Note

Largest family without $A \cup B \subseteq C \cap D$

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Abstract

Let $\mathscr{F}$ be a family of subsets of an n-element set not containing four distinct members such that $A \cup B \subseteq C \cap D$. It is proved that the maximum size of $\mathscr{F}$ under this condition is equal to the sum of the two largest binomial coefficients of order n. The maximum families are also characterized. A LYM-type inequality for such families is given, too.

Keywords: Families of subsets; Sperner; LYM

1. The inequalities

Let $[n] = \{1, \dots, n\}$ be a finite set and $\mathcal{F} \subset 2^{[n]}$ a family of its subsets. The well-known theorem of Sperner [9] says that if no member of $\mathcal{F}$ contains another member then $|\mathcal{F}| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}$, with equality iff $\mathcal{F}$ consists of all sets of size $\lfloor n/2 \rfloor$ or all sets of size $\lceil n/2 \rceil$.

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Moreover the LYM-type inequality [7,8,10] (see also [1])

$$\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} \leqslant 1$$

also holds for such a family. It is easy to see that the second inequality implies the first one. On the other hand, equality holds in the second inequality only when $\mathcal{F}$ consists of all sets of a fixed size. A family satisfying the conditions of Sperner's theorem (that is no member is contained in another member) is called an *antichain*.

The following theorem of Erdős [3] was the first generalization of Sperner's theorem. If the family $\mathcal{F}$ contains no *chain* $F_0 \subset F_1 \subset \cdots \subset F_k$ *(of length* $k+1$*)* of distinct subsets ($k \geqslant 1$ is an integer) then $|\mathcal{F}|$ cannot exceed the sum of the k largest binomial coefficients of order n. Katona and Tarján [6] determined the asymptotically largest family containing no 3 distinct members A, B, C satisfying $A \subset B, A \subset C$. The interested reader can find a large variety of related results in [2].

The main aim of the present note is to investigate an analogous problem, when $\mathcal{F}$ contains no four distinct sets A, B, C, D such that A is contained in both C and D, and at the same time B is contained in both C and D. In other words,

$$\text{there are no four distinct } A, B, C, D \in \mathcal{F} \text{ with } A \cup B \subseteq C \cap D. \qquad (*)$$

Following the suggestion of Professor J. Griggs, a family satisfying $(*)$ will be called *butterfly-free* or a *butterfly-free meadow*. It is easy to check that the family consisting of all k and $k+1$-element subsets satisfies $(*)$. We will see that this is the largest family for the appropriate choice of k. Observe that if $\mathcal{F}$ contains no butterfly then it cannot contain a chain of length 4, therefore Erdős's theorem implies that $|\mathcal{F}|$ is at most the sum of the 3 largest binomial coefficients of order n. On the other hand, our condition does not exclude chains of length 3, therefore Erdős's theorem does not imply the present result. Another little surprise is that the result, unlike the result in [6], is sharp.

Theorem 1. *Let* $n \geqslant 3$*. If the family* $\mathcal{F} \subseteq 2^{[n]}$ *satisfies* $(*)$ *then* $|\mathcal{F}|$ *cannot exceed the sum of the two largest binomial coefficients of order n, i.e.,* $|\mathcal{F}| \leqslant \binom{n}{\lfloor n/2\rfloor} + \binom{n}{\lfloor n/2\rfloor+1}$.

The LYM-type inequality holds only if $\emptyset$ and $[n]$ are excluded from the family.

Theorem 2. *Suppose* $n \geqslant 3$*. Let* $\mathcal{F} \subseteq 2^{[n]}$ *such that* $\emptyset, [n] \notin \mathcal{F}$*. If* $\mathcal{F}$ *satisfies* $(*)$*, then*

$$\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} \leqslant 2.$$

Let us first prove Theorem 2 by the method of cyclic permutations [5]. Consider the elements of $[n]$ to be arranged along a cycle. That is the elements are considered modulo n. An *interval* is a subset of form $\{k, k+1, \ldots, l\}$ where $1 \leqslant k, l \leqslant n$. (This is clear if $k \leqslant l$. If however $l < k$ then the interval is of the form $\{k, k+1, \ldots, n-1, n, 1, 2, \ldots l-1, l\}$.) Intervals will be denoted by $\hat{A}$, $\hat{B}$ etc. Families of intervals are denoted by $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$, etc. The proof starts with two lemmas.

