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Forbidden Intersection Patterns in the Families of Subsets (Introducing a Method)

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1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set, $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. In the present paper $\max |\mathcal{F}|$ will be investigated under certain conditions on the family $\mathcal{F}$. The well-known Sperner theorem ([14]) was the first such result.

Theorem 1.1. *If $\mathcal{F}$ is a family of subsets of $[n]$ without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$) then*

$$|\mathcal{F}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds, and this estimate is sharp as the family of all $\lfloor \frac{n}{2} \rfloor$-element subsets shows.

There is a very large number of generalizations and analogues of this theorem. (See e.g. [7]). Here we will consider only results when the condition on $\mathcal{F}$ excludes certain configurations what can be expressed by inclusion, only. That is, no intersections, unions, etc. are involved. The first such generalization was obtained by Erdős [8]. The family of k distinct sets with mutual inclusions, $F_1 \subset F_2 \subset \ldots F_k$ is called a *chain of length k*. It will be simply denoted by P_k. Let $\mathrm{La}(n, P_k)$ denote the largest family $\mathcal{F}$ without a chain of length k.

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Theorem 1.2 [8]**.** La(n, P_{k+1}) *is equal to the sum of the k largest binomial coefficients of order n.*

Let V_r denote the *r-fork*, that is the following family of distinct sets: $F \subset G_1, F \subset G_2, \dots F \subset G_r$. The quantity La$(n, V_r)$, that is, the largest family on n elements containing no V_r was first (asymptotically) determined for $r = 2$. We use the well-known notation $\Omega(n)$ where $f(n) = \Omega(n)$ means that there is a constant c such that $cn \le f(n)$ holds for all n.

Theorem 1.3 [11]**.**

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, V_2) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n}\right).$$

The first result for general r is contained in the following theorem.

Theorem 1.4 [15]**.**

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{r}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r^2}{n} + o\left(\frac{1}{n}\right)\right).$$

The constant in the second term in the upper estimate was recently improved.

Theorem 1.5 [3]**.**

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{r}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, V_{r+1})$$

$$\le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

See some remarks in Section 7 explaining why this second term is difficult to improve any more.

The aim of the present paper is to introduce some recent results and show a method, proving good upper estimates, developed recently.

2. Notations, Definitions

A *partially ordered set*, shortly *poset* P is a pair $P = (X, \leq)$ where X is a (in our case always finite) set and $\leq$ is a relation on X which is *reflexive* ($x \leq x$ holds for every $x \in X$), *antisymmetric* (if both $x \leq y$ and $x \geq y$ hold for $x, y \in X$ then $x = y$) and transitive ($x \leq y$ and $y \leq z$ always implies $x \leq z$). We say that y *covers* x if $x < y$ and there is no $z \in X$ such that $x < z < y$ holds. It is easy to see that if $X = 2^{[n]}$ and the $\leq$ is defined as $\subseteq$, then these conditions are satisfied, that is the family of all subsets of an n-element set ordered by inclusion form a poset. We will call this poset the *Boolean lattice* and denote it by B_n. Covering in this poset means "inclusion with difference 1".

The definition of a *subposet* is obvious: $R = (Y, \leq_2)$ is a subposet of $P = (X, \leq_1)$ iff there is an injection α of Y into X is such a way that $y_1, y_2 \in Y, y_1 \leq_2 y_2$ implies $\alpha(y_1) \leq_1 \alpha(y_2)$. On the other hand R is an *induced subposet* of P when $\alpha(y_1) \leq_1 \alpha(y_2)$ holds iff when $y_1 \leq_2 y_2$. If $P = (X, \leq)$ is a poset and $Y \subset X$ then the poset *spanned by* Y *in* P is defined as $(Y, \leq^*)$ where $\leq^*$ is the same as $\leq$, for all the pairs taken from Y. Given a "small" poset R, $\mathrm{La}(n, R)$ denotes the maximum number of elements of $Y \subset 2^{[n]}$ (that is, the maximum number of subsets of $[n]$) such that R is not a subposet of the poset spanned by Y in B_n.

Redefine our "small" configurations in terms of posets. The chain P_k contains k elements: $a_1, \ldots, a_k$ where $a_1 < \ldots < a_k$. The r-*fork* contains $r+1$ elements: $a, b_1, \ldots, b_r$ where $a < b_1, \ldots a < b_r$. It is easy to see that the definitions of $\mathrm{La}(n, P_k)$, $\mathrm{La}(n, V_r)$, in Sections 1 and 2 agree. In the rest of the paper we will use the two different terminology alternately. In the definition of $\mathrm{La}(n, R)$ we mean non-induced subposets, that is, if $R = V_2$ then P_3 is also excluded as a subposet.

A poset is *connected* if for any pair (z_0, z_k) of its elements there is a sequence $z_1, \ldots, z_{k-1}$ such that either $z_i < z_{i+1}$ or $z_i > z_{i+1}$ holds for $0 \leq i < k$. If the poset is not connected, maximal connected subposets are called its *connected components*. Given a family $\mathcal{F}$ of subsets of $[n]$, it spans a poset in B_n. We will consider its connected components Q in two different ways. First as posets themself, secondly as they are represented in B_n. In the latter case the sizes of in the sets are also indicated. This is called a realization of Q. A *full chain* in B_n is a family of sets $A_0 \subset A_1 \subset \ldots \subset A_n$ where $|A_i| = i$. We say that a (full) chain *goes through* a family (subposet)

$\mathcal{F}$ if their intersection is non-empty, that is if it "goes through" at least one member of the family.

3. Lubell's Proof of the Sperner Theorem

The number of full chains in $[n]$ is $n!$ since the choice of a full chain is equivalent to the choice of a permutation of the elements of $[n]$. On the other hand, the number of full chains going through a given set F of f elements is $f!(n-f)!$ since the chain "must grow" within F until it "hits" F and outside after that. Suppose that the family $\mathcal{F}$ of subsets of $[n]$ is without inclusion ($F, G \in \mathcal{F}$ implies $F \not\subset G$). Then a full chain cannot go through two members of $\mathcal{F}$. Therefore the set of full chains going through distinct members of $\mathcal{F}$ must be disjoint. Hence we have

$$\sum_{F\in\mathcal{F}} |F|!\big(n-|F|\big)! \le n!\,.$$

Dividing the inequality by $n!$

$$\sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}} \le 1 \tag{3.1}$$

is obtained. Replace $\binom{n}{|F|}$ by $\binom{n}{\lfloor \frac{n}{2}\rfloor}$. Then

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2}\rfloor}} = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2}\rfloor}} \le 1$$

follows, the theorem is proved.

