Sperner type theorems with excluded subposets

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Abstract

Let $\mathcal{F}$ be a family of subsets of an n-element set. Sperner's theorem says that if there is no inclusion among the members of $\mathcal{F}$ then the largest family under this condition is the one containing all $\lfloor \frac{n}{2} \rfloor$-element subsets. The present paper surveys certain generalizations of this theorem. The maximum size of $\mathcal{F}$ is to be found under the condition that a certain configuration is excluded. The configuration here is always described by inclusions. More formally, let P be a a poset. The maximum size of a family $\mathcal{F}$ which does not contain P as a (not-necessarily induced) subposet is denoted by $\mathrm{La}(n, P)$. The paper is based on a lecture of the author at the Jubilee Conference on Discrete Mathematics [Banasthali University, January 11-13, 2009], but it was somewhat updated in December 2010.

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1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set, $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. In the present paper $\max |\mathcal{F}|$ will be investigated under certain conditions on the family $\mathcal{F}$. The well-known Sperner theorem ([22]) was the first such result.

Theorem 1.1 *If $\mathcal{F}$ is a family of subsets of $[n]$ without inclusion ($F, G \in \mathcal{F}$*

implies $F \not\subset G$) then

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds, and this estimate is sharp as the family of all $\lfloor \frac{n}{2} \rfloor$-element subsets shows.

There is a very large number of generalizations and analogues of this theorem. (See e.g. [9]). Here we will consider only results when the condition on $\mathcal{F}$ excludes certain configurations what can be expressed by inclusion, only. That is, no intersections, unions, etc. are involved.

The first such generalization of Sperner theorem was obtained by Erdős [10]. The family of k distinct sets with mutual inclusions, $F_1 \subset F_2 \subset \ldots F_k$ is called a *chain of length* k. It will be simply denoted by P_k. Let $\mathrm{La}(n, P_k)$ denote the largest family $\mathcal{F}$ without a chain of length k.

Theorem 1.2 [10] $\mathrm{La}(n, P_{k+1})$ *is equal to the sum of the k largest bimomial coefficients of order n.*

Let V_r denote the *r-fork*, that is, the following family of distinct sets: $F \subset G_1, F \subset G_2, \ldots F \subset G_r$. The quantity $\mathrm{La}(n, V_r)$, that is, the largest family on n elements containing no V_r was first (asymptotically) determined for $r = 2$. We use the well-known notation $\Omega(n)$ where $f(n) = \Omega(n)$ means that there is a constant c such that $cn \leq f(n)$ holds for all n.

Theorem 1.3 [17]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n, V_2) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n}\right).$$

The first result for general r is contained in the following theorem.

Theorem 1.4 [23]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{r}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n, V_{r+1}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r^2}{n} + o\left(\frac{1}{n}\right)\right).$$

The constant in the second term in the upper estimate was recently improved.

Theorem 1.5 [4]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+2\frac{r}{n}+O\left(\frac{1}{n^2}\right)\right).$$

There is an unfortunate difference in the second terms of the lower and upper estimates, respectively. It seems to be very difficult to bridge this gap, since the construction in the lower estimate uses a certain construction from coding theory which gives only half of the known upper bound and it is unsolved since 1980 (see [12]).

The results listed above were proved by *ad hoc* methods. The theorems using the method of counting chains will be shown later.

Before formulating our general problem, let us show, for comparison, an old result what will not be included.

Theorem 1.6 (Kleitman [18]) *If the family $\mathcal{F}$ contains no three distinct members A, B, C satisfying $A \cap B = C$ then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor} + \frac{2^n}{n} = \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{c}{\sqrt{n}}+o\left(\frac{1}{\sqrt{n}}\right)\right).$$

The condition in this theorem is much weaker than that of Theorem 1.3, yet its upper bound is almost the same, they differ only in the second term. It is widely believed that the second term $\frac{2}{n}$ of the upper bound in Theorem 1.3 is also valid here. However, the best constructions (for large n) for the two problems are not the same. Recently M. Živković [24] found a pretty construction giving $\frac{\frac{4}{3}}{n}$ in the second term of the lower estimate for odd n what does not satisfy the strongest condition of Theorem 1.3.