Lemma 1. *Let $\hat{\mathcal{F}}$ be a family of intervals, $\emptyset, [n] \notin \hat{\mathcal{F}}$, such that any member $\hat{F} \in \hat{\mathcal{F}}$ is contained in at most one other member of $\hat{\mathcal{F}}$. If m denotes the number of the maximal members, a denotes the number of non-maximal members then*

$$m + \frac{a}{2} \leqslant n \tag{1}$$

holds.

Proof. A *full chain* is a family of subsets $L_1 \subset \cdots \subset L_n$ where $|L_i| = i$ for $1 \leqslant i \leqslant n$. We will count the number of pairs $(\hat{F}, \hat{\mathcal{L}})$ where $\hat{F} \in \hat{\mathcal{F}}$, $\hat{\mathcal{L}}$ is a full chain of intervals and $\hat{F} \in \hat{\mathcal{L}}$. The number of full chains of intervals containing $\hat{F}$ is $2^{|\hat{F}|-1}2^{n-|\hat{F}|-1} = 2^{n-2}$. Hence the total number of such pairs is $(m+a)2^{n-2}$.

Suppose that $\hat{A} \subset \hat{B}, \hat{A} \neq \hat{B}$. We give an upper bound on the number of full chains containing both of them. The number of choices of the new members of the full chains "between the two sets" is at most $2^{|\hat{B}|-|\hat{A}|-1}$ since, at least once, there is only one choice. Therefore the number of such full chains is at most $2^{|\hat{A}|-1}2^{|\hat{B}|-|\hat{A}|-1}2^{n-|\hat{B}|-1} = 2^{n-3}$. The total number of full chains is $n2^{n-2}$. Since a full chain contains one or two members, we obtain the inequality

$$(m+a)2^{n-2} \leqslant n2^{n-2} + a2^{n-3}$$

which is equivalent to the statement of the lemma. □

Lemma 2. *If $\hat{\mathcal{F}}$ is a family of intervals satisfying $(*)$, and $\emptyset, [n] \notin \hat{\mathcal{F}}$, then $|\hat{\mathcal{F}}| \leqslant 2n$.*

Proof. It is easy to see by complementation that the previous lemma holds for a family in which any member contains at most one other member. Divide $\hat{\mathcal{F}}$ into three subfamilies: the maximal $(\hat{\mathcal{M}}_1)$, the minimal $(\hat{\mathcal{M}}_2)$ and other members $(\hat{\mathcal{A}})$. Introduce the notations $|\hat{\mathcal{M}}_1| = m_1, |\hat{\mathcal{M}}_2| = m_2, |\hat{\mathcal{A}}| = a$. It is easy to see that $(*)$ implies that $\hat{\mathcal{M}}_1 \cup \hat{\mathcal{A}}$ satisfies the conditions of the previous lemma. Therefore we have $m_1 + \frac{a}{2} \leqslant n$. On the other hand, $\hat{\mathcal{M}}_2 \cup \hat{\mathcal{A}}$ satisfies the complementing of the previous lemma, we obtain the inequality $m_2 + \frac{a}{2} \leqslant n$. The sum of the two inequalities is $m_1 + m_2 + a \leqslant 2n$ as desired. □

Proof of Theorem 2. We will double-count the pairs (C, F) where C is a cyclic permutation of $[n]$, $F \in \mathcal{F}$ and F is an interval along C. For a fixed F the number of cyclic permutations is $|F|!(n-|F|)!$ therefore the number of pairs in question is

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)!.$$

For a fixed cyclic permutation C the number of possible F's is at most $2n$ by the previous lemma. We obtain the inequality

