Color the cycles

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Abstract

The cycles of length k in a complete graph on n vertices are colored in such a way that edge-disjoint cycles get distinct colors. The minimum number of colors is asymptotically determined.

Key Words: cycle, coloring, Turán type problem.

1 Introduction

Both authors have participated in the series of conferences *Cycles and Colorings* and decided that they should pose a nice new problem to its honor, that contains both concepts in the title. Of course there are other problems and results involving colors and cycles but as far as we know they are very different from the one we suggest below.

Let K_n be the complete graph on n vertices and denote the family of all cycles of length k in K_n by $\mathcal{C}(n,k)$ $(3 \le k \le n)$. Color the elements of $\mathcal{C}(n,k)$

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in such a way that the colors of two edge-disjoint cycles must be different. This is called a *proper coloring* of $\mathcal{C}(n,k)$. The minimum number of colors in a proper coloring is denoted by $\chi(n,k)$. Another way to define this number is the following: partition $\mathcal{C}(n,k)$ into the minimum number of classes in such a way that the cycles in each class are pairwise edge-intersecting.

The goal of the present paper is to determine this number. We were fully successful in the case of $k=3$ while our results for larger k are only asymptotic (fixed k, large n).

The problem is closely related, as we will see, to Turán type results. The basic problem of this area is to determine the maximum number of edges of a graph on n vertices containing no K_k as a subgraph. The Turán graph $T(n,k)$ has n vertices that are partitioned into k nearly equal classes (the sizes have difference at most one) and two vertices are joined by an edge if and only if they are in different classes. It is easy to see that $T(n,k-1)$ contains no K_k as a subgraph. The fundamental theorem of the area is the following one.

Theorem 1 (Turán [20], for $k=3$: Mantel [12]) *$T(n,k-1)$ has the largest number of edges among the graphs with n vertices containing no complete graph on k vertices.*

Several generalizations and extensions of the Turán theorem needed to our proofs will be presented later.

2 The results

Let us first show a construction giving $\chi(n,3)$. Take the complement $\overline{T}(n,2)$ of the Turán graph for $k=3$, that is, the vertex-disjoint union of two complete graphs with $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices, respectively. It is easy to see that this graph has $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ edges and contains no empty triangle (3 independent vertices). Color the edges of $\overline{T}(n,2)$ with $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ distinct colors. Color a triangle $C_3 \in \mathcal{C}(n,3)$ in K_n with the color of the edge contained in it. In this way some cycles of length 3 receive 3 colors. Delete 2 of them in an arbitrary way. It is easy to see that this is a proper coloring of $\mathcal{C}(n,3)$. Our first theorem claims that this is the best construction.

Theorem 2 *If $n \geq 16$ then $\chi(n,3) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$.*

The *Turán number* $\mathrm{ex}(n, H)$ of a "small" graph H is the maximum number of edges in a graph with n vertices containing no H as a subgraph. We will now generalize the construction for $k = 3$. Let $G(n, k)$ be a graph on n vertices containing no C_k. Then its complement $\overline{G}(n, k)$ has the following important property: if $C_k \in \mathcal{C}(n, k)$ then it has a common edge with $\overline{G}(n, k)$. (Of course the vertex sets of $G(n, k)$ and $\mathcal{C}(n, k)$ are the same.) Color the edges of $\overline{G}(n, k)$ with distinct colors. Let the color of the edge e be $c(e)$. If e is a common edge of the cycle C_k and $\overline{G}(n, k)$ then color the cycle with $c(e)$. In this way each cycle of length k receives at least one color. If it receives more than one, choose one arbitrarily. This is a proper coloring using as many colors as the number of edges of $\overline{G}(n, k)$. The following proposition is obtained.

Proposition 1 $\chi(n, k) \leq \binom{n}{2} - ex(n, C_k)$.

If $k = 3$ then $\mathrm{ex}(n, C_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$ by Theorem 1. Proposition 1 gives $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ as an upper estimate on $\chi(n, 3)$. It will be proved in Section 3 that this estimate is sharp.