2 Notations, definitions

A *partially ordered set*, shortly *poset* P is a pair $P = (X, \le)$ where X is a (in our case always finite) set and $\le$ is a relation on X which is *reflexive* ($x \le x$ holds for every $x \in X$), *antisymmetric* (if both $x \le y$ and $x \ge y$ hold for $x, y \in X$ then $x = y$) and transitive ($x \le y$ and $y \le z$ always implies $x \le z$). We say that y *covers* x if $x < y$ and there is no $z \in X$ such that $x < z < y$ holds. It is easy to see that if $X = 2^{[n]}$ and the $\le$ is defined as $\subseteq$, then these conditions are satisfied, that is the family of all subsets of an n-element set ordered by inclusion form a poset. We will call this poset the *Boolean lattice* and denote it by $\mathbb{B}_n$. Covering in this poset means "inclusion with difference 1".

The definition of a *subposet* is obvious: $R = (Y, \leq_2)$ is a subposet of $P = (X, \leq_1)$ iff there is an injection α of Y into X is such a way that $y_1, y_2 \in Y, y_1 \leq_2 y_2$ implies $\alpha(y_1) \leq_1 \alpha(y_2)$. On the other hand R is an *induced subposet* of P when $\alpha(y_1) \leq_1 \alpha(y_2)$ holds iff when $y_1 \leq_2 y_2$. If $P = (X, \leq)$ is a poset and $Y \subset X$ then the poset *spanned by Y in P* is defined as $(Y, \leq^*)$ where $\leq^*$ is the same as $\leq$, for all the pairs taken from Y. Given a "small" poset R, $\text{La}(n, R)$ denotes the maximum number of elements of $Y \subset 2^{[n]}$ (that is, the maximum number of subsets of $[n]$) such that R is not a subposet of the poset spanned by Y in $\mathbb{B}_n$.

Redefine our "small" configurations in terms of posets. The chain P_k contains k elements: $a_1, \ldots, a_k$ where $a_1 < \ldots < a_k$. The *r-fork* contains $r+1$ elements: $a, b_1, \ldots, b_r$ where $a < b_1, \ldots, a < b_r$. It is easy to see that the definitions of $\text{La}(n, P_k)$, $\text{La}(n, V_r)$, in Sections 1 and 2 agree. In the rest of the paper we will use the two different terminology alternately. In the definition of $\text{La}(n, R)$ we mean non-induced subposets, that is, if $R = V_2$ then P_3 is also excluded as a subposet.

A poset is *connected* if for any pair (z_0, z_k) of its elements there is a sequence $z_1, \ldots, z_{k-1}$ such that either $z_i < z_{i+1}$ or $z_i > z_{i+1}$ holds for $0 \leq i < k$. If the poset is not connected, maximal connected subposets are called its *connected components*. Given a family $\mathcal{F}$ of subsets of $[n]$, it spans a poset in $\mathbb{B}_n$. We will consider its connected components Q in two different ways. First as posets themselves, secondly as they are represented in $\mathbb{B}_n$. In the latter case the sizes of the sets are also indicated. This is called a realization of Q. A *full chain* in $\mathbb{B}_n$ is a family of sets $A_0 \subset A_1 \subset \ldots \subset A_n$ where $|A_i| = i$. We say that a (full) chain *goes through* a family (subposet) $\mathcal{F}$ if their intersection is non-empty, that is if it "goes through" at least one member of the family.

3 Lubell's proof of the Sperner theorem

The method using "counting chains" originates in the proof of Sperner theorem, given by Lubell.

