Most probably intersecting families of subsets

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Abstract

Let $\mathcal{F}$ be a family of an n-element set. It is called intersecting if every pair of its members have a non-disjoint intersection. It is well-known that an intersecting family satisfies the inequality $|\mathcal{F}| \leq 2^{n-1}$. Suppose that $|\mathcal{F}| = 2^{n-1} + i$. Choose the members of $\mathcal{F}$ independently with probability p (delete them with probability $1-p$). The new family is intersecting with a certain probability. We try to maximize this

1

probability by choosing $\mathcal{F}$ appropriately. The exact maximum is determined in this paper for some small i's. The analogous problem is considered for families consisting of k-element subsets, but the exact solution is obtained only when the size of the family exceeds the maximum size of the intersecting family only by one. A family is said to be inclusion-free if no member is a proper subset of another one. It is well known that the largest inclusion-free family is the one consisting of all $\lfloor \frac{n}{2} \rfloor$-element subsets. We determine the most probably inclusion-free family too, when the number of members is $\binom{n}{\lfloor \frac{n}{2} \rfloor} + 1$.

Key Words: families of subsets, intersecting family, Sperner, random family

1 Introduction

Let $[n] = \{1, 2, \ldots n\}$ be an n-element set and $\mathcal{F} \subset 2^{[n]}$ a family of its subset. We say that $\mathcal{F}$ is *intersecting* if $F_1 \cap F_2 \neq \emptyset$ holds for any two members $F_1, F_2 \in \mathcal{F}$. This condition implies that at most one of the sets F and $[n] - F$ can be a member of $\mathcal{F}$. Hence $|\mathcal{F}| \leq 2^{n-1}$ follows. On the other hand the family of all subsets containing 1 is obviously intersecting. We obtained that the largest intersecting family on n elements has exactly 2^{n-1} members as was noticed in [**?**].

Therefore, if $|\mathcal{F}| = 2^{n-1} + i$ where $0 < i$ then $\mathcal{F}$ is not intersecting. Choose the members of $\mathcal{F}$ independently with probability $p(0 < p < 1)$ and delete them with probability $1 - p$. Let $\mathcal{F}^{p\oplus}$ denote the so obtained random family. We want to maximize the probability of the event that $\mathcal{F}^{p\oplus}$ is intersecting, for families of given size. Define

$$I(i) = \max_{|\mathcal{F}| = 2^{n-1}+i} \Pr(\mathcal{F}^{p\oplus} \text{ is intersecting}). \tag{1}$$

The families attaining this maximum can be called *the most probably intersecting families.* The value of (1) for some small values of i is determined in Section 2.

We can consider the same problem for uniform families when all the members of $\mathcal{F}$ have a given size. Let $1 \leq k \leq n$ be a fixed integer and suppose $\mathcal{F} \subset \binom{[n]}{k}$. The celebrated theorem of Erdős, Ko and Rado states the following.

Theorem 1.1 ([**?**]) *If $1 \leq k \leq \frac{n}{2}$ and $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family*

then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

and this bound is sharp.

The sharpness of the theorem can be easily shown by taking all k-element subsets containing the element 1.

The same question can be asked here too: what are the most probably intersecting families of k-element subsets with a given size $\binom{n-1}{k-1}+i$ where $0 < i$. Unfortunately we were able to determine the most probably intersecting family of k-element subsets only for the case $i = 1$.

Another fundamental concept in the theory of extremal families is the following one. A family $\mathcal{F}$ is called *inclusion-free* or an *antichain* if $F_1 \subset F_2$ holds for no two distinct members of $\mathcal{F}$. The first results on this subject, Sperner's theorem has determined the largest inclusion-free families.

Theorem 1.2 ([**?**]) *If $\mathcal{F} \subset 2^{[n]}$ is inclusion-free then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

with equality only for the families $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ *and* $\binom{[n]}{\lceil \frac{n}{2} \rceil}$.

Here, again, one can try to determine the *most probably inclusion-free* families among the families of size $\binom{n}{\lfloor \frac{n}{2} \rfloor} + i$. Again, we were able to find the most probably inclusion-free family only for $i = 1$.

