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How many sums of vectors can lie in a circle of radius $\sqrt{2}$

by

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INTRODUCTION

Let $a_1, \dots, a_n$ be real numbers with the property $|a_i| \ge 1$ $(1 \le i \le n)$. Erdős [1] asked, what is the maximum number of sums $\sum_{i=1}^{n} \varepsilon_i a_i$ which can lie in an open interval of length h, where $\varepsilon_i = 0$ or 1. He proved, that this number is $\le$ sum of the largest binomial coefficients of order n. The example $a_i = 1$ $(1 \le i \le n)$ shows that this estimation is the best possible. Kleitman [2] and Katona [3] independently proved the same for two dimensions and for $h = 1$:

If $a_1, \dots, a_n$ are two-dimensional vectors such that $|a_i| \ge 1$, then at most $\binom{n}{[n/2]}$ sums $\sum_{i=1}^{n} \varepsilon_i a_i$ can lie in an open circle with unit diameter. Now we consider the case of diameter $\sqrt{2}$.

THEOREM 1. If $a_1, \dots, a_n$ are two-dimensional vectors with the property $|a_i| \ge 1$ $(1 \le i \le n)$, then at most $\binom{n}{[n/2]} + \binom{n}{[n/2]+1}$ sums $\sum_{i=1}^{n} \varepsilon_i a_i$ can lie in a circle of diameter $\sqrt{2}$, where $\varepsilon_i = 0$ or 1.

In the proof we use a Sperner type theorem, which is formulated in a more general language. The method of the proof is what we used in [4] and [5].

DEFINITIONS AND THEOREM 2

Let G be a partially ordered set with rank function. The i-th level is the set of elements $g \in G$ with rank $r(g) = i$. A chain of length h is a sequence $g_1, \dots, g_h \in G$, where $r(g_{i+1}) = r(g_i)+1 \quad (1 \le i \le h)$.

A chain is symmetrical if $r(g_1) + r(g_h) = n$, where

$$n = \max_{g \in G} r(g).$$

We say that a partially ordered set is a symmetrical chain set if we can split G into disjoint symmetrical chains. (It is defined in [4] under a different name.) It is easy to see, that the partially ordered set of the subsets of a finite set S is a partially ordered set with rank function $r(A) = |A| \quad$ ($|A|$ is the number of elements of A.)

If G and H are partially ordered sets, then the direct sum $G+H$ is the set of ordered pairs (g,h), $g \in G$, $h \in H$, with the ordering $(g_1,h_1) < (g_2,h_2)$ iff $g_1 \le g_2$ and $h_1 \le g_2$, but $(g_1,h_1) \ne (g_2,h_2)$. If G and H has rank function r and s, respectively, then we can define a rank function on $G+H$ as follows:

$$t((g,h)) = r(g) + s(h).$$

It is easy to see if G is the partially ordered set of the subsets of a set S_1 and H is the same of a set S_2 $(S_1 \cap S_2 = \emptyset)$, then $G+H$ is the partially ordered set of the subsets of $S_1 \cup S_2$.

Now we can formulate the following theorem:

THEOREM 2. Let G and H be symmetrical chain sets. If we have a set $(p_1,q_1), \dots, (p_m,q_m)$ of the elements of $G+H$, satisfying the following conditions:

no two different ones of them satisfy the conditions

$$p_i = p_j, \quad q_i < q_j, \quad s(q_i) < s(q_j) - 1$$

(C_1) or

$$p_i < p_j, \quad q_i = q_j, \quad r(p_i) < r(p_j) - 1,$$

no four different ones of them satisfy the conditions

(C_2)
$$p_i = p_j, \quad q_\ell = q_i, \quad p_k = p_\ell, \quad q_k = q_j,$$

$$q_i < q_j, \quad p_\ell > p_i, \quad q_k > q_\ell, \quad p_k > p_j,$$

then

$$m \le M_{[\frac{n}{2}]} + M_{[\frac{n}{2}]+1},$$

where M_i denotes the number of elements of the i-th level of $G+H$ and $n = \max_{(g,h)\in G+H} t((g,h))$. The estimation is the best possible.

PROOFS

The PROOF OF THEOREM 2 follows the ideas of the proof of the theorem in [4] and of Theorems 2, 3 in [5].

