Around the Complete Intersection Theorem

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1 Introduction

The underlying set will be $[n] = \{1, 2, \ldots, n\}$. The family of all k-element subsets of $[n]$ is denoted by $\binom{[n]}{k}$. Its subfamilies are called *uniform*. A family $\mathcal{F}$ of some subsets of $[n]$ is called *intersecting* if $F \cap G \neq \emptyset$ holds for every pair $F, G \in \mathcal{F}$. It is easy to determine the largest (non-uniform) intersecting family in $2^{[n]}$ since at most one of the complementing pairs can be taken.

Observation 1 (Erdős, Ko, Rado [2]) *If $\mathcal{F} \subset 2^{[n]}$ is intersecting then $|\mathcal{F}| \leq 2^{n-1} = 2^n/2$.*

The following trivial construction shows that the bound is sharp.

Construction 1 *Take all subsets of $[n]$ containing the element 1.*

However there are many other construction giving equality in Observation 1. The following one will be interesting for our further investigations.

Construction 2 *If n is odd take all sets of size at least $\frac{n+1}{2}$. If n is even then choose all the sets of size at least $\frac{n}{2} + 1$ and the sets of size $\frac{n}{2}$ not containing the element n.*

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The analogous problem when the intersecting subsets have size exactly $k (k \leq \frac{n}{2})$, that is the case of uniform families, is not so trivial.

Theorem 1 (Erdős, Ko, Rado [2]) *If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting where $k \leq \frac{n}{2}$ then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

For a shorter proof see [8]. In this case there is only one extremal construction, mimicking Construction 1.

Construction 3 *Take all subsets of $[n]$ having size k and containing the element 1.*

Already Erdős, Ko and Rado, in their seminal paper considered a more general problem. A family $\mathcal{F} \subset 2^{[n]}$ is *t-intersecting* if $|F \cap G| \geq t$ holds for every pair $F, G \in \mathcal{F}$. They posed a conjecture for the maximal size of non-uniform a t-intersecting family. This conjecture was justified in the following theorem.

Theorem 2 (Katona [7]) *If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+t}{2}}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \sum_{i=\frac{n+t+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+t-1}{2}} & \text{if } n+t \text{ is odd}\ . \end{cases}$$

Here the generalization of Construction 1 gives only 2^{n-t}, less than the upper bound in Theorem 2 (if $t > 1$). In order to obtain a sharp construction we have to mimic Construction 2.

Construction 4 *If $n+t$ is even, take all sets of size at least $\frac{n+t}{2}$. If $n+t$ is odd then choose all the sets of size at least $\frac{n+t+1}{2}$ and the sets of size $\frac{n+t-1}{2}$ not containing the element n.*

2 The Complete Intersection Theorem

The problem of t-intersecting families for the uniform case proved to be much more difficult than for the non-uniform case. Erdős, Ko and Rado were able to settle the problem when n is large with respect to k. The dependence of the threshold on t is not interesting here since $1 \leq t < k$ can be supposed.

Theorem 3 [2] *If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting and $n > n(k)$ then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}. \tag{1}$$

They also gave an example for small n when (1) does not hold. Let n be divisible by 4, $k = \frac{n}{2}$ and $t = 2$. The family

$$\mathcal{F} = \left\{ F : \ |F| = \frac{n}{2}, \left| F \cap \left[\frac{n}{2}\right] \right| \geq \left[\frac{n}{4}\right] + 1 \right\}$$

is 2-intersecting, since any two members meet in at leats two elements in $\left[\frac{n}{2}\right]$. On the other hand the size of this family is more than $\binom{n-2}{n/2-1}$, if $n > 4$. (See e.g. $n = 8$.) They believed that this construction was optimal.

The next step towards the better understanding the situation was when Frankl [4] and Wilson [10] determined the exact value of the threshold $n(k)$ in Theorem 3.

Let us now consider the following generalization of the construction above.

Construction 5 *Choose a non-negative integer parameter i and define the family*

$$\mathcal{A}(n, k, t, i) = \{A : \ |A| = k, |A \cap [t + 2i]| \geq t + i\}. \tag{2}$$

It is easy to see that $\mathcal{A}$ is t-intersecting for each i.

Introduce the following notation:

$$\max_{0 \leq i} |\mathcal{A}(n, k, t, i)| = \mathrm{AK}(n, k, t).$$

This is the size of the best of the constructions (2). Frankl [4] conjectured that this is construction gives the largest k-uniform t-intersecting family. Frankl and Füredi [5] proved the construction for a very large class of parameters but the full conjecture remained open until 1996 when it became a theorem.

Theorem 4 (**The Complete Intersection Theorem, Ahlswede and Khachatrian** [1]) *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a t-intersecting family. Then*

$$|\mathcal{F}| \leq AK(n, k, t)$$

holds.

This theorem was a very important step in the progress of the Extremal Set Theory. Its proof was a far-reaching generalization of the transformation method introduced in [2]. The author of the present paper must confess that he had mixed feeling when he learned about the result. On the one hand he was happy that a new important result of the theory came into life. On the other hand, however, he was a little disappointed because he had the plan to solve the conjecture later when he had time to devote all his energies to the solution.

Of course Theorem 4 has many consequences. We will exhibit only one new result of us, in Section 4, where this theorem is used and plays a role even in the formulation of the statement.

3 An Open Problem

Even the best theorems do not stop the progress in science. In contrary, they raise new questions. Let us show one.