Of course there is a natural common generalization behind all these problems. Let $G = (V, E)$ be a simple graph. We say that the set $A \subset V$ is *independent* if G has no edge between two elements of A. The maximum size of an independent set is denoted by $\alpha(G)$ and called the *independence number* of the graph. Of course, if $|A| > \alpha(G)$ then A is not independent in G. Choosing (independently) the elements of A with probability and deleting them with probability $1 - p$ a new random set $A^{p\oplus}$ is obtained. We can try to choose an A of a prescribed size to maximize the probability of the event that $A^{p\oplus}$ is independent.

Let the vertices of the graph be subsets of $[n]$ and two vertices be adjacent if the corresponding subsets are disjoint. Then our graph problem "determining the most probably independent vertex set of given size" becomes

the first problem mentioned above. On the other hand, if the adjacency of two vertices is defined by the inclusion of the corresponding subsets, then the problem becomes the "determination of the most probably inclusion-free family of given size".

Section 2 contains the definitions and the results while Section 3 gives the proofs.

2   Definitions and statements

Before formulating our theorem on the most probably intersecting families we have to make some comments on the largest intersecting families. We saw that one of these families is $\{F \subset [n] : 1 \in F\}$. However there are many. It was proved in [**?**] that any intersecting family can be extended by adding new members to make it of size 2^{n-1}, preserving the intersecting property. Yet, there is another important family, namely the one consisting of the "large" subsets. It can be easily described when n is odd. Take all the sets of size at least $\frac{n+1}{2}$. It is easy to see that that every pair of such subsets has a non-empty intersection and the number of such sets is exactly 2^{n-1}. If n is even, the construction is somewhat more complex. Take all the sets of size at least $\frac{n}{2} + 1$ and the sets of size $\frac{n}{2}$ containing the element 1. It is easy to see, again, that this family is intersecting. Its size is also 2^{n-1}.

The most probably intersecting families are extensions of this construction, at least for small i.

Theorem 2.1 *Suppose*

$$i \leq \binom{n-1}{\lceil \frac{n-3}{2} \rceil} = \begin{cases} \frac{1}{2}\binom{n}{\frac{n}{2}} & \text{if } n \text{ is even} \\ \binom{n-1}{\frac{n-3}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Then $I(i) = (1 - p^2)^i$ and the following families give equality.

n is even. Take all the sets of size at least $\frac{n}{2} + 1$, all the sets of size equal to $\frac{n}{2}$ and containing the element 1, and i other sets of size $\frac{n}{2}$.

n is odd. Take all the sets of size at least $\frac{n+1}{2}$ and i sets of size equal to $\frac{n-1}{2}$ containing the element 1.

We stated, a couple of years ago (conference on Random Structures and Algorithms, Poznań, 2005), the conjecture that the "continuation" of this construction always give the best value for $I(i)$. A series of families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_u \ldots$

is called *nested* if $\mathcal{F}_u \subset \mathcal{F}_{u+1}$ and $\mathcal{F}_{u+1} - \mathcal{F}_u$ has one member for every $1 \leq u$. In other words, a nested family can be defined by a sequence $F_1, F_2, \ldots, F_u, \ldots$ of subsets and $\mathcal{F}_u = \{F_1, F_2, \ldots, F_u\}$. The main point of our conjecture was that the most probably intersecting families are nested.

Conjecture 1 *There is a sequence of subsets of $[n]$ such that $|F_1| \geq |F_2| \geq \ldots \geq |F_u| \geq \ldots \geq |F_{2^n}|$ where $\{F_1, F_2, \ldots, F_u\}$ is one of the most probably intersecting families of size u.*

Paul Russell [**?**] informed us that he nearly proved the conjecture above. More precisely he proved that if $|\mathcal{F}| = 2^{n-1} + i = \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{r}$ for some integer r then the best construction is $\binom{[n]}{n} + \binom{[n]}{n-1} + \ldots + \binom{[n]}{r}$. Moreover, if

$$\binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{r} < |\mathcal{F}| < \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{r} + \binom{n}{r+1}$$

then the best construction consists of

$$\binom{[n]}{n} + \binom{[n]}{n-1} + \ldots + \binom{[n]}{r}$$

and some members of $\binom{[n]}{r+1}$ which are "left shifted".