The number of full chains in $[n]$ is $n!$ since the choice of a full chain is equivalent to the choice of a permutation of the elements of $[n]$. On the other hand, the number of full chains going through a given set F of f elements is $f!(n-f)!$ since the chain "must grow" within F until it "hits" F and outside after that. Suppose that the family $\mathcal{F}$ of subsets of $[n]$ is without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$). Then a full chain cannot go through two members of $\mathcal{F}$. Therefore the set of full chains going through distinct members of $\mathcal{F}$ must be disjoint. Hence we have

$$\sum_{F\in\mathcal{F}} |F|!(n-|F|)! \leq n!.$$

Dividing the inequality by $n!$

$$\sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1 \tag{3.1}$$

is obtained. Replace $\binom{n}{|F|}$ by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Then

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1$$

follows, the theorem is proved.

Let us remark that inequality (3.1) is important on its own right and is called the YBLM-inequality (earlier LYM, see [25], [3], [19], [20]). □

4 The method "counting chains"

Lubell's proof easily applies for Theorem 1.2, however, surprisingly it was not exploited for proving theorems of the present type. The reason might be that not the "excluded" configurations should be considered when using the idea, but the "allowed induced posets".

Take first the upper estimate of Theorem 1.3. This theorem already has two different proofs in [17] and [5], however each of these proofs needed an *ad hoc* idea, our new method also works here. It needs some tedious calculations, but the principal idea is easy.

The subposets V_2 are excluded. The family $\mathcal{F}$ spans a poset in $\mathbb{B}_n$. What are its connected components? First of all, a connected component cannot contain P_3 as a subposet, since V_2 is excluded in a "non-induced" way. It is easy to see that these components can only be the "upside-down" versions of V_r, that is, the *r-brushes* Λ_r consisting of $r+1$ elements: $a, b_1, \ldots, b_r$

where $a > b_1, \ldots, a > b_r$. Let Λ_r^* denote an r-brush in $\mathbb{B}_n$ that is a family of subsets $A, B_1, \ldots, B_r$ where $A \subset B_1, \ldots, A \subset B_r$. The number of chains going through Λ_r^* is denoted by $c(\Lambda_r^*)$. The trivial observation that no chain can go through two distinct components results in

$$\sum_{\Lambda_r^*} c(\Lambda_r^*) \leq n! \tag{4.1}$$

where the sum is taken for all connected components spanned by $\mathcal{F}$. There is a good lower estimate on

$$\frac{c(\Lambda_r^*)}{r+1},$$

namely

Lemma 4.1 *Suppose* $6 \leq n, 0 \leq r$. *Then*

$$u^*!u^*(n-u^*-1)! \leq \frac{c(\Lambda_r^*)}{|\Lambda_r|} \tag{4.2}$$

where $u^* = u^*(n) = \frac{n}{2} - 1$ *if* n *is even,* $u^* = \frac{n-1}{2}$ *if* n *is odd and* $r - 1 \leq n$*, while* $u^* = \frac{n-3}{2}$ *if* n *is odd and* $n < r - 1$.

The tedious part of the whole proof is the proof of this lemma. See [13] or [16]. (The title of [16] has a wrong word: "inclusion" should stand rather than "Intersection".) However, after having the lemma, (4.1) and (4.2) give

$$n! \geq \sum_{\Lambda_r^*} c(\Lambda_r^*) = \sum_{\Lambda_r^*} (r+1) \frac{c(\Lambda_r^*)}{r+1}$$

$$\geq \sum_{\Lambda_r^*} (r+1) u^*!u^*(n-u^*-1)! = |\mathcal{F}| u^*!u^*(n-u^*-1)!.$$

This really leads to the upper estimate in Theorem 1.3.

Now, this idea will be formulated for the general case. Let $\mathcal{P}$ be the set of forbidden subposets. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that the poset induced by $\mathcal{F}$ in $\mathbb{B}_n$ contains no member of $\mathcal{P}$ as a subposet. $\mathrm{La}(n, \mathcal{P})$ denotes the largest size of such a family. Consider the connected components of the poset induced by $\mathcal{F}$. The family of all possible components is denoted by $\mathcal{Q} = \mathcal{Q}(\mathcal{P})$.

