Combinatorics, Probability and Computing (2012) 00, 000–000. © 2012 Cambridge University Press
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

2-bases of quadruples

ZOLTÁN FÜREDI [1†] and GYULA O.H. KATONA [2‡]

[1]Rényi Institute of Mathematics of the Hungarian Academy of Sciences
Budapest, P. O. Box 127, Hungary-1364.
and
Department of Mathematics, University of Illinois at Urbana-Champaign
Urbana, IL61801, USA
(e-mail: furedi@renyi.hu, z-furedi@math.uiuc.edu)

[2]Rényi Institute of Mathematics of the Hungarian Academy of Sciences
Budapest, P. O. Box 127, Hungary-1364.
(e-mail: ohkatona@renyi.hu)

Let $\mathcal{B}(n, \leq 4)$ denote the subsets of $[n] := \{1, 2, \ldots, n\}$ of at most 4 elements. Suppose that $\mathcal{F}$ is a set system with the property that every member of $\mathcal{B}$ can be written as a union of (at most) two members of $\mathcal{F}$. (Then $\mathcal{F}$ is called a 2-base of $\mathcal{B}$.) Here we answer a question of Erdős proving that

$$|\mathcal{F}| \geq 1 + n + \binom{n}{2} - \lfloor \frac{4}{3} n \rfloor,$$

and this bound is best possible for $n \geq 8$.

1. 2-bases

The n-element set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. The family of all subsets of $[n]$ is called the Boolean lattice and is denoted by $\mathcal{B}(n)$. Its kth level is $\mathcal{B}(n,k) := \{B : B \subset [n] : |B| = k\}$, and $\mathcal{B}(n, \leq k) := \cup_{0 \leq i \leq k} \mathcal{B}(n,i)$. The set system $\mathcal{F}$ is called a **2-base** of $\mathcal{A}$ if every member $A \in \mathcal{A}$ can be obtained as a union of two members of $\mathcal{F}$, in other words $A = F_1 \cup F_2$, $F_1, F_2 \in \mathcal{F}$. Note that we allow $F_1 = F_2$ and we do not insist that the 2-base is a subset of the set system.

The interest is in how small a base one can find. Let $f(\mathcal{A}) := \min\{|\mathcal{F}| : \mathcal{F}$ is a 2-base of $\mathcal{A}\}$. This is known exactly in very few cases, even when the set system is a natural

† Research supported in part by a Hungarian National Science Foundation grant OTKA T 032452, T 037846 and by a National Science Foundation grant DMS 0140692.
‡ Research supported by the Hungarian National Science Foundation grants OTKA T 037846, T 038210, T 034702.

one. For example, it is not known even for the power-set itself (the discrete cube). In 1993 Erdős [2] proposed the problem of determining $f(\mathcal{B}(n))$ and also the problem of determining the minimum size of a 2-base of the small sets, $f(\mathcal{B}(n, \leq k))$. We also use $f_k(n)$ for $f(\mathcal{B}(n, \leq k))$. Erdős conjectured that

$$f(\mathcal{B}(n)) = 2^{\lfloor n/2 \rfloor} + 2^{\lceil n/2 \rceil} - 1,$$

and that the extremal family consists of all subsets of V_1 and V_2 where $V_1 \cup V_2 = [n]$ is a partition of $[n]$ into two almost equal parts. A lower bound $f(\mathcal{B}(n)) \geq (1 + o(1))2^{(n+1)/2}$ is obvious from the fact that

$$|\mathcal{A}| \leq \binom{|\mathcal{F}|}{2} + |\mathcal{F}|,$$

which holds for any 2-base $\mathcal{F}$ of $\mathcal{A}$.

The aim in this paper is to answer this question for the family $\mathcal{B}(n, \leq 4)$. The question of the smallest base for $\mathcal{B}(n, \leq k)$ is trivial for $k \leq 2$, and for $k = 3$ it turns out to be a question about graphs whose answer follows immediately from Turán's theorem. So the case $k = 4$ is the first non-trivial case. It boils down to an interesting question about 3-graphs (3-regular hypergraphs), and it might be somewhat surprising that it is possible to give an exact answer.