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \leqslant (n-1)!2n.$$

This is equivalent to the statement of the theorem. □

Proof of Theorem 1. If none of $\emptyset$ and $[n]$ is a member of $\mathcal{F}$ then the statement is an easy consequence of Theorem 2. If both of them are in $\mathcal{F}$ then $\mathcal{F}-\{\emptyset,[n]\}$ is an antichain, therefore we have the upper estimate $\binom{n}{\lfloor n/2\rfloor}+2$, which is less than our need, if $n\geqslant 3$. Suppose that exactly one of $\emptyset$ and $[n]$ is in $\mathcal{F}$. By complementation $\emptyset\in\mathcal{F}$ can be supposed. Then $\mathcal{F}'=\mathcal{F}-\{\emptyset\}$ contains no 3 distinct members A,B,C such that $A\subset B, A\subset C$. It was proved in [6] (our Corollary 2 in Section 2 is slightly weaker) that

$$|\mathcal{F}'|\leqslant\left(1+\frac{2}{n}\right)\binom{n}{\lfloor n/2\rfloor}$$

holds under this condition. This upper estimate is strong enough when $n\geqslant 3$. $\square$

Remark. Dániel Gerbner (student in Budapest) [4] noticed that there is no need to use the theorem from [6], since replacing $\emptyset$ by an arbitrarily chosen one-element set $\{i\}\notin\mathcal{F}$ reduces the problem to the case when $\emptyset,[n]\notin\mathcal{F}$. The case when $\emptyset$ and all one-element sets are in $\mathcal{F}$ is trivial.

2. Cases of equality

The methods of the previous section are not strong enough for finding the cases of equality. The conditions of Lemma 1 allow a large variety of families with equality. Therefore we have to consider the whole original family, rather than just the intervals.

Lemma 3. *Let $\mathcal{M}$ and $\mathcal{A}$ be two disjoint antichains in $2^{[n]}$ where $[n]\notin\mathcal{M}$. Suppose that for any $A\in\mathcal{A}$ there is a unique $f(A)\in\mathcal{M}$ with $A\subset f(A)$. Then*

$$\sum_{M\in\mathcal{M}}\binom{n}{|M|}^{-1}+\sum_{A\in\mathcal{A}}\binom{n}{|A|}^{-1}\left(1-\frac{1}{n-|A|}\right)\leqslant 1 \tag{2}$$

holds, with equality only when either $|f(A)|=n-1$ or $|f(A)|=|A|+1$ holds for each $A\in\mathcal{A}$.

Proof. The number of chains containing a set M is $|M|!(n-|M|)!$. Adding these numbers for all members of $\mathcal{M}$ and $\mathcal{A}$, a chain is counted once or twice. The latter can happen only if the chain contains an $A\in\mathcal{A}$ and $f(A)\in M$. The total number of chains is $n!$. The number of chains containing both A and $f(A)$ is $|A|!(|f(A)|-|A|)!(n-|f(A)|)!$. Hence, we have the following inequality:

$$\sum_{M\in\mathcal{M}}|M|!(n-|M|)!+\sum_{A\in\mathcal{A}}|A|!(n-|A|)!$$
$$\leqslant n!+\sum_{A\in\mathcal{A}}|A|!(|f(A)|-|A|)!(n-|f(A)|)!.$$

Dividing by $n!$ we obtain

$$\sum_{M\in\mathcal{M}}\binom{n}{|M|}^{-1}+\sum_{A\in\mathcal{A}}\binom{n}{|A|}^{-1}\left(1-\binom{n-|A|}{n-|f(A)|}^{-1}\right)\leqslant 1. \tag{3}$$

Since $|A| < |f(A)| < n$, the inequality $n - |A| \leqslant \binom{n-|A|}{n-|f(A)|}$ can be used in (3) to obtain (2). $\square$

We know that $n - |A| \geqslant 2$, which implies the following immediate corollary.

Corollary 1. *Under the conditions of Lemma 3*

$$\sum_{M \in \mathcal{M}} \binom{n}{|M|}^{-1} + \sum_{A \in \mathcal{A}} \frac{1}{2} \binom{n}{|A|}^{-1} \leqslant 1 \tag{4}$$

holds, where equality is possible only when $|A| = n - 2$ and $|f(A)| = n - 1$ for each $A \in \mathcal{A}$.