In the example above we had $\mathcal{P} = \{V_2\}$. Then $\mathcal{Q}(\{V_2\}) = \{\Lambda_r : 0 \leq r\}$.

If $Q \in \mathcal{Q}$ let Q_n^* be a realization of Q in the Boolean lattice $\mathbb{B}_n$, that is, Q is embedded into $\mathbb{B}_n$ and a size (of a subsets) is associated with each element $q \in Q_n^*$. Here $Q \to Q_n^*$ denotes that Q_n^* is a realization of Q. In our example, for instance, Λ_1 is just a poset with two comparable elements, Λ_1^* is a family of two subsets A, B satisfying $A \subset B$, with a and b elements, respectively.

Furthermore $c(Q_n^*)$ denotes the number of chains going through Q_n^*. In our example $c(\Lambda_1^*) = a!(b-a)!(n-b)!$.

Let $\min_{Q \to Q_n^*} c(Q_n^*) = c_n^*(Q)$ be the smallest number of chains respect to the realizations. In the example: $c_n^*(\Lambda_0) = \left\lfloor \frac{n}{2} \right\rfloor! \left\lceil \frac{n}{2} \right\rceil!, c_n^*(P_2) = n \left\lfloor \frac{n-1}{2} \right\rfloor! \left\lceil \frac{n-1}{2} \right\rceil!$.

The following theorem can be easily obtained by the method shown for V_2,

Theorem 4.2 [16]

$$\mathrm{La}(n, \mathcal{P}) \le \frac{n!}{\inf_{Q \in \mathcal{Q}(\mathcal{P})} \frac{c_n^*(Q)}{|Q|}}.$$

Now we show how to apply this theorem for the poset N which contains 4 distinct elements a, b, c, d satisfying $a < c, b < c, b < d$. In the Boolean lattice a subposet N consists of four distinct subsets satisfying $A \subset C, B \subset C, B \subset D$. It is somewhat surprising that excluding N the result is basically the same as in the case of V_2.

Let $\mathcal{F}$ be a family of subsets of $[n]$ containing no four distinct members forming an N. Consider the poset $P(\mathcal{F})$ spanned by $\mathcal{F}$ in $\mathbb{B}_n$. What can its components be? A component might be a P_3, but no component can contain a P_3 as a proper subposet, since adding one more element to P_3 an N is created no matter which element of P_3 is in relation with the new element. Let $a < b$ be two elements of a component. We claim that a and b cannot be both comparable within the component with some other distinct elements c, d (say, in this order), unless they are a part of a P_3. Indeed, the choices $c < a$ and $b < d$ lead to a P_3, therefore the only possibility is $a < c, d < b$. This is an N, contradicting the assumption. But one of them can be comparable with many others in the same direction. Therefore the following ones are the only possible components:

$$\mathcal{Q}(\mathcal{P}) = \{P_3, \Lambda_0, \Lambda_1, \Lambda_2, \ldots, \Lambda_r, \ldots, V_1, V_2, \ldots, V_r, \ldots\}.$$

In order to use Theorem 4.2 we have to give a good lower bound on the ratios

$$\frac{c_n^*(P_3)}{3}, \qquad \frac{c_n^*(\Lambda_r)}{r+1}, \qquad \frac{c_n^*(V_r)}{r+1}.$$

(4.2) is a good lower estimate on the middle one. By symmetry, the same applies for the last one. The only unknown one is the first ratio. Its exact value is determined in [13] (Lemma 3.1). We do not formulate the statement here, what is important for our purposes is that

$$u^*!u^*(n-u^*-1)! \leq \frac{c_n^*(P_3)}{3}$$

holds, therefore the denominator in Theorem 4.2 is the same as in the case of V_2. We obtained the following theorem almost "free".