Let us turn now to the uniform case. Analogously to (1) we can define

$$I_k(i) = \max_{\substack{\mathcal{F} \subset \binom{[n]}{k} \\ |\mathcal{F}| = \binom{n-1}{k-1} + i}} \Pr(\mathcal{F}^{\oplus} \text{ is intersecting}). \tag{2}$$

Theorem 2.2

$$I_k(1) = 1 - p + p(1-p)^{\binom{n-1-k}{k-1}}$$

with equality for the family consisting of all k-element sets containing the element 1, and one additional k-element set, say $\{2, 3, \ldots, k+1\}$.

The definition, analogous to (1) for the case of the most probably inclusion-free family is

$$A(i) = \max_{|\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor} + i} \Pr(\mathcal{F}^{\oplus} \text{ is inclusion-free}). \tag{3}$$

The corresponding result is the following.

5

Theorem 2.3

$$A(1) = 1 - p + p(1-p)^{\lfloor \frac{n}{2} \rfloor + 1}$$

with equality for the family consisting of all $\lfloor \frac{n}{2} \rfloor$-element sets and one additional $\lfloor \frac{n}{2} \rfloor + 1$-element set.

Finally we give the general definition of the *most probably independent vertex set* of given size in the graph $G = (V, E)$.

$$\text{mpi}(G, p, m) = \max_{\substack{A \subset V \\ |A| = m}} \Pr(A^{\oplus} \text{ is independent}). \tag{4}$$

As we mentioned in the introduction, (1), (2) and (3) are special cases of (4) for appropriately chosen graphs.

The vertex set A induces a subgraph of G, let us denote it by $G_A = (A, E_A)$ where E_A contains the edges of G joining two vertices both belonging to A. If $m = |V|$ in (4) then there is no choice, $\text{mpri}(G, p, |V|) = \text{pri}(G, p) = \Pr(V^{\oplus} \text{ is independent})$. Formula (4) can be written in the following slightly different form:

$$\text{mpri}(G, p, m) = \max_{\substack{A \subset V \\ |A| = m}} \text{pri}(G_A, p). \tag{5}$$

We will prove simple results on $\text{pri}(G, p)$ for some small graphs.

3 Proofs

Proof of Theorem 2.1. The family given at the end of the theorem contains exactly i pairs of disjoint sets: $F_1 \cap F_{i+1} = \emptyset$, $F_2 \cap F_{i+2} = \emptyset$, ..., $F_i \cap F_{2i} = \emptyset$ and no other pairs are disjoint. $\mathcal{F}^{p\oplus}$ is intersecting iff it does not contain both F_j and F_{j+i} ($1 \leq j \leq i$). The probability of this event, using the independence, is really $(1 - p^2)^i$. Hence we have $I(i) \geq (1 - p^2)^i$.

The following trivial lemmas are needed to the proof of the upper bound.

Lemma 3.1 *If G' is a subgraph of G then*

$$\text{pri}(G', p) \geq \text{pri}(G, p).$$

Let iK_2 denote the set of i pairwise vertex-disjoint edges.

Lemma 3.2 *If $G_i = (V, iK_2)$ then* $\mathrm{pri}(G_i, p) = (1 - p^2)^i$.

Suppose $|\mathcal{F}| = 2^{n-1} + i$ and consider the pairs $(F, [n] - F)$. Both must be in $\mathcal{F}$ for at least i such pairs. Let $G(\mathcal{F})$ denote the graph with vertex set $V = 2^{[n]}$ where two vertices are adjacent if the corresponding sets are disjoint. Then $G(\mathcal{F})$ contains at least i vertex-disjoint edges. By Lemmas 3.1 and 3.2 we have

$$\mathrm{pri}(G(\mathcal{F}), p) \leq \mathrm{pri}(G_i, p) = (1 - p^2)^i.$$

The left hand side is obviously equal to $\Pr(\mathcal{F}^{p \oplus}$ is intersecting). We found by (1) that $I(i) \leq (1 - p^2)^i$.

Note that the example given in the theorem shows equality.

Proof of Theorem 2.2. In the construction given at the end of the theorem the only member of the family which can be disjoint to other members of the family $\mathcal{F}$ is $\{2, 3, \ldots, k+1\}$. It is disjoint to $\binom{n-1-k}{k-1}$ other members. Consequently $G(\mathcal{F}) = S_{\binom{n-1-k}{k-1}}$ where S_r is the r-star, a graph with r edges, all containing one fixed vertex, the *center*.