By the definition of the symmetrical chain sets, G and H are divisible into disjoint symmetrical chains. Denote by G' and H' the partially ordered sets which have ordering relations only along these chains, that is $g_1 < g_2$ can hold only if g_1 and g_2 lie on the same chain. Thus, the set of relations in $G'(H')$ is a part of that in $G(H)$. It follows that the set of relations in $G'+H'$ is a part of that in $G+H$. So, it is sufficient to prove the statement of the theorem for $G'+H'$ instead of $G+H$. However, the direct sum of two chains $g_0, \dots, g_a$ and $h_0, \dots, h_b$ is a rectangular lattice of pairs (g_i, h_j), where $(g_i, h_j) < (g_\ell, h_k)$ iff $i \le \ell$, $j \le k$ but $(i,j) \ne (\ell, k)$.

We will prove in the Lemma, that the maximum number of the elements of such a rectangular, under conditions (C_1) and (C_2) is the number of elements of the two maximal levels, that is the number of pairs (g_i, h_j) with

(1) $$i+j = \left[\frac{a+b}{2}\right]$$

and

(2) $$i+j = \left[\frac{a+b}{2}\right]+1 .$$

However, by the symmetricity of the chains

(3) $$n_1 = r(g_0) + r(g_a) = 2r(g_0)+a$$
$$r(g_0) = \frac{n_1-a}{2}$$

and

(4) $$s(h_0) = \frac{n_2-b}{2}$$

follow, where $n_1 = \max_{g\in G} r(g)$ and $n_2 = \max_{h\in H} s(h)$.

Thus, in case (1), using (3) and (4) we get

$$t((g_i, h_j)) = r(g_0)+s(h_0)+i+j =$$
$$= \frac{n_1-a}{2} + \frac{n_2-b}{2} + \left[\frac{a+b}{2}\right] = \left[\frac{n_1+n_2}{2}\right]$$

Similarly, in the case (2)

$$t((g_i, h_j)) = \left[\frac{n}{2}\right] + 1$$

holds. So, if we choose the elements in a given maximal way from every rectangle, then we obtain elements of the $\left[\frac{n}{2}\right]$-th and $\left[\frac{n}{2}\right]+1$-th levels. It is easy to see, that we obtain every element of $M_{\left[\frac{n}{2}\right]}$ and $M_{\left[\frac{n}{2}\right]+1}$ in this way. This completes the proof.

LEMMA. Let R be the set of pairs (i,j) $(1\le i\le a,$ $1\le j\le b,$ a,b,i,j integers), and $(p_1,q_1),\dots,(p_m,q_m)$ a subset of it, such that

no two different ones of them satisfy the conditions

$$p_i = p_j , \quad q_i < q_j - 1$$

(C_1') or

$$p_i < p_j - 1 , \quad q_i = q_j ;$$

no four different ones of them satisfy the conditions

(C_2') $$p_i = p_j , \quad q_\ell = q_i , \quad p_k = p_\ell , \quad q_k = q_j$$

then the maximal m is given by the set of pairs satisfying

$$i+j = \left[\frac{a+b}{2}\right] \quad \text{or} \quad i+j = \left[\frac{a+b}{2}\right] + 1 .$$

PROOF. If $a \neq b$, assume $a > b$. By (C_1') at most two (p_i, q_i) can lie in every row (a row is the subset of elements in R with fixed second coordinate), thus the maximum is at most $2b$, however, it is easy to see that the maximal set given in the Lemma has exactly $2b$ element.

But if $a = b$, the given maximal set has $2a-1$ elements. We have to prove that we can not have $2a$ elements. We prove it in an indirect way. Let $(p_1, 0)$ and $(p_2, 0)$ be the elements chosen of the first row. We have two elements in the p_1-th column. By (C_1'), it must be $(p_1, 1)$. Similarly, we get $(p_2, 1)$, too. However, $(p_1, 0), (p_2, 0), (p_1, 1), (p_2, 1)$ form a configuration excluded in $(C_2($ in contradiction by our assumption. The proof is completed.

Let us return now to the PROOF OF THEOREM 1.

It is easy to see that we can reduce the problem to the case the first coordinates of the vectors are non-negative (transformating to $\varepsilon_i = \pm 1$, multiplying some a_i's by -1, and retransformating to $\varepsilon_i = 0,1$). Let S_1 and S_2 be the set of vectors a_i with nonnegative and negative second coordinates, respectively. We shall use Theorem 2 for G = partially ordered set of subsets of S_1 and H = partially ordered set of subsets of S_2. Let us fix an open circle of diameter $\sqrt{2}$ and consider the sums lying in the circle. We may correspond with each such sum a subset of $S_1 \cup S_2$, the subset of a_i's which have coefficient 1. We have only to verify, that the family of these subsets satisfies (C_1) and (C_2).

