le n_2$. If $\mathcal{F}$ is an intersecting subfamily of $\binom{X_1, X_2}{k, \ell}$ then*

$$|\mathcal{F}| \le \max\left\{\binom{n_1-1}{k-1}\binom{n_2}{\ell}, \binom{n_1}{k}\binom{n_2-1}{\ell-1}\right\}.$$

Actually his theorem is formulated for an arbitrary number of parts.

Theorem 11 [10] *Let $p \geq 2$ and suppose $n_1 \geq 2k_1, n_2 \geq 2k_2, \ldots, n_p \geq 2k_p$. Let $X_1, X_2, \ldots, X_p$ be p pairwise disjoint sets, where $|X_i| = n_i$ and let $X = \cup_{i=1}^k X_i$ be their union. Let $\mathcal{F}$ be an intersecting family of subsets of X, where each $F \in \mathcal{F}$ has exactly k_i elements in X_i for all $1 \leq i \leq p$. Then*

$$|\mathcal{F}| \leq \max_{1 \leq i \leq p} \frac{k_i}{n_i} \prod_{i=1}^{p} \binom{n_i}{k_i}.$$

Proof of Theorem 11 [10]. The (categorical) product $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph whose vertex set consists of the pairs (v_1, v_2) where v_i is a vertex of G_i and two vertices (u_1, u_2) and (v_1, v_2) are adjacent iff $\{u_1, v_1\}$ is an edge in G_1 and $\{u_2, v_2\}$ is an edge in G_2.

Let $A = (a_{ij})$ and B be $p \times r$ and $s \times t$ matrices, respectively. The *tensor product* $A \otimes B$ is an $ps \times rt$ matrix obtained by blocks of copies of B multiplied by the entries a_{ij}. It is easy to see that the adjacency matrix of $G_1 \times G_2$ is the tensor product of the respective adjacency matrices. Suppose that λ and μ are eigenvalues of A and B, respectively. That is $Ax = \lambda x$ and $By = \mu y$ hold for some non-zero vectors x and y. Then we have $(A \otimes B)(x \otimes y) = Ax \otimes By = \lambda x \otimes \mu y = \lambda\mu x \otimes y$ showing that $\lambda\mu$ is an eigenvalue associated with the eigenvector $x \otimes y$. One can see that all eigenvalues of $A \otimes B$ can be obtained in this way, see e.g. [17].

Using the (obvious extension of) our previous notation, the members of the family in the theorem are elements of

$$\binom{X_1, \ldots, X_p}{k_1, \ldots, k_p} = \biguplus_{i=1}^{p} \binom{X_i}{k_i}. \tag{4}$$

It is easy to see that the vertices of the product graph

$$\mathrm{K}(n_1, k_1) \times \mathrm{K}(n_2, k_2) \times \ldots \times \mathrm{K}(n_p, k_p) \tag{5}$$

are exactly the elements of (4). The number of vertices is $N^* = \prod_{i=1}^{p} \binom{n_i}{k_i}$. Two vertices are adjacent iff they are adjacent in every factor that is the corresponding subsets, elements of $\mathcal{F}$ are disjoint. Therefore the aim of the theorem is to determine the independence number of the graph (5).

The upper estimate (3) will be used. We made some easy remarks above, on the products of two graphs, their adjacency matrices and eigenvalues. These statements can be extended to the product of more graphs by induction. Hence the eigenvalues of the graph (5) will be products of eigenvalues of the Kneser graphs, one eigenvalue from each $\mathrm{K}(n_i, k_i)$. Therefore the largest eigenvalue of the product will be the product of the largest eigenvalues of $\mathrm{K}(n_i, k_i)$'s:

$$\lambda_1^* = d^* = \prod_{i=1}^{k} \binom{n_i - k_i}{k_i}. \tag{6}$$

The smallest eigenvalue must be negative, therefore the number of odd indices j_i in the corresponding product

$$\prod_{i=1}^{p} (-1)^{j_i} \binom{n_i - k_i - j_i}{k_i - j_i} \tag{7}$$

must be odd. If j_i is even then $\binom{n_i-k_i-j_i}{k_i-j_i}$ can be replaced by $\binom{n_i-k_i}{k_i}$ decreasing (making more negative) the product (7). If j_u and j_v are both odd then

$$(-1)^2 \binom{n_u - k_u - j_u}{k_u - j_u} \binom{n_v - k_v - j_v}{k_v - j_v}$$

can be replaced by

$$\binom{n_u - k_u}{k_u} \binom{n_v - k_v}{k_v},$$

decreasing the product, again. If all these changes are carried out, we have all j_i's 0 with one exception where it is 1. The smallest of these is the smallest eigenvalue:

$$\lambda_{N^*}^* = -\max_{1 \le i \le p} \left\{ \frac{k_i}{n_i - k_i} \right\} \prod_{i=1}^{p} \binom{n_i - k_i}{k_i} = -\max_{1 \le i \le p} \left\{ \frac{k_i}{n_i - k_i} \right\} d^*. \tag{8}$$