We will show that the upper estimate is asymptotically sharp for other values of k. Since the asymptotical behavior of $\mathrm{ex}(n, C_k)$ is very different for even and odd k's, the meaning of the "asymptotics" largely depends on the parity of k. The reason is the following theorem.

Theorem 3 (Erdős-Stone-Simonovits [7], [5]) *For a fixed graph H*

$$\lim_{n\to\infty} \frac{ex(n, H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H) - 1}$$

where $\chi(H)$ is the chromatic number of H.

If k is odd then $\chi(C_k) = 3$, the theorem above implies that $\mathrm{ex}(n, C_k) \sim \frac{1}{4}n^2$, hence the proposition gives the asymptotical lower bound $\frac{1}{4}n^2$. This is completed by the following theorem.

Theorem 4 *If k is odd then*

$$\lim_{n\to\infty} \frac{\chi(n, k)}{\binom{n}{2}} = \frac{1}{2}.$$

For $H = C_k$ (k is odd) there is a theorem stronger than Theorem 3.

Theorem 5 (Simonovits [18]) *If k is odd, $n \geq n_0(k)$ then $ex(n, C_k) = \left\lfloor \frac{n^2}{4} \right\rfloor$.*

This statement encourages us to pose the following conjecture.

Conjecture 1 *If k is odd, $n \geq n_1(k)$ then $\chi(n, k) = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ holds.*

The situation is very different for even k. In that case Theorem 3 does not determine the exponent of n in $\mathrm{ex}(n, C_k)$ it tells only that it is less than 2. Then the second term in Proposition 1 is negligible in comparison with the first term, hence the asymptotic value of $\chi(n, k)$ is the first term.

Theorem 6 *If $k \geq 4$ is even then*

$$\lim_{n \to \infty} \frac{\chi(n, k)}{\binom{n}{2}} = 1$$

holds.

3 Proof of Theorem 2

We have seen that Proposition 1 implies that

$$\chi(n, 3) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor.$$

To prove the other direction an old theorem will be used.

Theorem 7 (Nordhaus and Steward [15], Moon and Moser [13]) *Let $G = (V, E)$ be a graph where $|V| = n, |E| = m$. The number of copies of C_3 in G is at least*

$$\frac{4}{3}\frac{m^2}{n} - \frac{mn}{3}.$$

The other ingredient of the proof is the following easy lemma. A family of triangles is called *edge-intersecting* or shortly *intersecting* if any two members have an edge in common. It is *trivially intersecting* or *book* if the intersection of all of them contains an edge. The edge in the intersection of the triangles of a book is called its *spine*. (If the book consists of one triangle then choose one of its edges as a spine.) On the other hand if the vertices of all triangles in the family are subsets of a set of 4 elements, and the number of triangles is at least 3 then we say that the family is a *quadruplet*.

Lemma 1 *A family of intersecting triangles (C_3's) is either a book or a quadruplet.*

Proof. Choose two triangles from the intersecting family. They have a common edge e, so their edge sets are, say, $\{a, b, e\}$ and $\{c, d, e\}$. If the family is not trivially intersecting then there is a third triangle containing one edge from $\{a, b\}$ and one from $\{c, d\}$. Suppose that it is $\{a, c, f\}$. It is easy to see that if the family has a fourth member it can only be the triangle $\{b, d, f\}$. The 4 endpoints of these 6 edges cover all 3 or 4 triangles therefore it is a quadruplet. $\boxed{\text{L}}$

Consider a proper coloring of $\mathcal{C}(n, 3)$. By Lemma 1 the color classes are either books or quadruplets. Let their numbers be s and q, respectively. If $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor \leq s$, we are done, so

$$s < \left\lfloor \frac{(n-1)^2}{4} \right\rfloor \tag{1}$$

can be supposed. Let G be the graph with n vertices and the spines of the books as edges. $\overline{G}$ has $\binom{n}{2} - s$ edges. Using Theorem 10 for $\overline{G}$ we obtain that it contains at least

$$\frac{4}{3} \cdot \frac{\left(\binom{n}{2} - s\right)^2}{n} - \frac{\left(\binom{n}{2} - s\right) n}{3} \tag{2}$$

triangles. G contains at least this many empty triangles. These triangles do not belong to any of the books. They must belong to quadruplets. Since a quadruplet contains at most 4 triangles, the number of quadruplets is at least one fourth of (2). Therefore the total number of color classes is at least

$$s + \frac{1}{3} \frac{\left(\binom{n}{2} - s\right)^2}{n} - \frac{\left(\binom{n}{2} - s\right) n}{12}. \tag{3}$$