Let $f_4(n) := 1 + n + \binom{n}{2} - h(n)$. The main result of this paper can be summarized in the following table.

n	0	1	2	3	4	5	6	7	$n \geq 8$
$h(n)$	0	0	1	2	4	5	7	8	$\lfloor \frac{4}{3}n \rfloor$

Theorem 1.1. *For $n \geq 8$ $f_4(n) = 1 + n + \binom{n}{2} - \lfloor \frac{4}{3}n \rfloor$.*

Let $g_k(n) := f(\mathcal{B}(n, 4))$, the size of a minimum 2-base for the k-tuples. We will deduce from Theorem 1.1 that $g_4(n) + n + 1 = f_4(n)$ for $n \geq 5$.

Theorem 1.2. *We have $g_4(5) = 5$, $g_4(6) = 8$, $g_4(7) = 13$ and for $n \geq 8$ $g_4(n) = \binom{n}{2} - \lfloor \frac{4}{3}n \rfloor$.*

In the following section we discuss $f_k(n)$ in the (easy) case $k \leq 3$. Then give constructions for $f_4(n)$ separating the cases $n \leq 7$ and $n \geq 8$ and thus providing lower bounds for $h(n)$. In Chapter 2 the structure of minimal bases of $\mathcal{B}(n, \leq 4)$ is investigated, namely those with minimum deficiency with at least 2, and then (the upper bounds for) the values of $h(n)$ in the above table is proved in Chapter 3. In Chapter 4 the uniform case (the case of g_4) is considered, and in Chapter 5 we close with a few remarks on the case $k > 4$.

1.1. The case $\mathcal{B}(n, \leq 3)$

For $k \geq 1$ every 2-base of $\mathcal{B}(n, k)$ must contain the $\emptyset$ and all singletons. This easily leads to

$$f_0(n) = 1, \quad f_1(n) = 1 + n, \quad f_2(n) = 1 + n.$$

Suppose that $\mathcal{F}$ is a 2-base of $\mathcal{B}(n, \leq k)$, $1 < k \leq n$, such that $|\mathcal{F}| = f_k(n)$ and $\sum_{F\in\mathcal{F}} |F|$ is minimal. Such bases are called **minimal**. Then

(i) $\emptyset \in \mathcal{F}$, $\mathcal{B}(n,1) \subset \mathcal{F}$,

(ii) for every $F \in \mathcal{F}$ we have $|F| \leq k-1$.

Indeed, one only need to observe that in case of $F \in \mathcal{F}$, $|F| = k$, $x \in F$ one can replace F by $F' := F \setminus \{x\}$, i.e., $\mathcal{F} \setminus \{F\} \cup \{F'\}$ is also a 2-base.

Construction 1.3. *Consider a 2-partition $V_1 \cup V_2$ of $[n]$ with $\lfloor n/2 \rfloor \leq |V_1| \leq |V_2| \leq \lceil n/2 \rceil$ and let $\mathcal{F}$ be all the subsets of V_1 and V_2 of size at most 2. Every triple from $[n]$ meets a V_i in at least 2 elements so it also contains a 2-element member of $\mathcal{F}$. Hence $\mathcal{F}$ is a 2-base of $\mathcal{B}(n, \leq 3)$.*

Claim 1.4. $f_3(n) = 1 + n + \binom{\lfloor n/2 \rfloor}{2} + \binom{\lceil n/2 \rceil}{2}$.