Let us remark that inequality (3.1) is important on its own right and is called the YBLM-inequality (earlier LYM, see [17], [2], [12], [13]). ∎

4. The Method, Illustrated with an Old Result

Lubell's proof easily applies for Theorem 1.2, however, surprisingly it was not exploited for proving theorems of the present type. The reason might be that not the "excluded" configurations should be considered when using the idea, but the "allowed induced posets". (See later.)

Following the definition of the r-fork, let us define the r-*brush* (in a poset) which contains $r+1$ elements: $a, b_1, \dots, b_r$ where $a > b_1, \dots, a > b_r$ and is the "dual" of the r-fork. (Here and in what follows we use the expression "dual" when the complements of the sets involved are considered.) Theorem 1.3 gives the best expected asymptotic upper bound up to the second term for V_2 in the Boolean lattice. It is easy to see that it implies the same solution for Λ_2. However the result is very different when both of them are excluded. Our notation $\mathrm{La}(n, R)$ is extended in an obvious way for the case when two subposets R_1 and R_2 are excluded: $\mathrm{La}(n, R_1, R_2)$.

Theorem 4.1 [11]**.**

$$\mathrm{La}(n, V_2, \Lambda_2) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

Proof. The construction giving the equality is the following:

$$\left\{F \subset [n]\colon 1 \notin F,\ |F| = \left\lfloor \frac{n-1}{2} \right\rfloor\right\} \cup \left\{F \subset [n]\colon 1 \in F,\ |F| = \left\lfloor \frac{n+1}{2} \right\rfloor\right\}.$$

The non-trivial part of the proof is the verification of the upper bound.

Let $\mathcal{F}$ be a family of subsets of $[n]$ which contains neither a V_2 nor a Λ_2 as subposet. Therefore it cannot contain a P_3 either. Consider the connected components of the poset spanned by $\mathcal{F}$. It is obvious that a connected component can be either a one element poset P_1 or a P_2. Let α_1 and α_2 be their respective numbers. Then

$$|\mathcal{F}| = \alpha_1 + 2\alpha_2. \tag{4.1}$$

We will now determine the minimum number of full chains going through a one or two-element component. Let $P(1;a)$ be a one-element component which is an a-element set. The number $c\big(P(1;a)\big)$ of full chains going through $P(1;a)$ is $a!(n-a)!$. Therefore

$$\frac{c\big(P(1;a)\big)}{n!} = \frac{1}{\binom{n}{a}}.$$

It takes on its minimum at the value $a = \lfloor \frac{n}{2} \rfloor$. Hence we obtained

$$\left\lfloor \frac{n}{2} \right\rfloor! \left\lceil \frac{n}{2} \right\rceil! \le c\big(P(1;a)\big). \tag{4.2}$$

The two-element component consisting of an a-element subset A and a b-element subset B $(A \subset B)$ $(a < b)$ is denoted by $P(2; a, b)$. The number $c\big(P(2; a, b)\big)$ of full chains going through (at least one element of) $P(2; a, b)$ is

$$c\big(P(2; a, b)\big) = a!(n-a)! + b!(n-b)! - a!(b-a)!(n-b)!\,. \tag{4.3}$$

Divide it by $n!$.

$$\frac{c\big(P(2; a, b)\big)}{n!} = \frac{1}{\binom{n}{a}} + \frac{1}{\binom{n}{b}} - \frac{1}{\binom{n}{b}\binom{b}{a}} = \frac{1}{\binom{n}{a}} + \frac{1}{\binom{n}{b}}\left(1 - \frac{1}{\binom{b}{a}}\right). \tag{4.4}$$

Suppose first that a is fixed and is $\le \left\lfloor \frac{n-1}{2} \right\rfloor$. Then (4.4) takes on its minimum for $b = B\left\lfloor \frac{n+1}{2} \right\rfloor$. Fix b here and consider the following variant of (4.4):

$$\frac{c\big(P(2; a, b)\big)}{n!} = \frac{1}{\binom{n}{a}} + \frac{1}{\binom{n}{b}} - \frac{1}{\binom{n}{a}\binom{n-a}{n-b}} = \frac{1}{\binom{n}{b}} + \frac{1}{\binom{n}{a}}\left(1 - \frac{1}{\binom{n-a}{n-b}}\right). \tag{4.5}$$

This is a monotone decreasing function of a in the interval $0 \le a \le \left\lceil \frac{n}{2} \right\rceil$. Therefore the pair giving the minimum in this case is $a = \left\lfloor \frac{n-1}{2} \right\rfloor$, $b = \left\lfloor \frac{n+1}{2} \right\rfloor$.

Suppose now that $a \ge \left\lfloor \frac{n}{2} \right\rfloor$. Then b can be chosen to be $a+1$ by (4.4), and (4.3) becomes $na!(n-a-1)!$ It achieves its minimum at $a = \left\lfloor \frac{n-1}{2} \right\rfloor$, again. We obtained

$$n \left\lfloor \frac{n-1}{2} \right\rfloor! \left\lceil \frac{n-1}{2} \right\rceil! \le c\big(P(2; a, b)\big). \tag{4.6}$$

Observe that a full chain cannot go through two distinct components, therefore

$$\sum_{P_1 \text{ is a component}} c(P_1) + \sum_{P_2 \text{ is a component}} c(P_2) \le n!$$

holds. The left hand side can be lower estimated by (4.2) and (4.6):

$$\alpha_1 \left\lfloor \frac{n}{2} \right\rfloor! \left\lceil \frac{n}{2} \right\rceil! + \alpha_2 n \left\lfloor \frac{n-1}{2} \right\rfloor! \left\lceil \frac{n-1}{2} \right\rceil! \le n!\,. \tag{4.7}$$