Lemma 3.3 $\mathrm{pri}(S_r, p) = 1 - p + p(1 - p)^r$.

Proof. All subsets A of vertices not containing the center of the star are independent. The probability of the event that the center is deleted is $1 - p$. On the other hand, if the center is in an independent A, it cannot contain any other element. The probability of this event is $p(1 - p)^r$.

We have proved $I_k(1) \geq 1 - p + p(1 - p)^{\binom{n-1-k}{k-1}}$.

Now suppose that $\mathcal{F}$ is an arbitrary family satisfying $\mathcal{F} \subset \binom{[n]}{k}, |\mathcal{F}| = \binom{n-1}{k-1} + 1$ and prove $\Pr(\mathcal{F}^{p \oplus}$ is intersecting$) \leq 1 - p + p(1 - p)^{\binom{n-1-k}{k-1}}$. This will prove $I_k(1) \leq 1 - p + p(1 - p)^{\binom{n-1-k}{k-1}}$.

Lemma 3.4 *If $G = (V, E)$ where $|E| = r$ then* $\mathrm{pri}(G, p) \leq 1 - p + p(1 - p)^r$.

Proof. We use induction on r. If $r = 0$ then the statement is trivially true with equality. Let now G have $r + 1$ edges and choose a vertex a of degree $d(a) \geq 1$. Start with the equation

$$\Pr(V^{p\clubsuit} \text{ is independent}) = \tag{6}$$

$$\Pr(V^{p\clubsuit} \text{ is independent, } a \in V^{p\clubsuit}) + \Pr(V^{p\clubsuit} \text{ is independent, } a \notin V^{p\clubsuit}).$$

In the first case when $a \in V^{p\clubsuit}$ and $V^{p\clubsuit}$ is independent then the neighbors of a are not in $V^{p\clubsuit}$. Let $N(a)$ denote the set of neighbors of a and take $W = V - N(a) - \{a\}$. Define the reduced graph $G_1 = (W, E_1)$ where E_1 is the set of edges in E joining vertices in W. The event "$a \in V^{p\clubsuit}$ and $V^{p\clubsuit}$ is independent" can be equivalently given in the form "$a \in V^{p\clubsuit}$, $N(a) \cap V^{p\clubsuit} = \emptyset$ and $W^{p\clubsuit}$ is independent (in G_1)". Hence

$$\Pr(V^{p\clubsuit} \text{ is independent , } a \in V^{p\clubsuit}) = p(1-p)^{d(a)} \Pr(W^{p\clubsuit} \text{ is independent}) \leq$$

$$p(1-p)^{d(a)} \tag{7}$$

In the other case "$a \notin V^{p\clubsuit}$ and $V^{p\clubsuit}$ is independent" let $G_0 = (V - \{a\}, E_0)$ be the graph deleting a and the edges adjacent to it in G. Then "$a \notin V^{p\clubsuit}$ and $V^{p\clubsuit}$ is independent" is equivalent to "$a \notin V^{p\clubsuit}$ and $(V - \{a\})^{p\clubsuit}$ is independent". The inductional hypothesis can be used for G_0 which has exactly $r + 1 - d(a)$ edges.

$$\Pr(V^{p\clubsuit} \text{ is independent}, a \notin V^{p\clubsuit}) = (1-p) \Pr((V - \{a\})^{p\clubsuit} \text{ is independent}) \leq$$

$$(1-p)(1 - p + p(1-p)^{r+1-d(a)}) \tag{8}$$

By (6), (7) and (8) we have

$$\Pr(V^{p\clubsuit} \text{ is independent}) \leq p(1-p)^{d(a)} + (1-p)(1 - p + p(1-p)^{r+1-d(a)}).$$

We need to prove

$$p(1-p)^{d(a)} + (1-p)(1 - p + p(1-p)^{r+1-d(a)}) \leq 1 - p + p(1-p)^{r+1}. \tag{9}$$

After some elementary steps (divide by $1 - p$, subtract 1, divide by p) the equivalent, trivial inequality

$$0 \leq (1 - (1-p)^{d(a)-1})(1 - (1-p)^{r-d(a)+1})$$

is obtained, proving (9) and the lemma. $\boxed{\text{L}}$

We will need the following old theorem to complete the proof. A family $\mathcal{F}$ is called *trivially intersecting* if the intersection of all members is non-empty.