Corollary 2 (*Katona and Tarján [6]*). *Let $n \geqslant 4$. Suppose that the family $\mathcal{F}$ contains no three distinct members A, B, C such that $A \subset B, C$. Then*

$$|\mathcal{F}| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n-3}\right) \tag{5}$$

holds.

Proof. If $[n] \in \mathcal{F}$ then the rest of $\mathcal{F}$ satisfies the conditions of Sperner's theorem. So we can suppose $[n] \notin \mathcal{F}$. If we see that

$$\binom{n}{|A|} \frac{n - |A|}{n - |A| - 1} \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor} \left(1 + \frac{2}{n-3}\right)$$

holds for every $0 \leqslant |A| \leqslant n - 2$, then Lemma 3 implies (5). That is, we have to find the maximum of the function $g(i) = \binom{n}{i} \frac{n-i}{n-i-1}$ in the interval $0 \leqslant i \leqslant n-2$. Here $g(i-1) \leqslant g(i)$ holds if and only if $i(n-i-1) \leqslant (n-i)^2$. The discriminant $\sqrt{n^2 - 6n + 1}$ of this quadratic inequality can be bounded from below and above by $n-4$ and $n-3$, respectively, provided $n > 7.5$. Using these estimates it is easy to see that the smaller root of the equation $i(n-i-1) = (n-i)^2$ is in the interval $(\frac{n}{2} + \frac{1}{2}, \frac{n}{2} + 1)$ while the larger root is larger than $n-2$. Hence, $g(i-1) < g(i)$ holds if and only if $1 \leqslant i < \frac{n}{2} + 1$. The function $g(i)$ takes on its maximum in the interval $1 \leqslant i \leqslant n-2$ at $\lfloor \frac{n+1}{2} \rfloor$. This is also true for $n = 4, 5, 6, 7$ which can be checked separately. $\square$

This corollary is slightly weaker than the statement in [6] , but its proof is much shorter.

Theorem 3. *If $n = 3$ or $n \geqslant 5$ then equality holds in Theorem 2 only if the family consist all k and $k+1$-element sets for some k. The same is true for Theorem 1, with k equal to $\lfloor \frac{n-1}{2} \rfloor$ or $\lceil \frac{n-1}{2} \rceil$. If $n = 4$ then there is up to isomorphism one more extremal family for both theorems:*

$$\binom{[4]}{2} \cup \{\{1\}, \{2, 3, 4\}, \{2\}, \{1, 3, 4\}\}.$$

Proof. First suppose that $\emptyset, [n] \notin \mathcal{F}$. We will give a new proof of Theorem 2 which does not use the cyclic permutations. Proceed similarly to the proof of Lemma 2. Define $\mathcal{M}_1$ and $\mathcal{M}_2$ as the families of maximal and minimal members of $\mathcal{F}$, respectively. $\mathcal{A} = \mathcal{F} - \mathcal{M}_1 - \mathcal{M}_2$. It is easy to see that $\mathcal{M}_1 \cup \mathcal{A}$ satisfies the conditions of Corollary 1. On the other hand, the complements of the members of $\mathcal{M}_2 \cup \mathcal{A}$ also satisfy it. The sum of the two inequalities again yield the statement of Theorem 2. Let us check the possibilities of equality. If $n > 4$ there is no A satisfying the conditions of equality in Corollary 1 for both (direct and complementing) cases. Therefore in this case the equality in Theorem 2 implies $\mathcal{A} = \emptyset$. So $\mathcal{F}$ is the union of two antichains. It is well-known that if such an $\mathcal{F}$ satisfies the inequality of Theorem 2 with equality then it may consist of two full levels, only. It is easy to see that if these levels are not neighboring then the family contains a butterfly. We are done with the case $n \geqslant 5$ and Theorem 2.

Since Theorem 1 is a consequence of Theorem 2 in the case $\emptyset, [n] \notin \mathcal{F}$, we are also done with this case.

Let us consider now the equality in Theorem 1 in the cases when one or both of $\emptyset, [n]$ are in $\mathcal{F}$. The proof of Theorem 1 can be repeated. To make the paper self-contained, one can use Corollary 2 instead of the result from [6].

The cases of small n have to be checked separately, by case analysis. □

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