(4.1) has to be maximized with respect to (4.7). Rewrite (4.7) a little bit:

$$\alpha_1 \left\lfloor \frac{n}{2} \right\rfloor! \left\lceil \frac{n}{2} \right\rceil! + 2\alpha_2 \frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor! \left\lceil \frac{n-1}{2} \right\rceil! \le n!\,. \tag{4.8}$$

Compare the coefficients of α_1 and $2\alpha_2$ in (4.8).

$$\left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! \geq \frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \tag{4.9}$$

holds (with equality for even n). Replacing the coefficient of α_1 in (4.8) using (4.9), the inequality

$$(\alpha_1 + 2\alpha_2)\frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \leq n!$$

is obtained, what results in

$$|\mathcal{F}| = \alpha_1 + 2\alpha_2 \leq \frac{n!}{\frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil !} = 2\binom{n-1}{\left\lfloor \frac{n-1}{2} \right\rfloor}. \qquad \blacksquare$$

5. The Method, in General

Let $\mathcal{P}$ be the set of forbidden subposets. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that the poset induced by $\mathcal{F}$ in B_n contains no member of $\mathcal{P}$ as a subposet. $\mathrm{La}(n, \mathcal{P})$ denotes the largest size of such a family. Consider the connected components of the poset induced by $\mathcal{F}$. The family of all possible components is denoted by $\mathcal{Q} = \mathcal{Q}(\mathcal{P})$.

In our Section 4 we had $\mathcal{P} = \{V_2, \Lambda_2\}$. Then $\mathcal{Q}\big(\{V_2, \Lambda_2\}\big) = \{P_1, P_2\}$.

If $Q \in \mathcal{Q}$ let Q_n^* be a realization of Q in the Boolean lattice B_n, that is, Q is embedded into B_n and a size (of a subsets) is associated with each element $q \in Q_n^*$. Here $Q \to Q_n^*$ denotes that Q_n^* is a realization of Q. In Section 4, for instance, P_2 is a path containing two elements, while P_2 is a labelled path, labelled with two integers a and b.

Furthermore $c(Q_n^*)$ denotes the number of chains going through Q_n^*. In our example these numbers are $a!(n-a)!$ and $a!(b-a)!(n-b)!$, respectively.

Let $\min_{Q \to Q_n^*} c(Q_n^*) = c_n^*(Q)$ be the smallest number of chains respect to the realizations. In the example: $c_n^*(P_1) = \lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !$, $c_n^*(P_2) = n \lfloor \frac{n-1}{2} \rfloor ! \lceil \frac{n-1}{2} \rceil !$.

Theorem 5.1.

$$\mathrm{La}(n, \mathcal{P}) \leq \frac{n!}{\inf_{Q \in \mathcal{Q}(\mathcal{P})} \frac{c_n^*(Q)}{|Q|}}.$$

Proof. Let $\mathcal{F}$ be a family without a copy of any of the posets in $\mathcal{P}$. The connected components of the poset induced by $\mathcal{F}$ all belong to $\mathcal{Q}(\mathcal{P})$. Since no chain can go through two distinct components, the sum of the numbers of chains cannot exceed the total number of chains.

$$\sum_{Q\in\mathcal{Q}(\mathcal{P})}\ \sum_{Q_n^*:\, Q\to Q_n^*} c(Q_n^*) \le n!\,. \tag{5.1}$$

Since

$$Q \to Q_n^* \quad \text{implies} \quad c_n^*(Q) \le c(Q_n^*)$$

(5.1) can be replaced by

$$\sum_{Q\in\mathcal{Q}(\mathcal{P})} \big|\{Q_n^*\colon Q \to Q_n^*\}\big|\, c_n^*(Q) \le n!\,. \tag{5.2}$$

Easy manipulations on the left hand side give

$$\sum_{Q\in\mathcal{Q}(\mathcal{P})} \big|\{Q_n^*\colon Q \to Q_n^*\}\big|\,|Q|\frac{c_n^*(Q)}{|Q|}$$

$$\ge \sum_{Q\in\mathcal{Q}(\mathcal{P})} \big|\{Q_n^*\colon Q \to Q_n^*\}\big|\,|Q| \inf_{Q\in\mathcal{Q}(\mathcal{P})} \frac{c_n^*(Q)}{|Q|}$$

and a new form of (5.2):

$$\inf_{Q\in\mathcal{Q}(\mathcal{P})} \frac{c_n^*(Q)}{|Q|} \sum_{Q\in\mathcal{Q}(\mathcal{P})} \big|\{Q_n^*\colon Q \to Q_n^*\}\big|\,|Q| \le n!\,. \tag{5.3}$$

Here

$$\sum_{Q\in\mathcal{Q}(\mathcal{P})} \big|\{Q_n^*\colon Q \to Q_n^*\}\big|\,|Q| = \mathcal{F},$$

and (5.3) gives

$$|\mathcal{F}| \inf_{Q\in\mathcal{Q}(\mathcal{P})} \frac{c_n^*(Q)}{|Q|} \le n!$$

what proves the theorem. ■

6. The Upper Estimate in Theorem 1.3

This theorem already has two different proofs in [11] and [4], however each of these proofs needed an *ad hoc* idea, our new method also works here. It needs some tedious calculations, but the principal idea is as easy as in Section 4. Especially if the concise form, Theorem 5.1 is used.

Suppose that $\mathcal{F}$ contains no V_2 as a subposet. Then it cannot contain a P_3 either. It is easy to deduce that the components of the poset spanned by $\mathcal{F}$ are all of type Λ_r where $0 \le r$. This is a new phenomenon! The sizes of the components are unbounded. Yet, the method works. This is why we had to write "inf" in Theorem 5.1.

In terms of Section 5: $\mathcal{P} = \{V_2\}$ and $\mathcal{Q}(\mathcal{P}) = \{\Lambda_0, \Lambda_1, \Lambda_2, \dots, \Lambda_r \dots\}$. The following lemma gives a good lower estimate on $c_n^*(\Lambda_r)$. For the sake of completeness the proof from [10] is repeated.

Lemma 6.1. *Suppose* $6 \le n, 1 \le r$. *Then*

$$u^*!(n-u^*)! + ru^*!u^*(n-u^*-1)! \le c_n^*(\Lambda_r)$$

holds where $u^* = u^*(n) = \frac{n}{2} - 1$ *if* n *is even,* $u^* = \frac{n-1}{2}$ *if* n *is odd and* $r - 1 \le n$, *while* $u^* = \frac{n-3}{2}$ *if* n *is odd and* $n < r - 1$.