Substituting (6) and (8) into (3) gives:

$$\prod_{i=1}^{p} \binom{n_i}{k_i} \frac{\max_{1 \le i \le p} \left\{ \frac{k_i}{n_i - k_i} \right\}}{1 + \max_{1 \le i \le p} \left\{ \frac{k_i}{n_i - k_i} \right\}}.$$

The last factor is equal to

$$\max_{1 \le i \le p} \left\{ \frac{k_i}{n_i} \right\},$$

completing the proof of the theorem. $\square$

Theorems 10 and 11 could be formulated in such a way that the largest subfamily of (4) is one of the trivially intersecting families. It is natural to ask what is the largest non-trivially intersecting subfamily. For sake of simplicity let us first consider the case of two parts.

Take a Hilton-Milner family (Construction 4) in X_1, denote it by $\mathrm{HM}(X_1, k)$. Extend its members in all possible ways by ℓ-element subsets chosen from X_2:

$$\mathrm{HM}_1(X_1, k; X_2, \ell) = \{F \cup G : F \in \mathrm{HM}(X_1, k), G \subset X_2, |G| = \ell\}.$$

Define, similarly,

$$\mathrm{HM}_2(X_1, k; X_2, \ell) = \{F \cup G : F \subset X_1, |F| = k, G \in \mathrm{HM}(X_2, \ell)\}.$$

It was conjectured in [22] that either $\mathrm{HM}_1(X_1, k; X_2, \ell)$ or $\mathrm{HM}_2(X_1, k; X_2, \ell)$ is the largest nontrivially intersecting subfamily of $\binom{X_1, X_2}{k, \ell}$. Kwan, Sudakov and Vieira [26] showed that this is not true: there are other, "mixed" Hilton-Milner families which are better in some cases.

Fix an element $a \in X_1$, a set $A \subset X_1$ such that $a \notin A$, $|A| = k$ and a set $B \subset X_2$ such that $|B| = \ell$ and define

$$\mathrm{HM}_1^{\mathrm{mix}}(X_1, k; X_2, \ell) = \{F : |F \cap X_1| = k, |F \cap X_2| = \ell, a \in F, F \cap (A \cup B) \neq \emptyset\}.$$

$\mathrm{HM}_2^{\mathrm{mix}}(X_1, k; X_2, \ell)$ is the symmetric construction.

Theorem 12 (Kwan, Sudakov, Vieira [26]) *If both $|X_1|$ and $|X_2|$ are large enough then the largest non-trivially intersecting subfamily of $\binom{X_1, X_2}{k, \ell}$ is one of $\mathrm{HM}_1(X_1, k; X_2, \ell)$, $\mathrm{HM}_2(X_1, k; X_2, \ell)$, $\mathrm{HM}_1^{\mathrm{mix}}(X_1, k; X_2, \ell)$ and $\mathrm{HM}_2^{\mathrm{mix}}(X_1, k; X_2, \ell)$.*

Their result actually claims the analogous statement for more parts. The proof uses the shifting method.

Let us consider now the case when other sizes are also allowed that is the family consists of sets satisfying $|F \cap X_1| = k_i, |F \cap X_2| = \ell_i$ for certain pairs (k_i, ℓ_i) of positive integers. Using the notation above, we will consider subfamilies of

$$\bigcup_{i=1}^{m} \binom{X_1, X_2}{k_i, \ell_i}.$$

The generalization is however a little weaker at one point. In Theorem 10 the thresholds $2k \leq n_1, 2\ell \leq n_2$ for validity are natural. If either n_1 or n_2 is smaller then the problem becomes trivial, all such sets can be selected in $\mathcal{F}$. In the generalization below there is no such natural threshold. There will be another difference in the formulation. We give the construction of the extremal family rather than the maximum number of sets.

Theorem 13 [22] *Let X_1, X_2 be two disjoint sets of n_1 and n_2 elements, respectively. Some positive integers $k_i, \ell_i (1 \leq i \leq m)$ are given. Define $b = \max_i\{k_i, \ell_i\}$. Suppose that $9b^2 \leq n_1, n_2$. If $\mathcal{F}$ is an intersecting subfamily of*

$$\bigcup_{i=1}^{m} \binom{X_1, X_2}{k_i, \ell_i} \tag{9}$$

then $|\mathcal{F}|$ cannot exceed the size of the largest trivially intersecting family satisfying the conditions.