Theorem 3.5 (Hilton-Milner, [**?**]) *If* $1 \leq k \leq \frac{n}{2}$ *and* $\mathcal{F} \subset \binom{[n]}{k}$ *is a non-trivially intersecting family then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 1$$

and this bound is sharp.

Lemma 3.6 *If* $\mathcal{F} \subset \binom{[n]}{k}, |\mathcal{F}| = \binom{n-1}{k-1} + 1$ *then there are at least* $\binom{n-1-k}{k-1}$ *disjoint pairs in* $\mathcal{F}$, *that is,* $G(\mathcal{F})$ *has at least* $\binom{n-1-k}{k-1}$ *edges.*

Proof. Let $\mathcal{F}^* \subset \mathcal{F}$ be the largest intersecting subfamily of $\mathcal{F}$. Two cases will be distinguished.

Case 1.
$$|\mathcal{F}^*| > \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 1.$$

By Theorem 3.5 $\mathcal{F}^*$ is a trivially intersecting family, by symmetry it can be supposed that every member contains the element 1. Let $|\mathcal{F}| - |\mathcal{F}^*| = i$ where

$$1 \leq i < \binom{n-1-k}{k-1}. \tag{10}$$

Choose a member F of $\mathcal{F} - \mathcal{F}^*$. Of course, $1 \notin F$. There are exactly $\binom{n-1-k}{k-1}$ sets of size k containing 1 and disjoint to F. Since $|\mathcal{F}^*| = |\mathcal{F}| - i = \binom{n-1}{k-1} + 1 - i$, the number of sets containing 1, but not in $\mathcal{F}^*$ is at most $i - 1$. Therefore there are at least $\binom{n-1-k}{k-1} - (i - 1)$ sets $G \in \mathcal{F}^*$ containing 1 and disjoint to F. Our conclusion is that there are at least

$$i \left(\binom{n-1-k}{k-1} - (i - 1) \right) \tag{11}$$

disjoint pairs of members of $\mathcal{F}$. It is easy to see that $i(a - i + 1) \geq a$ when $1 \leq i \leq a$. This implies that (11) is at least $\binom{n-1-k}{k-1}$, by (10), as desired.

Case 2.
$$|\mathcal{F}^*| \leq \binom{n-1}{k-1} - \binom{n-1-k}{k-1} + 1.$$

Then $|\mathcal{F} - \mathcal{F}^*| \geq \binom{n-1-k}{k-1}$. Choose an $F \in \mathcal{F} - \mathcal{F}^*$. Here F cannot be added to $\mathcal{F}^*$ because of its maximality, therefore there is a $G \in \mathcal{F}^*$ such that $F \cap G = \emptyset$. The number of disjoint pairs is at least $\binom{n-1-k}{k-1}$. $\boxed{\text{L}}$

9

Now it is easy to prove the second part of Theorem 2.2. We have to give a sharp upper bound on $\Pr(\mathcal{F}^{\oplus}$ is intersecting) under the conditions $\mathcal{F} \subset \binom{[n]}{k}, |\mathcal{F}| = \binom{n-1}{k-1} + 1$. By Lemma 3.6 $G(\mathcal{F})$ has at least $\binom{n-1-k}{k-1}$ edges, then Lemma 3.4 implies $\Pr(\mathcal{F}^{\oplus}$ is intersecting) $= \operatorname{pri}(G(\mathcal{F}), p) \leq 1 - p + p(1-p)^{\binom{n-1-k}{k-1}}$. Hence $I_k(1) \leq 1 - p + p(1-p)^{\binom{n-1-k}{k-1}}$ follows. $\boxed{\text{T}}$

Proof of Theorem 2.3. The following statement will be used that is a special case of a more general theorem.