In the case $r = 0$ *the inequality* $\lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil ! \le c_n^*(\Lambda_0)$ *holds.*

Proof. By symmetry we can consider V_r instead of Λ_r. Since it was done in this form in [10] it is more convenient to use this form for the proof. Let $V(r; u, u_1, \dots, u_r)$ $(u < u_1, \dots, u_r)$ be a realization of Λ_r (in notation $V_r \to V(r; u, u_1, \dots, u_r)$) where the subset of u elements is included in all other ones of sizes $u_1, \dots, u_r$, respectively.

1. One can easily show by using the sieve that

$$c\big(V(r; u, u_1, \dots, u_r)\big)$$
$$= u!(n-u)! + \sum_{i=1}^{r} u_i!(n-u_i)! - \sum_{i=1}^{r} u!(u_i-u)!(n-u_i)!\,.$$

This will actually be used in the form

$$c\big(V(r; u, u_1, \dots, u_r)\big) \tag{6.1}$$
$$= \sum_{i=1}^{r} \left(\frac{1}{r} u!(n-u)! + u_i!(n-u_i)! - u!(u_i-u)!(n-u_i)! \right).$$

Dividing one term by $n!$ two useful forms are obtained for the summand in (6.1):

$$\frac{1}{r\binom{n}{u}}+\frac{1}{\binom{n}{u_i}}-\frac{1}{\binom{n}{u_i}\binom{u_i}{u}}=\frac{1}{r\binom{n}{u}}+\frac{1}{\binom{n}{u_i}}\left(1-\frac{1}{\binom{u_i}{u}}\right) \tag{6.2}$$

and

$$\frac{1}{r\binom{n}{u}}+\frac{1}{\binom{n}{u_i}}-\frac{1}{\binom{n}{u}\binom{n-u}{n-u_i}}=\frac{1}{\binom{n}{u_i}}+\frac{1}{\binom{n}{u}}\left(\frac{1}{r}-\frac{1}{\binom{n-u}{n-u_i}}\right). \tag{6.3}$$

2. First we will show that (6.2)–(6.3) attains its minimum under the condition $u<u_i$ for some pair $u, u_i=u+1$.

If $\frac{n}{2}-1\le u$, fix u and consider changing u_i in (6.2). Here, $\binom{n}{u_i}$ is a decreasing fuction of u_i in the interval $\left[\left\lfloor\frac{n}{2}\right\rfloor, n\right]$, while $\binom{u_i}{u}$ is increasing. Therefore, one can suppose that $u_i=u+1$, and we are done.

Else, $\frac{n}{2}-1>u$ and the method in (6.2) above leads to $u_i\le\left\lfloor\frac{n}{2}\right\rfloor$. Fix this value and increase u using (6.3). It will not increase by moving u to $u=\left\lfloor\frac{n}{2}\right\rfloor-1$.

Hence, we obtained the lower estimate

$$\min_u\left(\frac{1}{r}u!(n-u)!+(u+1)!(n-u-1)!-u!1!(n-u-1)!\right)$$
$$=\min_u\left(\frac{1}{r}u!(n-u)!+u!u(n-u-1)!\right)$$

for (6.2)–(6.3) and therefore we have

$$\min_u\big(u!(n-u)!+ru!u(n-u-1)!\big)\le c\big(V(r;u,u_1,\dots,u_r)\big) \tag{6.4}$$

This minimum will be determined in the rest of the proof.

3. Suppose now that $2\le r$. Take the "derivative" of $f_r(u)=u!(n-u)!+ru!u(n-u-1)!$, that is, compare two consecutive places of $f_r(u)$. When does the inequality

$$\begin{aligned} f_r(u-1)&=(u-1)!(n-u+1)!+r(u-1)!(u-1)(n-u)! \\ &<f_r(u)=u!(n-u)!+ru!u(n-u-1)!\,. \end{aligned} \tag{6.5}$$

hold? It is equivalent to

$$0 < 2(r-1)u^2 - \big(n(r-3)+r-1\big)\,u - n^2 + (r-1)n.$$

The discriminant of the corresponding quadratic equation in u is

$$\big(n(r-3)+r-1\big)^2 + 8(r-1)\big(n^2-(r-1)n\big)$$
$$= (r+1)^2n^2 - 2(r-1)(3r-1)n + (r-1)^2.$$

The latter expression can be strictly upper estimated by

$$\big((r+1)n-(r-1)\big)^2,$$

if $r+1 < 3r-1$ holds, that is, if $r > 1$. Hence, the larger root α_2 of the quadratic equation is less than

$$\frac{n(r-3)+r-1+(r+1)n-(r-1)}{4(r-1)} = \frac{n}{2}.$$

On the other hand, as it is easy to see, $\big(n(r+1)-3(r-1)\big)^2$ is a lower bound for the discriminant if $r-1 \le n$ holds. Using this estimate we obtain that $\frac{n-1}{2} \le \alpha_2$ in this case. Substituting this lower estimate into the formula for the smaller root α_1 we obtain $\alpha_1 \le 0$ when $n \ge r-1$. Since (6.5) holds exactly below α_1 and above α_2, we can state that $f_r(u)$ attains its minimum at $u = \lfloor \alpha_2 \rfloor$. By the inequalities above we can conclude that this is at $\frac{n}{2}-1$ if n is even and $\frac{n-1}{2}$ if n is odd. The statement of the lemma is proved in the case of $n \ge r-1$.

Else suppose $n < r-1$. The inequality $\alpha_2 < \frac{n-1}{2}$ can be proved in the same way as in the previous case. On the other hand, $6 \le n$ implies that $\big(n(r+1)-5(r-1)\big)^2$ is a lower estimate on the discriminant, hence we have $\frac{n}{2}-1 < \alpha_2$. This gives that $\alpha_1 < \frac{3}{2}$. The if n is even, $\lfloor\alpha_2\rfloor$ is again $\frac{n}{2}-1$, while $\lfloor\alpha_2\rfloor = \frac{n-3}{2}$ when n is odd. Although $f_r(0) < f_r(1)$ is allowed by this estimate, it is easy to check that $f_r(0) > f_r(1)$ holds in reality. By (6.4) the proof is finished for $r \ge 2$.