Theorem 4.3 [13]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \leq \mathrm{La}(n,N) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)$$

holds.

It is interesting to mention that the "La" function will jump if the excluded poset contains one more relation. The *butterfly* $\bowtie$ contains 4 elements: a, b, c, d with $a < c, a < d, b < c, b < d$.

Theorem 4.4 [5] *Let* $n \geq 3$. *Then* $\mathrm{La}(n, \bowtie) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.

It was proved by a different method.

5 A further improvement of the method

Observe that the main part of a large family is near the middle, the total number of sets far from the middle is small. More precisely, let $0 < \alpha < \frac{1}{2}$ be a fixed real number. The total number of sets F (for a given n) of size satisfying

$$|F| \notin \left[n\left(\frac{1}{2}-\alpha\right), n\left(\frac{1}{2}+\alpha\right)\right] \tag{5.1}$$

is very small. It is well-known (see e.g. [1], page 214) that for a fixed constant $0 < \beta < \frac{1}{2}$

$$\sum_{i=0}^{\beta n} \binom{n}{i} = 2^{n(h(\beta)+o(1))}$$

holds where $h(x) = -x \log_2 x - (1-x) \log_2(1-x)$. Therefore the total number of sets satisfying (5.1) is at most

$$2 \sum_{i=0}^{\lfloor n\left(\frac{1}{2}-\alpha\right)\rfloor} \binom{n}{i} = 2^{n\left(h\left(\frac{1}{2}-\alpha\right)+o(1)\right)} = \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right)$$

where $0 < h\left(\frac{1}{2}-\alpha\right) < 1$ is a constant.

In view of this observation we can improve our main tool, Theorem 4.2. First we have to generalize $c_n^*(Q)$. Let $c_n^{*\alpha}(Q)$ denote $\min_{Q \to Q_n^*} c(Q_n^*)$ where only those realizations Q_n^* are considered whose member subsets are of size in the interval (5.1). It is obvious that $c_n^*(Q) \leq c_n^{*\alpha}(Q)$. We actually believe that they are equal for large n, but we cannot prove this statement.

Theorem 5.1 [16] *Let $0 < \alpha < \frac{1}{2}$ be a real number. Then*

$$\mathrm{La}(n, \mathcal{P}) \leq \frac{n!}{\inf_{Q \in \mathcal{Q}(\mathcal{P})} \frac{c_n^{*\alpha}(Q)}{|Q|}} + \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right).$$

Using this theorem, we were able to prove the following rather general one.

Theorem 5.2 [16] *Let $1 \leq r$ be a fixed integer, independent on n. Suppose that every element $Q \in \mathcal{Q}(\mathcal{P})$ has the following property: if $a \in Q$ then a covers at most r elements of Q. Then*

$$\mathrm{La}(n, \mathcal{P}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

If the family $\mathcal{F}$ contains no V_{r+1} then the components cannot contain a set which is contained in $r+1$ other sets. Therefore the conditions of Theorem 5.2 are satisfied (in the dual form). This is why Theorem 5.2 implies Theorem 1.5.

6 Results determining the main term only

Some recent results determine only the main term, that is the coefficient of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, and there are only rough estimates of the coefficient of $\frac{1}{n}$. Two elements a and b of a poset are called *neighboring* if $a < b$ or $a > b$ and there is no c satisfying $a < c < b$ or $a > c > b$. A poset T is called a tree if the graph obtained by joining the neighboring elements is a tree. Griggs and Linyuan solved the problem for trees of two levels.

Theorem 6.1 [15] *Let T be a tree and suppose that it has two levels, then*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\Omega\left(\frac{1}{n}\right)\right) \leq \mathrm{La}(n,T) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+O\left(\frac{1}{n}\right)\right).$$

Bukh found the first term for arbitrary trees independently, about the same time. Let $h(P)$ denote the hight (maximal length of a chain) in a poset.