We have to prove that (3) is at least $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$. To make the algebra easier, introduce the notation $A = \binom{n}{2} - s$. Equation (1) implies

$$\left\lfloor \frac{n^2}{4} \right\rfloor < A. \tag{4}$$

The desired inequality, by (3), becomes

$$\left\lfloor \frac{(n-1)^2}{4} \right\rfloor \leq \binom{n}{2} - A + \frac{A^2}{3n} - \frac{An}{12}$$

or equivalently

$$0 \le \left\lfloor \frac{n^2}{4} \right\rfloor + \frac{A^2}{3n} - \frac{An}{12} - A. \tag{5}$$

For fixed n the right hand side as a function of A takes it minimum at $A = \frac{n^2+12n}{8}$. If $12 \le n$ holds then this is less than or equal to our range of A by (4). Hence the right hand side of (5) is at least

$$\frac{n^2-1}{4} + \frac{\left(\frac{n^2+3}{4}\right)^2}{3n} - \frac{\left(\frac{n^2+3}{4}\right)n}{12} - \left(\frac{n^2+3}{4}\right) = \frac{3n^2-48n+9}{48n},$$

that is positive for $n \ge 16$.

$\boxed{\text{T}}$

4 Proof of Theorem 4

All of our other proofs follow the pattern of the proof of Theorem 2. The upper bounds are consequences of Proposition 1. The proofs of the lower bounds use the following idea: it is known from certain theorems that if the number of edges in a graph is more than the Turán number $\mathrm{ex}(n, C_k)$ (the graph is *oversaturated*) then there are many copies of C_k in the graph. On the other hand it will be shown that if a color class is not a *book* (a collection of k-element sets sharing one pair of elements) then its size has a smaller order of magnitude.

The following theorem was proved in a much more general context, we state it only in the special case needed here. Let $h_k(n,t)$ denote the minimum number of copies of C_k in a graph with n vertices and $\mathrm{ex}(n, C_k) + t$ edges.

Theorem 8 (Mubayi [14]) *Let $k \ge 3$ be an odd integer. Then there is a constant $\alpha_k > 0$ such that if the number of edges in a graph with n vertices is at least $ex(n, C_k) + t$ where $0 < t \le \alpha_k n$, and $n > n(k)$ then*

$$h_k(n,t) \ge tc(n, C_k)$$

where

$$c(n, C_k) = \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1 \right) \ldots \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \left(\left\lceil \frac{n}{2} \right\rceil - 2 \right) \ldots \left(\left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{k}{2} \right\rfloor \right)$$

and the second product is interpreted as empty for $k = 3$.

For quadratic t another well-known theorem will be used what is an easy consequence of Szemerédi's regularity lemma [19]. First implicitly used in [16], for a more general version see [2].

Theorem 9 (Graph Removal Lemma) *Let F be a graph with f vertices. Suppose that an n vertex graph H has at most $o(n^f)$ copies of F. Then there is a set of edges of H of size $o(n^2)$ whose removal from H results in a graph with no copy of F where $n \to \infty$.*

It will be actually used in the following usual way. Given a real number $\varepsilon > 0$ there is another real number $\delta(\varepsilon) > 0$ such that if the number of copies of F in H is at most $\delta(\varepsilon)n^f$ then one can delete at most εn^2 edges removing all copies of F.

Using these theorems the following lemma can be proved that ensures the necessary number of copies of C_k for supersaturated graphs. It is probably well-known, but we could not find it in the literature in the present form.