The case $r = 1$ is much easier. The comparison (6.5) leads to a linear inequality which is an equality for $u = \frac{n}{2}$. The formula $f_1(u)$ also has its minimum at $\lfloor \frac{n-1}{2} \rfloor$. (But it has the same value at $\frac{n}{2}-1$ and $\frac{n}{2}$.) $\blacksquare_{\mathrm{L}}$

The inequality

$$\text{(6.6)}\qquad u^*!u^*(n-u^*-1)! \le \frac{u^*!(n-u^*)! + ru^*!u^*(n-u^*-1)!}{r+1} \qquad (0 \le r)$$

is a consequence of the lemma and the remark on the case $r = 0$.

Lemma 6.1 and (6.6) gives

$$u^*!u^*(n-u^*-1)! \leq \frac{c_n^*(\Lambda_r)}{|\Lambda_r|}. \tag{6.7}$$

Substituting this into Theorem 5.1 we obtain

$$|\mathcal{F}| \leq \frac{n!}{u^*!u^*(n-u^*-1)!}.$$

This right hand side is equal to

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{\frac{n}{2}}{\frac{n}{2}-1}, \qquad \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{\frac{n+1}{2}}{\frac{n-1}{2}}, \qquad \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{\frac{n-1}{2}}{\frac{n-3}{2}}$$

in the cases $u^* = \frac{n}{2} - 1, \frac{n-1}{2}$ and $\frac{n-3}{2}$, respectively. These are all equal to

$$= \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right). \qquad \blacksquare$$

7. A Construction = A Lower Estimate

Although we concentrate in this paper on the upper estimates, it seems to be important to show the construction serving as a lower estimate in Theorem 1.3

The construction for a family avoiding a V_2 is the following. Take all the sets of size $\lfloor \frac{n}{2} \rfloor$ and a family $A_1, \dots A_m$ of $\lfloor \frac{n}{2} \rfloor + 1$-element sets satisfying the condition $|A_i \cap A_j| < \lfloor \frac{n}{2} \rfloor$ for every pair $i < j$. It is easy to see that this family contains no V_2. We only have to maximize m. Since the $\lfloor \frac{n}{2} \rfloor$-element subsets of the A_is are all distinct, we have

$$m\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

This gives the upper estimate

$$m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{2}{n}. \tag{7.1}$$

There is a very nice construction of such sets A_i with

$$m = \binom{n}{\lfloor \frac{n}{2} \rfloor + 1} \frac{1}{n} = \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(\frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right). \tag{7.2}$$

Let $\lfloor \frac{n}{2} \rfloor + 1$ be denoted by k in this proof for saving space. Consider the sets $\{x_1, x_2, \ldots, x_k\}$ where these xs are distinct integers in the interval $[n]$ satisfying

$$x_1 + x_2 + \ldots + x_k \equiv a \pmod{n} \tag{7.3}$$

for some fixed $a \in [n]$. It is easy to see that the intersection of two such sets is $< k$, as we needed. The total number of k-element sets is $\binom{n}{k}$, therefore there is an a for which the number of sets satisfying (7.3) is at least $\frac{1}{k}\binom{n}{k}$. We found the necessary number of "good" sets.

This construction can be found in this form in [9] (the paper contains a much more general form), but the basic idea have appeared in [5] and [16], too.

It is a longstanding conjecture of coding theory what the right constant is here, 1 or 2. Or if the limit exists at all? This is why there is a disturbing factor 2 between the second terms of the lower and upper estimates in Theorem 1.3. This gap cannot be bridged without making progress in the problem in coding theory mentioned above.

8. Excluding the N

The poset N contains 4 distinct elements a, b, c, d satisfying $a < c$, $b < c$, $b < d$. In the Boolean lattice a subposet N consists of four distincts subsets satisfying $A \subset C$, $B \subset C$, $B \subset D$. It is somewhat surprising that excluding N the result is basically the same as in the case of V_2. The goal of the present section is to prove the following theorem.

Theorem 8.1 [10]**.**

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \le \mathrm{La}(n, N) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right)$$

holds.

The lower estimate is obtained from Theorem 1.3, since $\mathrm{La}(n, V_2) \leq \mathrm{La}(n, N)$.

The upper estimate will be proved by Theorem 5.1, again. Here $\mathcal{P} = \{N\}$.

Let $\mathcal{F}$ be a family of subsets of $[n]$ containing no four distinct members forming an N. Consider the poset $P(\mathcal{F})$ spanned by $\mathcal{F}$ in B_n. What can its components be? A component might be a P_3, but no component can contain a P_3 as a proper subposet, since adding one more element to P_3 an N is created no matter which element of P_3 is in relation with the new element. Let $a < b$ be two elements of a component. We claim that a and b cannot be both comparable within the component with some other distinct elements c, d (say, in this order), unless they are a part of a P_3. Indeed, the choices $c < a$ and $b < d$ lead to a P_3, therefore the only possibility is $a < c, d < b$. This is an N, contradicting the assumption. But one of them can be comparable with many others in the same direction. Therefore the following ones are the only possible components:

$$\mathcal{Q}(\mathcal{P}) = \{P_3, \Lambda_0, \Lambda_1, \Lambda_2, \ldots, \Lambda_r, \ldots, V_1, V_2, \ldots, V_r, \ldots\}.$$

In order to use Theorem 5.1 we have to give a good lower bound on the ratios

$$\frac{c_n^*(P_3)}{3}, \qquad \frac{c_n^*(\Lambda_r)}{r+1}, \qquad \frac{c_n^*(V_r)}{r+1}. \tag{8.1}$$

(6.7) is a good lower estimate on the middle one. By symmetry, the same applies for the last one. The only unknown one is the first ratio. Its exact value is determined in [10] (Lemma 3.1). We do not repeat the proof, since both the statement and the proof are obvious.

Let $P(3; u, v, w)$ $(u < v < w)$ be a realization of P_3 by sets of sizes $u < v < w$.