The family (9), in general, cannot be given in a product form. This is why the eigenvalues cannot be as easily determined as in the case above. The proof of Theorem 13 is based on the cycle method, more precisely on lemmas on direct products of cycles.

Theorems 1, (10), 11 and 13 state that the largest intersecting subfamily of a certain uniform family is a trivially intersecting one.

Open Problem 2 *Find a general sufficient condition for uniform families $\mathcal{F}$ which ensures that the largest intersecting subfamily of $\mathcal{F}$ is trivially intersecting.*

We do not even have a conjecture of this type, unlike in the case of non-uniform families. A family $\mathcal{F}$ is called *hereditry* (or *downset*) if $G \subset F \in \mathcal{F}$ implies $G \in \mathcal{F}$.

Conjecture 1 (Chvátal, [6]) *If $\mathcal{F} \subset 2^{[n]}$ is a hereditary family then its largest intersecting subfamily is a trivially intersecting one.*

Many special cases are settled, but the conjecture is still open in its full generality.

5 Minimum shadows of t-intersecting families

Suppose that $\mathcal{F}$ is a k-uniform family: $\mathcal{F} \subset \binom{[n]}{k}$. Its shadow is a $k-1$-uniform family obtained by removing single elements from the members of $\mathcal{F}$.

$$\sigma(\mathcal{F}) = \{G : \ |G| = k-1, G \subset F \text{ for some } F \in \mathcal{F}\}.$$

The shadow problem asks for the minimum of $|\sigma(\mathcal{F})|$, given n, k and $|\mathcal{F}|$. The shadow theorem ([25], [19]) determines the exact minimum for all cases, but here we give only a special case.

Theorem 14 (Special case of the Shadow Theorem, [25], [19].) *Let n, k be integers and suppose that $\mathcal{F} \in \binom{[n]}{k}$ has the size $\binom{a}{k}$ for some integer a. Then*

$$\min |\sigma(\mathcal{F})| = \binom{a}{k-1}.$$

The construction giving equality is simply $\mathcal{F} = \binom{[a]}{k}$ that is the family of all k-element subsets of an a-element part of $[n]$. (It does not depend on n.)

It is natural to ask what is the minimum of $|\sigma(\mathcal{F})|$ under the condition that $\mathcal{F}$ is t-intersecting. The old construction does not work here if $a > 2k$. The answer is somewhat disappointing: we do not know this minimum value. But we can answer a more modest question, we are able to determine the minimum of the ratio

$$\frac{|\sigma(\mathcal{F})|}{|\mathcal{F}|}.$$

Theorem 15 [18] *If $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting family, then*

$$\frac{|\sigma(\mathcal{F})|}{|\mathcal{F}|} \geq \frac{\binom{2k-t}{k-1}}{\binom{2k-t}{k}} = \frac{k-1}{k-t+1}.$$

The problem is not just for itself. The proof of Theorem 3 is based on (a more general form of) Theorem 14. This estimate is sharp. If $\mathcal{F}$ consists of all k-element subsets of a $2k-t$-element set then the size of the shadow is $\binom{2k-t}{k-1}$, the ratio is exactly the above one. In this construction however the size $|\mathcal{F}|$ of the family is "small", does not depend on n. What happens if we suppose that $|\mathcal{F}|$ is large? We have a slight improvement in this case.

Theorem 16 [23]. *If $\mathcal{F} \subset \binom{[n]}{k}$ is a t-intersecting family, $1 \leq t$ then*

$$|\sigma(\mathcal{F})| \geq |\mathcal{F}|\frac{k-1}{k-t} - c(k,t)$$

where $c(k,t)$ does not depend on n and $|\mathcal{F}|$.

This is an improvement only when $t > 1$. A better multiplicative constant cannot be expected as the following example shows.

Divide $[n]$ into two parts, X_1, X_2 where $|X_1| = 2k - t - 2, |X_2| = n - 2k + t + 2$ and define $\mathcal{F}$ as the family of all k-element sets F such that $|F \cap X_1| = k - 1, |F \cap X_2| = 1$. Here $|\mathcal{F}| = \binom{2k-t-2}{k-1}(n - 2k + t + 2), |\sigma(\mathcal{F})| = \binom{2k-t-2}{k-2}(n - 2k + t + 2) + \binom{2k-t-2}{k-1}$. Their ratio tends to $\frac{k-1}{k-t}$.

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