Proof of Claim 1.4. Suppose that $\mathcal{F}$ is a minimal 2-base of $\mathcal{B}(n, \leq 3)$ satisfying the (i) and (ii). Split it into subfamilies according to the sizes of its members, $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_i := \mathcal{F} \cap \mathcal{B}(n,i)$. Then $\mathcal{F}_2$ is a graph (i.e., a 2-graph) with the property that every triple contains an edge, so its complement $\mathcal{H}_2$ is triangle-free. ($\mathcal{H}_2 := \mathcal{B}(n,2) \setminus \mathcal{F}_2$.) Then Turán's theorem [7] implies that $|\mathcal{H}_2| \leq \lfloor n^2/4 \rfloor$, hence

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}_1| + |\mathcal{F}_2| \geq 1 + n + \binom{n}{2} - \lfloor \frac{n^2}{4} \rfloor. \qquad \square$$

1.2. Constructions for $\mathcal{B}(n, \leq 4)$ if $n \leq 7$

Let $\mathcal{F}$ be a minimal 2-base of $\mathcal{B}(n, \leq 4)$ satisfying (i) and (ii). Let $\mathcal{F}_i := \mathcal{F} \cap \mathcal{B}(n,i)$, then $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ where $\mathcal{F}_0 = \{\emptyset\}$, $\mathcal{F}_1 = \mathcal{B}(n,1)$. Use the notation $\mathcal{H}_2 := \mathcal{B}(n,2) \setminus \mathcal{F}_2$. Then

$$|\mathcal{F}| = 1 + n + \binom{n}{2} - |\mathcal{H}_2| + |\mathcal{F}_3| := 1 + n + \binom{n}{2} - h(n).$$

Since $\mathcal{B}(n, \leq 2)$ is a 2-base of $\mathcal{B}(n, \leq 4)$ we have $h(n) \geq 0$.

Let us summarize the properties of $\mathcal{F}_2 \cup \mathcal{F}_3$.

For every triple $T \subset [n]$	either T contains a pair from $\mathcal{F}_2$	(1.1)
	or $T \in \mathcal{F}_3$	(1.2)
For every quadruple $Q \subset [n]$	either Q contains a triple from $\mathcal{F}_3$	(1.3)
	or Q is a union of two edges from $\mathcal{F}_2$.	(1.4)

Construction 1.5. *For $n \geq 4$ let $\mathcal{H}_2$ be a Hamilton cycle, $|\mathcal{F}_3| = 0$.*

It is easy to show that this family $\mathcal{F}_2$ satisfies (1.1) and (1.4) so (together with $\mathcal{B}(n, \leq 1)$) it is a 2-base. This construction shows that $h(n) \geq n$ (for ≥ 4), and one can see that this is the best possible for $n = 4$ and $n = 5$.

Claim 1.6. *$h(0) = h(1) = 0$, $h(2) = 1$, $h(3) = 2$, $h(4) = 4$ and $h(5) = 5$.*

The proof of this (and the following two claims concering $n = 6$ and 7) is a short, finite process. For completeness we sketch them in Section 3.

Construction 1.7. *For $n = 6$ let $\mathcal{F}_3$ be two disjoint triples F_1, F_2 and let $\mathcal{F}_2$ be the six pairs contained in either of F_1 or F_2.*

Another construction of the same size can be obtained by considering a Hamilton cycle $\mathcal{F}_2 := \{12,\ 23,\ 34,\ 45,\ 56,\ 16\}$ with two triples $\mathcal{F}_3 := \{135, 246\}$.

Claim 1.8. $h(6) = 7$.

Construction 1.9. *For $n = 7$ label the seven elements by two coordinates, $V := \{v(1,1),\ v(1,2),\ v(1,3),\ v(2,1),\ v(2,2),\ v(3,1)\}$. Let $\mathcal{F}_2$ be the ten pairs $v(\alpha,\beta)v(\alpha',\beta')$ with $\alpha \neq \alpha'$ and $\beta \neq \beta'$, and let $\mathcal{F}_3$ be formed by the three triples having a constant coordinate i.e., $\{v(1,1), v(1,2), v(1,3)\}$, $\{v(2,1), v(2,2), v(2,3)\}$ and $\{v(1,1), v(2,1), v(3,1)\}$. (This is a truncated version of Construction 1.13 for $n = 9$.)*

Claim 1.10. $h(7) = 8$.