Lemma 2 *Let $k \geq 3$ be an odd integer. There is a threshold $n(k, N)$ (not depending on t) such that*

$$\frac{h_k(n,t)}{tn^{k-3}} \geq N \qquad \left(1 \leq t \leq \binom{n}{2} - ex(n, C_k)\right)$$

holds for every $n > n(k, N)$.

Proof. Three cases will be distinguished.

If $t \leq \alpha_k n$ then Theorem 8 implies that

$$\frac{h_k(n,t)}{t} \geq c(n, C_k) \geq \beta_k n^{k-2}$$

holds for some constant β_k. Hence

$$\frac{h_k(n,t)}{tn^{k-3}} \geq \beta_k n$$

follows where the right hand side is at least N if $n \geq \frac{N}{\beta_k}$.

If $\alpha_k n < t \leq \frac{\alpha_k \beta_k}{2N} n^2$ then the same theorem gives

$$\frac{h_k(n,t)}{t} \geq \frac{h_k(n, \lfloor \alpha_k n \rfloor)}{\frac{\alpha_k \beta_k}{2N} n^2} \geq \frac{\lfloor \alpha_k n \rfloor}{\frac{\alpha_k \beta_k}{2N} n^2} \beta_k n^{k-2} \geq N n^{k-3},$$

proving the desired inequality in this case, if $n \geq \frac{2}{\alpha_k}$.

Finally suppose $t > \frac{\alpha_k \beta_k}{2N} n^2$ and take a graph H with n vertices and $t_k(n, C_k) + \lfloor \frac{\alpha_k \beta_k}{2N} n^2 \rfloor$ edges. We claim that H contains at least $\delta(\frac{\alpha_k \beta_k}{4N}) n^k$ copies of C_k. Otherwise the graph removal lemma could be used: deleting at most $\frac{\alpha_k \beta_k}{4N} n^2$ edges of H the so obtained graph H', with more edges than the Turán number $\text{ex}(n, C_k)$ would contain no copy of C_k, a contradiction. Now, if

$$n > \frac{N}{\delta(\frac{\alpha_k \beta_k}{4N})}$$

then

$$\frac{h_k(n,t)}{t n^{k-3}} \geq \frac{h_k(n, \lfloor \frac{\alpha_k \beta_k}{2N} n^2 \rfloor)}{n^2 n^{k-3}} \geq \frac{\delta(\frac{\alpha_k \beta_k}{4N}) n^k}{n^{k-1}} \geq N$$

holds, as desired.

A possible choice for $n(k, N)$ is

$$\max \left\{ n(k), \frac{N}{\beta_k}, \frac{2}{\alpha_k}, \frac{N}{\delta(\frac{\alpha_k \beta_k}{4N})} \right\}.$$

$\boxed{\text{L}}$

We did give this detailed way of the proof to avoid the appearance of misusage of the notation $o(n)$.

The next lemma shows that the order of magnitude of the size of a color class that is not a book is $O(n^{k-3})$. For further use the lemma is stated and proved in a more general form.

Let $H = (V, E)$ be a graph where $|V| = a, |E| = b$. The family of all subgraphs of K_n isomorphic to H is denoted by $\mathcal{H}(n, H)$. We say that $\mathcal{G} \subset \mathcal{H}(n, H)$ that is *edge-intersecting* or shortly *intersecting* if any two members have an edge of K_n in common. It is called *non-trivially intersecting* if the intersection of the edge sets of the members is empty.

Lemma 3 *If $\mathcal{G} \subset \mathcal{H}(n, H)$ is non-trivially intersecting then $|\mathcal{G}| \leq \gamma(H) n^{a-3}$ holds with a positive constant $\gamma(H)$, independent of n.*

Proof. Let R denote the set of edges of K_n. Define the b-uniform hypergraph $\mathcal{F}$ on the vertex set R where $F \in \mathcal{F}$ if and only if $F \subset R$ is the set of edges of a graph in $\mathcal{G}$. Here $\mathcal{F}$ is obviously a non-trivially intersecting hypergraph.