Lemma 8.2. *$c\big(P(3; u, v, w)\big)$ $(u < v < w)$ takes its minimum for the values $u = \lfloor \frac{n}{2} \rfloor - 1$, $v = \lfloor \frac{n}{2} \rfloor$, $w = \lfloor \frac{n}{2} \rfloor + 1$, that is,*

$$\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)! \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)! \left(\left\lfloor \frac{n}{2} \right\rfloor^2 - n \left\lfloor \frac{n}{2} \right\rfloor + n^2 - 1\right) \leq c\big(P(3; u, v, w)\big).$$

Hence we have

$$\frac{1}{3}\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)! \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)! \left(\left\lfloor \frac{n}{2} \right\rfloor^2 - n \left\lfloor \frac{n}{2} \right\rfloor + n^2 - 1\right) \leq \frac{c_n^*(P_3)}{3} \tag{8.2}$$

If we are lucky, the left hand side of (8.2) is not smaller than the left hand side of (6.7). Indeed, the inequality

$$u^*!u^*(n-u^*-1)! \le \frac{1}{3}\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)!\left(\left\lceil\frac{n}{2}\right\rceil - 1\right)!\left(\left\lfloor\frac{n}{2}\right\rfloor^2 - n\left\lfloor\frac{n}{2}\right\rfloor + n^2 - 1\right)$$

can be easily checked (for $2 \le n$) by distinguishing the three cases of u^*.

Since $u^*!u^*(n-u^*-1)!$ is a lower estimate for all ratios in (8.1) the proof of Theorem 7.1 can be finished as in the case of Theorem 1.3. ■

Remarks. Knowing the estimate of Lemma 6.1 we obtained the proof of Theorem 7.1 almost free, we only had to show that P_3 does not decrease the infimum in the denominator of Theorem 5.1. Of course we cannot state that $\mathrm{La}(n, V_2) = \mathrm{La}(n, N)$, they are equal only asymptotically.

It is interesting to mention that the "La" function will jump if the excluded poset contains one more relation. The *butterfly* $\bowtie$ contains 4 elements: a, b, c, d with $a < c$, $a < d$, $b < c$, $b < d$.

Theorem 8.3. [4] *Let* $n \ge 3$. *Then* $\mathrm{La}(n, \bowtie) = \binom{n}{\lfloor n/2\rfloor} + \binom{n}{\lfloor n/2\rfloor+1}$.

9. A Further Generalization

Observe that the main part of a large family is near the middle, the total number of sets far from the middle is small. More precisely, let $0 < \alpha < \frac{1}{2}$ be a fixed real number. The total number of sets F (for a given n) of size satisfying

$$|F| \notin \left[n\left(\frac{1}{2}-\alpha\right), n\left(\frac{1}{2}+\alpha\right)\right] \tag{9.1}$$

is very small. It is well-known (see e.g. [1], page 214) that for a fixed constant $0 < \beta < \frac{1}{2}$

$$\sum_{i=0}^{\beta}\binom{n}{i} = 2^{n\left(h(\beta)+o(1)\right)}$$

holds where $h(x) = -x\log_2 x - (1-x)\log)2(1-x)$. Therefore the total number of sets satisfying (9.1) is at most

$$2\sum_{i=0}^{\lfloor n(\frac{1}{2}-\alpha)\rfloor}\binom{n}{i} = 2^{n\left(h(\frac{1}{2}-\alpha)+o(1)\right)} = \binom{n}{\lfloor\frac{n}{2}\rfloor}O\left(\frac{1}{n^2}\right) \tag{9.2}$$

where $0 < h\left(\frac{1}{2} - \alpha\right) < 1$ is a constant.

In view of this observation we can improve our main tool, Theorem 5.1. First we have to generalize $c_n^*(Q)$. Let $c_n^{*\alpha}(Q)$ denote $\min_{Q \to Q_n^*} c(Q_n^*)$ where only those realizations Q_n^* are considered whose member subsets are of size in the interval (9.1). It is obvious that $c_n^*(Q) \le c_n^{*\alpha}(Q)$. We actually believe that they are equal for large n, but we cannot prove this statement.

Theorem 9.1. *Let $0 < \alpha < \frac{1}{2}$ be a real number. Then*

$$\mathrm{La}(n, \mathcal{P}) \le \frac{n!}{\inf_{Q \in \mathcal{Q}(\mathcal{P})} \frac{c_n^{*\alpha}(Q)}{|Q|}} + \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right).$$

Proof. Let $\mathcal{F}$ be a family containing no subposet belonging to $\mathcal{P}$. Define $\mathcal{F}_\alpha \subset \mathcal{F}$ consisting of the sets F of sizes in the interval (9.1). The rest, $\mathcal{F} - \mathcal{F}_\alpha$ is denoted by $\mathcal{F}_{\overline{\alpha}}$.

$$|\mathcal{F}_{\overline{\alpha}}| \le \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right) \tag{9.3}$$

is a consequence of (9.2). We have to prove only that

$$|\mathcal{F}_\alpha| \le \frac{n!}{\inf_{Q \in \mathcal{Q}(\mathcal{P})} \frac{c_n^{*\alpha}(Q)}{|Q|}}. \tag{9.4}$$

The proof of Theorem 5.1 can be repeated. The only difference is that since all the sets in $\mathcal{F}_\alpha$ are of size in the interval (9.1), the components have the same property therefore c_n^* can be really replaced by $c_n^{*\alpha}$. The sum of (9.3) and (9.4) gives the statement of the theorem. ■

Theorem 9.2. *Let $1 \le r$ be a fixed integer, independent on n. Suppose that every element $Q \in \mathcal{Q}(\mathcal{P})$ has the following property: if $a \in Q$ then a covers at most r elements of Q. Then*

$$\mathrm{La}(n, \mathcal{P}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

Proof. Theorem 9.1 will be used with $0 < \alpha < \frac{1}{8r+12}$. Suppose that $Q \in \mathcal{Q}(\mathcal{P})$, $Q \to Q^*_n$, that is, Q^*_n is a realization of Q and its member sets are of size in the interval (9.1). Using the first two terms of the sieve

(9.5)
$$c(Q^*_n) \geq \sum_{F\in Q^*_n} |F|!\big(n-|F|\big)! - \sum_{F,G\in Q^*_n, G\subset F} |G|!\big(|F|-|G|\big)!\big(n-|F|\big)!$$
$$= \sum_{F\in Q^*_n} \Bigg(|F|!\big(n-|F|\big)! - \sum_{G\in Q^*_n:\, G\subset F} |G|!\big(|F|-|G|\big)!\big(n-|F|\big)!\Bigg).$$