Theorem 6.2 [6] *Let T be a tree. Then*

$$h(T)\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\Omega\left(\frac{1}{n}\right)\right) \leq \mathrm{La}(n,T) \leq h(T)\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+O\left(\frac{1}{n}\right)\right).$$

However [11] contains another, surprising result. Let $G = (V,E)$ be a graph. $P(G)$ is the poset on two levels, V is the level below, $v < e(v \in V, e \in E)$ iff $v \in e$.

Theorem 6.3 [15]

$$\mathrm{La}(n,P(G)) \leq \left(1+\sqrt{1-\frac{1}{\chi(G)-1}}+o(1)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Of course this is sharp if $\chi(G) = 2$. For instance when G is cycle of length ℓ then $P(G)$ is a cycle of length 2ℓ on two levels. By this the posets on two levels forming a cycle are settled, up to the first term.

7 Excluding induced posets, only

One can ask what happens if we exclude the posets R belonging to $\mathcal{P}$ only in a strict form, that is, there is no induced copy in the poset induced in $\mathbb{B}_n$ by the family. Given a "small" poset R, $\mathrm{La}^\sharp(n,R)$ denotes the maximum number of elements of $Y \subset 2^{[n]}$ (that is, the maximum number of subsets of $[n]$) such that R is not an induced subposet of the poset spanned by Y in $\mathbb{B}_n$. This obviously generalizes for $\mathrm{La}^\sharp(n,\mathcal{P})$ where $\mathcal{P}$ is a set of posets.

For instance, calculating $\mathrm{La}(n, V_2)$ the path of length 3, P_3 is also excluded, while in the case of $\mathrm{La}^\sharp(n, V_2)$ this is allowed, three sets A, B, C are excluded from the family only when $A \subset B, A \subset C$ but B and C are incomparable. As we saw in the proof of Theorem 1.3, $\mathcal{Q}(V_2)$ consists of Λ_rs $(0 \leq r)$. The set $\mathcal{Q}^\sharp(V_2)$ of possible components when only the induced V_2s are excluded is much richer. $\mathcal{Q}^\sharp(V_2)$ contains all posets whose graph is a "descending" tree with one maximal vertex. That is, not only the sizes of these posets are unbounded, but their depths, as well. Yet, this case can be also be treated, on the basis of Theorem 5.2. To be precise we have to modify the formulations of our previous theorems. These modifications need no proofs, since the original proofs did not really depended on $\mathcal{P}$, only on $\mathcal{Q}(\mathcal{P})$ and this is simply replaced by $\mathcal{Q}^\sharp(\mathcal{P})$.

Theorem 7.1 [16]

$$\mathrm{La}^\sharp(n, \mathcal{P}) \leq \frac{n!}{\inf_{Q \in \mathcal{Q}^\sharp(\mathcal{P})} \frac{c_n^*(Q)}{|Q|}}.$$

Theorem 7.2 [16] *Let $0 < \alpha < \frac{1}{2}$ be a real number. Then*

$$\mathrm{La}^\sharp(n, \mathcal{P}) \leq \frac{n!}{\inf_{Q \in \mathcal{Q}^\sharp(\mathcal{P})} \frac{c_n^{*\alpha}(Q)}{|Q|}} + \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right).$$

Theorem 7.3 [16] *Let $1 \leq r$ be a fixed integer, independent on n. Suppose that every element $Q \in \mathcal{Q}^\sharp(\mathcal{P})$ has the following property: if $a \in Q$ then a covers at most r elements of Q. Then*

$$\mathrm{La}^\sharp(n, \mathcal{P}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

It is quite obvious that if $Q \in \mathcal{Q}^\sharp(V_{r+1})$ then no element of Q is covered by more than r other elements. Theorem 7.3 can be applied in a dual (upside-down) form.

Theorem 7.4 [16]

$$\mathrm{La}^\sharp(n, V_{r+1}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

This is a stronger form of Theorem 1.5. The special case $r = 1$ was solved in [7].