We will repeatedly perform a certain operation on $\mathcal{F}$ obtaining a new hypergraph after every step. They will not be necessarily uniform any more, but the sizes of the members (in other terminology hyperedges) will be at most b. Set $\mathcal{F}_0$ to be $\mathcal{F}$, assume that $\mathcal{F}_0, \dots \mathcal{F}_i$ are defined. Construct $\mathcal{F}_{i+1}$ as follows. Suppose that there is a pair $(x, F), x \in F$ where $F \in \mathcal{F}_i$ such that $F - \{x\}$ meets all hyperedges of $\mathcal{F}_i$. Then apply the following *shrinking operation*:

$$\mathcal{F}_{i+1} = (\mathcal{F}_i - \{F\}) \cup \{F - \{x\}\}. \tag{6}$$

Note that if $\{F - \{x\}\} \in \mathcal{F}_i$ then $|\mathcal{F}_{i+1}| < |\mathcal{F}_i|$. It is easy to see that if $\mathcal{F}_i$ is a non-trivially intersecting hypergraph then so is $\mathcal{F}_{i+1}$. Let the final hypergraph be $\mathcal{F}_u$. A hypergraph is called *intersecting critical* if every application of the operation (6) makes the hypergraph non-intersecting. Of course an intersecting critical hypergraph is non-trivially intersecting (except the uninteresting case when $\mathcal{F}$ consists of one 1-element set). supp($\mathcal{F}$) denotes the union of all members of $\mathcal{F}$. The following theorem of Lovász will be used.

Theorem 10 (Lovász [10]) *For every positive integer b there is an integer $f(b)$ such that if $\mathcal{F}$ is an intersecting critical hypergraph with hyperedges of size at most b then $|supp(\mathcal{F})| \le f(b)$.*

For the (up to now) best estimates on the smallest possible values of $f(b)$ see [21] and [22]. The theorem above is formulated in the more general case of ν-critical hypergraphs, where ν is the maximum number of pairwise disjoint hyperedges and the application of operation (6) increases ν. However we need here the case $\nu = 1$ only.

This theorem can be applied to $\mathcal{F}_u$: $|\text{supp}(\mathcal{F}_u)| \le f(b)$. Obviously, $\mathcal{F}_u$ contains no empty or one-element member. (Otherwise it would be trivially intersecting.) Therefore the members of $\mathcal{F}_u$ are at least two-element subsets of a set $R_0 \subset R$ of edges of size at most $f(b)$. Take all two-element subsets of the members of $\mathcal{F}_u$. This 2-uniform family is denoted by $\mathcal{F}^*$. If $F \in \mathcal{F}$ then there is an $F_u \subset F$ with $F_u \in \mathcal{F}_u$ and an $F^* \subset F_u$ such that $F^* \in \mathcal{F}^*$. Define $\mathcal{F}(F^*)$ as the family of all members of $\mathcal{F}$ containing a given F^*. Then

$$\mathcal{F} = \bigcup_{F^* \in \mathcal{F}^*} \mathcal{F}(F^*)$$

implies

$$|\mathcal{F}| \le \sum_{F^* \in \mathcal{F}^*} |\mathcal{F}(F^*)|. \tag{7}$$

The number of terms in (7) is at most $\binom{f(b)}{2}$. The two elements of an F^* are two edges in K_n.

Suppose that these two edges meet and their vertices in K_n are x, y, z. If $F^* \subset F \in \mathcal{F}$ then the elements of F as edges in K_n form a copy of the graph H. Two adjacent edges of H are equal to the elements of F^* therefore the vertices of this copy of H are x, y, z and $a-3$ other, different vertices out of the $n-3$ possibilities. Hence

$$|\mathcal{F}(F^*)| \leq v(H)\binom{n-3}{a-3} = \Omega(n^{a-3}) \tag{8}$$

where $v(H)$ is the number of adjacent pairs of edges in H, independent on H.