Consider one term of (9.5) and divide it by $n!$:

$$\frac{1}{\binom{n}{|F|}}\Bigg(1 - \sum_{G\in Q^*_n:\, G\subset F} \frac{1}{\binom{|F|}{|G|}}\Bigg). \tag{9.6}$$

$n\big(\frac{1}{2}-\alpha\big) \leq |G| < |F| \leq n\big(\frac{1}{2}+\alpha\big)$ implies $|F|-|G| \leq 2\alpha n$, therefore $|G|$ can be $|F|-1, |F|-2, \ldots, |F|-\lfloor 2\alpha n\rfloor$. The number of sets G with $|G| = |F|-1$ is at most r by the assumption of the theorem. It is easy to see that the number of sets G with $|G| = |F|-i$ is at most r^i. Hence the following lower estimate on (9.6) is obtained:

$$\frac{1}{\binom{n}{|F|}}\Bigg(1 - \sum_{i=1}^{\lfloor 2\alpha n\rfloor} \frac{r^i}{\binom{|F|}{i}}\Bigg). \tag{9.7}$$

We will show that these negative terms are increasing in absolute value, when $n \geq 4r$. Compare two neighboring terms.

$$\frac{r^i}{\binom{|F|}{i}} \geq \frac{r^{i+1}}{\binom{|F|}{i+1}}$$

holds iff

$$|F| \geq (r+1)i + r. \tag{9.8}$$

Since $|F| \geq n\big(\frac{1}{2}-\alpha\big)$ and $i \leq 2\alpha n$, it is sufficient to show

$$n\left(\frac{1}{2}-\alpha\right) \geq (r+1)2\alpha n + r$$

rather than (9.8). However this is an easy consequence of $0 < \alpha < \frac{1}{8r+12}$ and $n \geq 4r$. By the monotonity a new lower estimate is obtained for (9.7):

$$\frac{1}{\binom{n}{|F|}}\left(1 - \frac{r}{|F|} - \frac{r^2}{\binom{|F|}{2}} - 2\alpha n \frac{r^3}{\binom{|F|}{3}}\right). \tag{9.9}$$

A further decrease can be obtained by choosing the appropriate, but different values of $|F|$ in the two factors of (9.9). Choose $|F| = \left\lfloor \frac{n}{2} \right\rfloor$ in the first factor and $|F| = \left\lceil n\left(\frac{1}{2} - \alpha\right)\right\rceil$ in the second one. The lower estimate

$$\frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}\left(1 - 2\frac{r}{n(1-2\alpha)} + O\left(\frac{1}{n^2}\right)\right) \tag{9.10}$$

is obtained.

(9.6)–(9.10) are lower estimates on one term of (9.5). Since (9.10) does not depend on F, we have the lower estimate

$$n!|Q_n^*|\frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}\left(1 - 2\frac{r}{n(1-2\alpha)} + O\left(\frac{1}{n^2}\right)\right)$$

on (9.5). Using the trivial $|Q_n^*| = |Q|$

$$\frac{c_n^{*\alpha}(Q)}{|Q|} \geq |Q|\frac{n!}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}\left(1 - 2\frac{r}{n(1-2\alpha)} + O\left(\frac{1}{n^2}\right)\right)$$

can be obtained what is independent on Q. Substitute this lower estimate into the first term of the statement of Theorem 9.1 to obtain an upper estimate:

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{1}{1 - \frac{2r}{n(1-2\alpha)} - O\left(\frac{1}{n^2}\right)} = \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{2r}{n(1-2\alpha)} + O\left(\frac{1}{n^2}\right)\right).$$

Theorem 9.1 gives

$$\mathrm{La}(n, \mathcal{P}) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n(1-2\alpha)} + O\left(\frac{1}{n^2}\right)\right).$$

Since this holds for arbitrary small α, the proof is complete. ■

If the family $\mathcal{F}$ contains no V_{r+1} then the components cannot contain a set which is contained in $r+1$ other sets. Therefore the conditions of Theorem 9.2 are satisfied (in the dual form). This is why Theorem 9.2 implies Theorem 1.5. However, as we will see in the next section, a stronger statement follows.

10. Excluding Induced Posets, Only

One can ask what happens if we exclude the posets R belonging to $\mathcal{P}$ only in a strict form, that is, there is no induced copy in the poset induced in B_n by the family. Given a "small" poset R, $\mathrm{La}^\sharp(n, R)$ denotes the maximum number of elements of $Y \subset 2^{[n]}$ (that is, the maximum number of subsets of $[n]$) such that R is not an induced subposet of the poset spanned by Y in B_n. This obviously generalizes for $\mathrm{La}^\sharp(n, \mathcal{P})$ where $\mathcal{P}$ is a set of posets.

For instance, calculating $\mathrm{La}(n, V_2)$ the path of length 3, P_3 is also excluded, while in the case of $\mathrm{La}^\sharp(n, V_2)$ this is allowed, three sets A, B, C are excluded from the family only when $A \subset B$, $A \subset C$ but B and C are incomparable. As we saw in the proof of Theorem 1.3, $\mathcal{Q}(V_2)$ consists of Λ_rs $(0 \le r)$. The set $\mathcal{Q}^\sharp(V_2)$ of possible components when only the induced V_2s are excluded is much richer. $\mathcal{Q}^\sharp(V_2)$ contains all posets whose graph is a "descending" tree with one maximal vertex. That is, not only the sizes of these posets are unbounded, but their depths, as well. Yet, this case can be also be treated, on the basis of Theorem 9.2. To be precise we have to modify the formulations of our previous theorems. These modifications need no proofs, since the original proofs did not really depended on $\mathcal{P}$, only on $\mathcal{Q}(\mathcal{P})$ and this is simply replaced by $\mathcal{Q}^\sharp(\mathcal{P})$.