Indeed, if two different subsets $(p_i, q_i) = P_i \cup Q_i$, $(p_j, q_j) = P_j \cup Q_j$ of $S_1 \cup S_2$ satisfy e.g.

$$P_i = P_j, \qquad Q_i \subset Q_j, \qquad |Q_i| < |Q_j| - 1, \tag{5}$$

then for the corresponding sums

$$\begin{aligned} \Big| \sum_{u=1}^{n} \varepsilon_u a_u - \sum_{u=1}^{n} \varepsilon'_u a_u \Big| &= \Big| \sum_{u \in P_j \cup Q_j} a_u - \sum_{u \in P_i \cup Q_i} a_u \Big| = \\ &= \Big| \sum_{u \in Q_j - Q_i} a_u \Big| \end{aligned} \tag{6}$$

holds. The members of the sum are vectors with nonnegative first and negative second coordinates and with absolute value ≥ 1. The number of members is at least 2 by (5). It is easy to see, that the sum of such vectors has absolute value $\geq \sqrt{2}$. Thus the difference (6) is at least $\sqrt{2}$ which contradicts our assumption that both sums lie in the same open circle of diameter $\sqrt{2}$. The proof of holding of (C_1) is completed.

In order to prove the same for (C_2) we have to show that in the case

$$\begin{aligned} &P_i = P_j, \quad Q_\ell = Q_i \quad P_k = P_\ell \quad Q_k = Q_j \\ &Q_i \subset Q_j, \quad P_\ell \supset P_i, \quad Q_k \supset Q_\ell, \quad P_k \supset P_j \qquad \text{(proper subsets)} \end{aligned} \tag{7}$$

at least two of the sums

$$\sum_{u \in P_i \cup Q_i} a_u, \quad \sum_{u \in P_j \cup Q_j} a_u, \quad \sum_{u \in P_\ell \cup Q_\ell} a_u, \quad \sum_{u \in P_k \cup Q_k} a_u \tag{8}$$

differ with at least $\sqrt{2}$. The difference of the 4-th and 1-st sum is

$$\sum_{u \in (P_k - P_i) \cup (Q_k - Q_i)} a_u = \sum_{u \in P_\ell - P_i} a_u + \sum_{u \in Q_j - Q_i} a_u = v_2 + v_1 .$$

The difference of the 3-rd and 2-nd sum is

$$\sum_{u\in P_\ell - P_j} a_u - \sum_{u\in Q_j - Q_\ell} a_u = \sum_{u\in P_\ell - P_i} a_u - \sum_{u\in Q_j - Q_i} a_u = v_2 - v_1 .$$

Here v_1 (and v_2) is a (nonvoid) sum of vectors lying in the same quadrant, with absolute values ≥ 1. Thus $|v_1| \geq 1$, $|v_2| \geq 1$.

If the angle of v_1 and v_2 is $\leq \frac{\pi}{4}$, then $|v_1+v_2| \geq \sqrt{2}$, conversely, if the angle $\geq \frac{\pi}{4}$ then $|v_2 - v_1| \geq \sqrt{2}$. In other words there are always two sums in (8) with absolute difference $\geq \sqrt{2}$. They cannot lie in the same circle what contradicts our supposition.

The conditions (C_1) and (C_2) are really satisfied in this case. We may apply Theorem 2, thus, the two middle levels of the partially ordered set of subsets of $S_1 \cup S_2$ give an optimal set. The two middle levels consists of the subsets with elements $\left[\frac{n}{2}\right]$ or $\left[\frac{n}{2}\right] + 1$. The number of such subsets is $\binom{n}{[n/2]} + \binom{n}{[n/2]+1}$.

The proof is completed.

CONCLUDING REMARKS

The estimation of Theorem 1 is the best possible in the sense that for $a_i = 1$ $(1 \leq i \leq n)$ the maximum is attained.

It is also true that the theorem does not hold for a larger number instead of $\sqrt{2}$, since the vectors $(0,1)$ and $(1,0)$ would provide a counterexample. However we have the following

CONJECTURE. Theorem 1 holds with 2 instead of $\sqrt{2}$ in the case $n > 2$.

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