If the two edges in F^* are vertex-disjoint, the situation is similar. Then

$$|\mathcal{F}(F^*)| \leq p(H)\binom{n-4}{a-4} = \Omega(n^{a-4}) \tag{9}$$

holds where $p(H)$ is the number of vertex-disjoint pairs of edges in H. (7)-(9) result in

$$|\mathcal{F}| \leq \binom{f(b)}{2}\Omega(n^{a-3})$$

proving the statement, since $|\mathcal{F}| = |\mathcal{G}|$. $\boxed{\text{L}}$

Let us finish the proof of the lower bound in the theorem. Consider a proper coloring of $\mathcal{C}(n, k)$ with the minimum number of colors. A family of cycles (of length k) is a *book* if they all have at least one common edge. If there is exactly one common edge, it is called the *spine* of the book. If there are more than one common edges, choose one arbitrarily. Let $s(n)$ denote the number of color classes forming a book and $r(n)$ the number of other color classes. Suppose

$$s(n) \leq \left(\frac{1}{2} - \varepsilon\right)\binom{n}{2} \tag{10}$$

holds, with a fixed $\varepsilon > 0$. Let G be the graph with n vertices and the spines of the books as edges. $\overline{G}$ has $\binom{n}{2} - s(n)$ edges. Hence the number of edges exceeding the Turán number is

$$t(n) = \binom{n}{2} - \mathrm{ex}(n, C_k) - s(n).$$

Theorem 3 and (10) imply that

$$t(n) \geq \frac{\varepsilon}{2}\binom{n}{2} \tag{11}$$

when n is larger than a certain $n(\varepsilon)$. By Lemma 2 and (11) $\overline{G}$ contains at least

$$\frac{\varepsilon}{2}\binom{n}{2}n^{k-3}N$$

copies of C_k. In other words G contains at least this many empty C_k's. They do not belong to any of the books. They must belong to color classes of other type. By Lemma 3 their sizes are at most $\gamma(C_k)n^{k-3} = \gamma_k n^{k-3}$ therefore

$$r(n) \geq \frac{\frac{\varepsilon}{2}\binom{n}{2}n^{k-3}N}{\gamma_k n^{k-3}} = \frac{\varepsilon}{2}\binom{n}{2}\frac{N}{\gamma_k}$$

Choose

$$N > \frac{2\gamma_k}{\varepsilon}.$$

Then

$$r(n) > \binom{n}{2}$$

holds, showing that if (10) holds for $n \geq n(\varepsilon)$ then the number of colors must be too large. $\boxed{\mathrm{T}}$

5 Proof of Theorem 6

The following theorem will be used.

Theorem 11 (Erdős and Simonovits [6], Sidorenko [17]) *Let $k \geq 4$ be an even number, $\varepsilon > 0$. There is a constant λ_k such that the number of copies of C_k in a graph on n vertices and at least $\varepsilon\binom{n}{2}$ edges is at least*

$$\lambda_k \varepsilon^k n^k$$

for $n \geq n(k, \varepsilon)$.

This is a special case of the Erdős-Simonovits-Sidorenko conjecture stated for bipartite graphs.

Consider a proper coloring of $\mathcal{C}(n, k)$ with the minimum number of colors. Let $s(n)$ denote the number of color classes forming a book and $r(n)$ the number of other color classes. If $s(n) > (1-\varepsilon)\binom{n}{2}$ holds for every positive ε

and for all $n > n(\varepsilon)$ then we are done. Therefore we can suppose that there is a fixed $\varepsilon > 0$ such that

$$s(n) \leq (1-\varepsilon)\binom{n}{2} \tag{12}$$

holds for an infinite sequence Σ of n's. Let G be the graph with n vertices and the spines of the books as edges. $\overline{G}$ has $\binom{n}{2} - s(n) \geq \varepsilon\binom{n}{2}$ edges. Theorem 11 implies that $\overline{G}$ contains at least $\lambda_k \varepsilon^k n^k$ copies of C_k if $n \in \Sigma$. In other words G contains at least this many empty C_k's. They do not belong to any of the books. They must belong to color classes of other type. By Lemma 3 their sizes are at most $\gamma(C_k)n^{k-3} = \gamma_k n^{k-3}$ therefore

$$r(n) \geq \frac{\lambda_k \varepsilon^k n^k}{\gamma_k n^{k-3}} = \frac{\lambda_k \varepsilon^k}{\gamma_k} n^3$$

holds which implies

$$r(n) > \binom{n}{2}$$

if $n \in \Sigma, n > n_1(\varepsilon)$. This contradicts the minimality of the coloring, completing the proof. $\boxed{\mathrm{T}}$