Theorem 3.7 (Kleitman [?]) *If $\mathcal{F} \subset 2^{[n]}, |\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor} + 1$ then there are at least $\lfloor \frac{n}{2} \rfloor + 1$ pairs $F, G \in \mathcal{F}$ such that $F \subset G, F \neq G$.*

The graph $G(\mathcal{F})$ formed from the family $\mathcal{F}$ given in the theorem is a star with exactly $\lfloor \frac{n}{2} \rfloor + 1$ edges. Lemma 3.3 gives $\operatorname{pri}(G(\mathcal{F}), p) = 1 - p + p(1-p)^{\lfloor \frac{n}{2} \rfloor + 1}$ proving $A(1) \geq 1 - p + p(1-p)^{\lfloor \frac{n}{2} \rfloor + 1}$.

The proof of the upper bound follows the logic of the proof of the previous theorem. Kleitman's theorem ensures the existence of at least $\lfloor \frac{n}{2} \rfloor + 1$ edges in $G(\mathcal{F})$. Then Lemmas 3.1 and 3.4 prove $\operatorname{pri}(G, p) \leq 1 - p + p(1-p)^{\lfloor \frac{n}{2} \rfloor + 1}$. Hence $A(1) \leq 1 - p + p(1-p)^{\lfloor \frac{n}{2} \rfloor + 1}$ follows. $\boxed{\text{T}}$

4 Remarks

Problem 2 *Determine $I_k(i)$ when $i > 1$.*

The following series of families forms a good candidate for a nested optimal solution. Take all k-element sets containing the element 1 in any order, then take all k-element sets containing 2 but not containing 1 in any order, then take all k-element sets containing 3 but disjoint to $\{1, 2\}$, and so on ...

Problem 3 *Determine $A(i)$ when $i > 1$.*

The following series of families forms a good candidate for a nested optimal solution. Take all sets of size $\lfloor \frac{n}{2} \rfloor$ in any order, then take all sets of size $\lfloor \frac{n}{2} \rfloor + 1$ in any order, then take all sets of size $\lfloor \frac{n}{2} \rfloor - 1$, and so on ...

We can see that the order of the values $\operatorname{pri}(G, p)$ for graphs G plays an important role in Section 3. Of course it is easier to study these inequalities for $p = \frac{1}{2}$, so let us start with that. Let P_k denote the path with k vertices.

Proposition 4.1 *If G is different from the graphs $P_2, P_3, P_4, 2K_2, S_r (3 \leq r)$ and K_3 then $\operatorname{pri}(G, \frac{1}{2}) < \frac{1}{2}$.*

Their order is

$$\mathrm{pri}\left(P_2, \tfrac{1}{2}\right) > \mathrm{pri}\left(P_3, \tfrac{1}{2}\right) > \mathrm{pri}\left(2K_2, \tfrac{1}{2}\right) = \mathrm{pri}\left(S_3, \tfrac{1}{2}\right) > \mathrm{pri}\left(S_4, \tfrac{1}{2}\right) > \ldots >$$

$$\mathrm{pri}\left(S_r, \tfrac{1}{2}\right) > \ldots > \mathrm{pri}\left(K_3, \tfrac{1}{2}\right) = \mathrm{pri}\left(P_4, \tfrac{1}{2}\right) = \tfrac{1}{2}.$$

Some of these inequalities remain true for general p, some of them do change. For instance if $p > \frac{\sqrt{5}-1}{2}$ then

$$\mathrm{pri}\left(P_2, p\right) > \mathrm{pri}\left(P_3, p\right) > \mathrm{pri}\left(S_3, p\right) > \mathrm{pri}\left(S_4, p\right) > \ldots > \mathrm{pri}\left(S_r, p\right) > \ldots$$

$$> 1 - p > \mathrm{pri}\left(2K_2, p\right) > \mathrm{pri}\left(K_3, p\right) = \mathrm{pri}\left(P_4, p\right).$$

Compare $2K_2$ and S_3 with different p's. It is easy to check that $\mathrm{pri}\left(2K_2, p\right) > \mathrm{pri}\left(S_3, p\right)$ holds if $p < \tfrac{1}{2}$ while $\mathrm{pri}\left(2K_2, p\right) < \mathrm{pri}\left(S_3, p\right)$ when $p > \tfrac{1}{2}$. Consider the graph G having the edges $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{4, 5\}$. Then $\alpha(G) = 3$ and both $2K_2$ and S_3 can be the most probably independent vertex set in G for $m = 4$, depending on the value of p.

Remark added on October 20, 2011. Paul Russell informed us that he and Mark Walters disproved our Conjecture 1.

5 Acknowledgements

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