8 The diamond and the snakes

At the end of the first talk given by the author on this subject around 2004, someone from the audience asked: "What about the cycle of length 4 on 3 levels?" (I am sorry I cannot remember who he was, therefore I cannot give credit to him.) The answer: "This is a good question, I think the present method will work on this problem too."

The problem is still unsolved, but there are some papers attacking it. A *diamond* is a poset of four elements a, b, c, d such that $a < b, c$ and $b, c < d$. Denote it by D_2. The first published result on the diamond is due to Axenovich, Manske and Martin.

Theorem 8.1 [2]

$$\text{La}(D_2) \leq (2.283 + o(1)) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Griggs, Li and Lu [14] have improved the constant to $2 + \frac{3}{11}$. However their main achievement is the solution of the problem for the generalized diamond D_k what is a poset of the elements $a, b_1, \ldots, b_k, d$ where $a < b_i < d$ holds for every i. They have solved the problem for most of the k's (but unfortunately not for $k = 2$).

The k-*snake* S_k is the following poset consisting of k elements: $a_1 < a_2 > a_3 < a_4 > a_5 < \ldots$. Theorems 1.3 and 4.3 give

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq \text{La}(n, S_k) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right) \tag{8.1}$$

for $k = 3, 4$. How far can this go?

Now we give a simple general construction which shows that (8.1) cannot hold for too big k.

Proposition 8.2 *If P is not a subposet of $\mathbb{B}_\ell$ (e.g. connected and $|P| > 2^\ell$) then*

$$2^\ell \binom{n-\ell}{\lfloor \frac{n-\ell}{2} \rfloor} \leq \text{La}(P). \tag{8.2}$$

Proof. Take an ℓ-element subset L of $[n]$ and let

$$\mathcal{F}_L = \{A \cup B : \ A \subset L, B \subset [n] - L, |B| = \lfloor \frac{n-\ell}{2} \rfloor\}.$$

This family induces $\binom{n}{\lfloor\frac{n-\ell}{2}\rfloor}$ copies of $\mathbb{B}_\ell$ therefore cannot contain a P. $\square$

It is easy to see that the left hand side of (8.2) is

$$\binom{n}{\lfloor\frac{n}{2}\rfloor}\left(1+\frac{x}{n}+O\left(\frac{1}{n^2}\right)\right) \tag{8.3}$$

where $x = \lfloor\frac{\ell}{2}\rfloor$ if n is even and $\lceil\frac{\ell}{2}\rceil$ if n is odd (ℓ is fixed, n tends to infinity).

S_{33} is not a subposet of $\mathbb{B}_5$, therefore Proposition 8.2 and (8.3) give the second term $\frac{3}{n}$ in the lower estimate of $\mathrm{La}(S_{33)}$ for odd n. Hence (8.1) cannot hold for $k = 33$. But already S_9 is suspicious, since the same reasoning gives $\frac{2}{n}$ for the second term in the lower estimate for odd n, what "almost" contradicts (8.1).

Open Problem. Find the best possible second terms in the lower and upper estimate of $\mathrm{La}(S_k)$ for $k = 5, 6, 7, \ldots$..

We believe that the only way to use Proposition 8.2 for an S_k is when $k > 2^\ell$, because otherwise S_k is a subposet of $\mathbb{B}_\ell$. Actually we think that a slightly stronger statement is also true. Let the vertices of the directed graph $G(\mathbb{B}_n)$ be the subsets of $[n]$, and (A, B) is an edge iff $A \subset B$.

Conjecture. If $n > 2$ then there is a Hamiltonian cycle in $G(\mathbb{B}_n)$ such that the directions of its edges alternatingly agree and do not agree with the direction of the cycle.

This problems is closely related to the famous problem if there is a Hamiltonian cycle between the "two middle levels". This was generalized for the symmetric (kth and $(n-k)$th) levels. For results see [8], [11] and [21]. The solution of this problem of every pair of symmetric levels would solve the present problem for even n, but our problem might be easier.

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