6 Remarks

1. Our problem is closely related to the *Kneser-Lovász problem.* Let $[n] = \{1, 2, \ldots, n\}$ be the set of the first n natural numbers and let $\mathcal{N} = \binom{[n]}{k}$ denote the family of all k-element subsets of an n-element set. Kneser suggested to color the members of $\mathcal{N}$ in such way that disjoint subsets must get distinct colors. One such coloring is the following. Color the sets with minimum element i with color i if $1 \leq i \leq n-2k+1$. The sets which are subsets of $\{n-2k+2, \ldots, n\}$ are pairwise intersecting, they are colored with the last color, $n-2k+2$. Kneser conjectured and Lovász ([11]) proved that this $n-2k+2$ is the minimum number of colors. The problem suggested in the present paper is very similar, since we are coloring k-element subsets of the set of edges of K_n and the definition of proper coloring is the same. Here, however we color only those subsets of edges which form a cycle.

2. The problem can be asked for any other graph not only for cycles. Let H be a graph, color all the copies of H in K_n with the minimum number of colors in such a way that edge-disjoint copies must have different colors. Let this number be denoted by $\chi(n, H)$. The ingredients of the proof of Theorem 4 are formulated for a general H except Theorem 8. But this is also known ([14]) for general graphs H under the additional condition that it contains an edge whose removal reduces the chromatic number.

Theorem 12 *Let H be graph with chromatic number $\chi(H)$ at least 3 and suppose that it contains an edge whose removal reduces the chromatic number. Then*

$$\lim_{n\to\infty} \frac{\chi(n, H)}{\binom{n}{2}} = \frac{1}{\chi(H) - 1}$$

holds.

Theorem 6 can also be generalized for those bipartite graphs for which the Erdős-Simonovits-Sidorenko conjecture holds.

3. There are good lower and upper estimates on $\mathrm{ex}(n, C_k)$ when k is even.

Theorem 13 (Erdős and Rényi [3], Bondy and Simonovits [1])

$$\Omega(n^{1+\frac{1}{k}}) \leq ex(n, C_k) \leq O(n^{1+\frac{2}{k}}).$$

The lower estimate and Proposition 1 result in a lower estimate on $\chi(n, k)$. This suggests the following open problem.

Open problem 1 *Let k be even. Does*

$$\Omega(n^{1+\frac{1}{k}}) \leq \binom{n}{2} - \chi(n, k) \leq O(n^{1+\frac{2}{k}})$$

hold?

The problem can be more precisely formulated in the case of $k = 4$ when the value of $\mathrm{ex}(n, C_4)$ is fairly well known.

Theorem 14 (Erdős, Rényi and Sós [4], Kővári, Sós and Turán [9], Füredi [8]) *$ex(n, C_4) \sim \frac{1}{2}n^{\frac{3}{2}}$ and $ex(n, C_4) = \frac{1}{2}q(q + 1)^2$ when q is a prime power and $n = q^2 + q + 1$.*

Let us formulate the problem above for this special case.

Open problem 2 *Is the following statement true?*

$$\chi(n,4) = \binom{n}{2} - \frac{1}{2}n^{\frac{3}{2}} + o(n^{\frac{3}{2}}).$$

For certain special values of n, based on Füredi's theorem there a little hope to find the exact value of $\chi(n,4)$.

Open problem 3 *If q is a prime power, $n = q^2 + q + 1$ does*

$$\chi(n,4) = \binom{q^2+q+1}{2} - \frac{1}{2}q(q+1)^2$$

hold?

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