Theorem 10.1.

$$\mathrm{La}^\sharp(n, \mathcal{P}) \le \frac{n!}{\inf_{Q \in \mathcal{Q}^\sharp(\mathcal{P})} \frac{c_n^*(Q)}{|Q|}}.$$

Theorem 10.2. *Let $0 < \alpha < \frac{1}{2}$ be a real number. Then*

$$\mathrm{La}^\sharp(n, \mathcal{P}) \le \frac{n!}{\inf_{Q \in \mathcal{Q}^\sharp(\mathcal{P})} \frac{c_n^{*\alpha}(Q)}{|Q|}} + \binom{n}{\lfloor \frac{n}{2} \rfloor} O\left(\frac{1}{n^2}\right).$$

Theorem 10.3. *Let $1 \le r$ be a fixed integer, independent on n. Suppose that every element $Q \in \mathcal{Q}^\sharp(\mathcal{P})$ has the following property: if $a \in Q$ then a covers at most r elements of Q. Then*

$$\mathrm{La}^\sharp(n, \mathcal{P}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

It is quite obvious that if $Q \in \mathcal{Q}^\sharp(V_{r+1})$ then no element of Q is covered by more than r other elements. Theorem 10.3 can be applied in a dual (upside-down) form.

Theorem 10.4.

$$\mathrm{La}^\sharp(n, V_{r+1}) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + 2\frac{r}{n} + O\left(\frac{1}{n^2}\right)\right).$$

This is a stronger form of Theorem 1.5. The special case $r = 1$ was solved in [6].

11. Concluding Remarks

1. We obtained Theorem 8.1 almost free after having the proof of Theorem 1.3 with our method. This probably will often happen. The solution for a given excluded configuration can be obtained by putting together estimates for "allowed" posets, which have been already solved for other excluded patterns.

2. In most of our results the extremal configuration consist of some full levels plus a thinned out level. Theorem 4.1 shows that this is not necessary true in all cases.

3. We do not see the limits of our method. We hope we will be able to generalize for more levels, like Theorems 8.3 and 1.2. However we are quite sure that it will not give the solution for every family $\mathcal{P}$ of posets.

4. The problem $\mathcal{P} = \{V_2, \Lambda_3\}$ is very instructive. It is easy to see that $\mathcal{Q}(\mathcal{P}) = \{P_1, P_2, \Lambda_2\}$. The cases of even and odd n are somewhat different. Consider first the case when n is even. Then $\frac{1}{3}c_n^*(\Lambda_2) = \frac{n}{2}!\frac{n}{2}!\left(1 - \frac{2}{3n}\right) < \frac{1}{2}c_n^*(P_2) = c_n^*(P_1)$. Therefore Theorem 5.1 implies

Theorem 11.1. *If n is even then*

$$\mathrm{La}(n, V_2, \Lambda_3) \le \frac{n!}{\frac{n}{2}!\frac{n}{2}!\left(1 - \frac{2}{3n}\right)} = \binom{n}{\frac{n}{2}}\left(1 + \frac{2}{3n} + O\left(\frac{1}{n^2}\right)\right).$$

Observe that Theorem 9.2 would only give $\frac{2}{n}$ in the second term. The conclusion is that, although Theorem 9.2 is rather strong in general, it could be week in special cases. On the other hand, it seems that the obvious construction, taking all $\frac{n}{2}$-element subsets is not optimal. We found a better construction for $n = 4$: $\big\{\{1,2\}, \{1,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}\big\}$. But we do not know such a construction for infinitely many n.

Suppose now that n is odd. Then $\frac{1}{3}c_n^*(\Lambda_2) = \frac{n-1}{2}!\frac{n-1}{2}!\left(\frac{n}{2} - \frac{1}{6}\right) < \frac{1}{2}c_n^*(P_2) < c_n^*(P_1)$. Theorem 5.1 gives the following upper estimate. The lower estimate comes from Theorem 4.1.

Theorem 11.2. *If n is odd then*

$$\binom{n}{\frac{n-1}{2}}\left(1 + \frac{1}{n}\right) \le \mathrm{La}(n, V_2, \Lambda_3) \le \binom{n}{\frac{n-1}{2}}\left(1 + \frac{4}{3n-1}\right).$$

Since the right hand side is an integer (42) for $n = 7$, it might cherish the hope that there are some nice constructions (similar to the construction of Theorem 4.1) giving equality in the upper bound, at least for some n. This equality can be achieved only by 14 Λ_2s: consisting of 14 four-element subsets $F_1, \dots, F_{14}$ and their 3-element subset pairs: T_i^1, T_i^2 where these are 28 distinct 3-element subsets, $T_i^1, T_i^2 \subset F_i$ but $T_i^j \not\subset F_\ell$ holds whenever $i \ne \ell$. Let $R_1, \dots, R_7$ denote the remaining seven 3-element sets. Since every F contains exactly four 3-element subsets, it must contain two of the Rs. Consequently there are 28 pairs $R \subset F$. Since each R has four 4-element supersets, they must be all in the family of Fs. However the minimum number of 4-element supersets of seven 3-element sets is 15. (This can be shown by taking the complements. The minimum size of the shadow of a family of seven 4-element sets can be obtained from $7 = \binom{5}{4} + \binom{3}{3} + \binom{2}{2}$. It is $\binom{5}{3} + \binom{3}{2} + \binom{2}{1} = 15$.) Our hopes failed.

But there might be other (perhaps infinitely many) n giving equality in the upper bound of Theorem 11.2. The next candidate is $n = 15$.

5. One feels that the optimal construction uses sets in the middle, only.

Conjecture 1. *For every poset $\mathcal{P}$ there is a constant $c(\mathcal{P})$ (independent on n) such that $\mathrm{La}(n, \mathcal{P})$ can be achieved by a family containing sets of size in the interval*

$$\left[\frac{n}{2} - c(\mathcal{P}), \frac{n}{2} + c(\mathcal{P})\right].$$

Let us remark that we have proved a much weaker version in Section 9: $\mathrm{La}(n, \mathcal{P})$ can be asymptotically achieved by sets from the interval

$$\left[n\left(\frac{1}{2} - \alpha\right), n\left(\frac{1}{2} + \alpha\right)\right]$$

where $0 < \alpha$ is arbitrarily small.

The conjecture is not true for the induced case. For instance if the induced Λ_2 is excluded (P_3 is allowed, determination of $\mathrm{La}^\sharp(n, \Lambda_2)$ then the optimal construction always contains the $\emptyset$.

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