If $\mathcal{F} \subset 2^{[n]}$ is a family of subsets, let $p_i(\mathcal{F})$ denote the number of i-element members of $\mathcal{F}$, that is, $p_i(\mathcal{F}) = \left|\mathcal{F} \cap \binom{[n]}{i}\right|$. Then the vector $p(\mathcal{F}) = (p_0(\mathcal{F}), p_1(\mathcal{F}), \ldots, p_n(\mathcal{F}))$ is called the *profile vector* of $\mathcal{F}$.

Take all profile vectors of t-intersecting families. They will form a set of points with integer coordinates in the $n+1$-dimensional Euclidean space. The vertices of the convex hull of this set of points are called the *extreme points* of the class of t-intersecting families. If some sets are deleted from a t-intersecting family then the remaining family will also be t-intersecting. Hence if $(p_0, p_1, \ldots, p_n)$ is the profile vector of a t-intersecting family and $q_i \leq p_i$ holds then $(q_0, q_1, \ldots, q_n)$ is also a profile vector of a t-intersecting family. An extreme point $(p_0, p_1, \ldots, p_n)$ is called *essential* if there is no other essential point $(r_0, r_1, \ldots, r_n)$ satisfying $p_i \leq r_i$ for all i. Let $E_n(t)$ denote the set of essential extreme points of the set of profile vectors of all t-intersecting families.

It is easy to see that if $\alpha_j \geq 0$ are fixed constants then

$$\max_{\mathcal{F} \text{ is } t\text{-intersecting}} \sum_{i=0}^{n} \alpha_j p_j(\mathcal{F}) \tag{3}$$

is attained for at least one essential extreme point. Therefore if we want to determine the maximum in (3) it is sufficient to calculate the linear combination of each of the vectors in $E_n(t)$ with the given α_js and find the largest one among these values. Observe that if the coefficients are all zero except for a fixed k for which $\alpha_k = 1$ then (3) gives the size of the largest k-uniform t-intersecting family. On the other hand if the coefficients $\alpha_i = 1 (0 \leq i \leq n)$ are taken then (3) gives the total number of sets in the family.

The essential extreme points were determined for the case $t = 1$ in [3] (Theorem 6). (For an easier treatment of the theory see the paper of Gerbner [6].) We have no place to give the full form of the statement of this theorem. But it is easy to check that if $k \leq \frac{n}{2}$ then the largest kth coordinate in the essential extreme points is $\binom{n-1}{k-1}$ giving Theorem 1. On the other hand, calculating the sums of the coordinates of the essential extreme points we obtain the formula in Observation 1.

Open problem 1 *Determine the essential extreme points of the t-intersecting families ($t > 1$).*

Of course we know some of the extreme points. The one that maximizes the linear combination $\sum_{i=0}^{n} p_i(\mathcal{F})$. It is determined by Construction 4 for Theorem 2. This point is the "farthest" one from the origin. The difficulty lies in the determination of the extreme points near the axes. Yet, the extreme points along the axes are given by Theorem 4 and Construction 5.

4 Union-intersecting families

The following problem was asked by János Körner.

Let $\mathcal{F} \subset 2^{[n]}$ and suppose that if $F_1, F_2, G_1, G_2 \in \mathcal{F}, F_1 \neq F_2, G_1 \neq G_2$ holds then

$$(F_1 \cup F_2) \cap (G_1 \cup G_2) \neq \emptyset.$$

What is the maximum size of such a family?

He conjectured that the following construction gives the largest one.

Construction 6 *If n is odd then take all sets of size at least $\frac{n-1}{2}$. If n is even then choose all the sets of size at least $\frac{n}{2}$ and the sets of size $\frac{n}{2} - 1$ containing the element 1.*

We solved the problem in a more general setting. A family $\mathcal{F} \subset 2^{[n]}$ is called a *union-t-intersecting* if

$$|(F_1 \cup F_2) \cap (G_1 \cup G_2)| \geq t$$

holds for any four members such that $F_1 \neq F_2, G_1 \neq G_2$.

Theorem 5 (Katona-D.T. Nagy [9]) *If $\mathcal{F} \subset 2^{[n]}$ is a union-t-intersecting family then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+t}{2}-1}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \sum_{i=\frac{n+t-1}{2}}^{n} \binom{n}{i} + AK\left(n, \frac{n+t-3}{2}, t\right) & \text{if } n+t \text{ is odd .} \end{cases}$$

The following construction shows that the estimate is sharp.

Construction 7 *If $n+t$ is even, take all the sets with size at least $\frac{n+t}{2} - 1$. Otherwise choose all the sets of size at least $\frac{n+t-1}{2}$ and the sets of size $\frac{n+t-3}{2}$ following Construction 5 where $k = \frac{n+t-1}{2}$ and i chosen to maximize (3).*

Since the result contains the AK-function, it is obvious that Theorem 4 must be used in the proof of this theorem.

As before, the uniform case is more difficult. Yet, we will treat it in an even more general form. A family $\mathcal{F} \subset 2^{[n]}$ is called a (u, v)-*union-intersecting* if for different members $F_1, \ldots, F_u, G_1, \ldots, G_v$ the following holds:

$$\left(\cup_{i=1}^{u} F_i\right) \cap \left(\cup_{j=1}^{v} G_j\right) \neq \emptyset.$$

Theorem 6 (Katona-D.T. Nagy [9]) *Let $1 \leq u \leq v$ and suppose that the family $\mathcal{F} \subset \binom{[n]}{k}$ is a (u, v)-union–intersecting family then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} + u - 1$$

holds if $n > n(k, v)$.

The following construction shows that the estimate is sharp.

Construction 8 *Take all k-element subsets containing the element 1, and choose $u - 1$ distinct sets non containing 1.*

The theorem does not give a solution for small values.

Open problem 2 *Is there an Ahlswede-Khachatrian type theorem here, too?*

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