Construction 1.11. *Let n_1, n_2 be nonnegative integers, $V^1 \cup V^2$ a partition of $[n]$ with $|V^i| = n_i$, $\mathcal{F}^i$ a minimal 2-base on V_i. Define $\mathcal{F}$ as $\mathcal{F}^1 \cup \mathcal{F}^2$ together with all pairs joining V^1 and V^2.*

It is easy to see that this construction satisfies (1.1)–(1.4), it is a 2-base. Indeed, it is sufficient to check a triple T and a quadruple Q meeting both V_1 and V_2. Then T contains a pair joining V^1 and V^2 thus it satisfies (1.1). If $|Q \cap V^1| = |Q \cap V^2| = 2$, then it is a union of two crossing pairs. Finally, if $Q = \{a,b,c,d\}$ and $Q \cap V^1 = \{a,b,c\}$, then since $\mathcal{F}^1$ is a 2-base, $Q \cap V^1$ satisfies either (1.1) or (1.2). In the first case $Q \cap V^1$ it contains a pair, say ab from $\mathcal{F}^1$, then $\{a,b\} \cup \{c,d\}$ is a partition of Q satisfying (1.4). In the second case $Q \cap V^1 \in \mathcal{F}^1$, so Q satisfies (1.3). We obtained:

Claim 1.12. *For n_1, n_2 nonnegative integers $h(n_1 + n_2) \geq h(n_1) + h(n_2)$.* □

1.3. Constructions for $n \geq 8$

Construction 1.13. *Suppose that $\mathcal{F}_3$ is a triple system on $[n]$ of* girth at least 4, *i.e., $|F' \cap F''| \leq 1$ for $F', F'' \in \mathcal{F}_3$ and $F_1, F_2, F_3 \in \mathcal{F}_3$ and $F_1 \cap F_2 \neq \emptyset$, $F_1 \cap F_3 \neq \emptyset$, $F_2 \cap F_3 \neq \emptyset$ imply $F_1 \cap F_2 \cap F_3 \neq \emptyset$. Suppose further that every* degree *of $\mathcal{F}_3$ is at most two, i.e., every singleton is contained in at most two triples. Define $\mathcal{H}_2$ as the pairs covered by the members of $\mathcal{F}_3$.*

This construction (together with $\mathcal{B}(n, \leq 1)$) form a 2-base. Indeed, if a triple $T \subset [n]$ contains no edge from $\mathcal{F}_2$, then it belongs to $\mathcal{F}_3$, so either (1.1) or (1.2) holds. Moreover, if $Q = \{a,b,c,d\} \subset [n]$ is a quadruple and contains no triple from $\mathcal{F}_3$, then the induced graph $\mathcal{H}_2|Q$ contains no triangle. So $\mathcal{F}_2|Q$ contains two disjoint edges (and thus fulfills (1.4)) unless $\mathcal{H}_2|Q$ has a vertex of degree 3, say, $ab, ac, ad \in \mathcal{H}_2$. Since the degree of $\mathcal{F}_3$ at the vertex a is at most two and the edges of $\mathcal{H}_2$ are obtained from the triples of $\mathcal{F}_3$ we get that there exists a triple $T \in \mathcal{F}_3$ with $a \in T \subset Q$. We obtained that Construction 1.13 indeed defines a 2-base.

For $n = 3k$, $k \geq 3$ we obtain $h(3k) \geq 4k$ as follows. Let $[n] = \{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_k\} \cup \{c_1, c_2, \dots, c_k\}$. Define $\mathcal{F}_3$ as all triples of the form $a_i b_i c_i$ and $a_i b_{i+1} c_{i+2}$ (indices are taken modulo k). This satisfies the constraint of Construction 1.13. Since $|\mathcal{H}_2| = 3|\mathcal{F}_3|$, we get $h(n) \geq 2|\mathcal{F}_3| = 4k$.

If we leave out from the above construction the 2 triples of $\mathcal{F}_3$ and the 4 pairs of $\mathcal{H}_2$ containing the element $3k$ we obtain that $h(3k-1) \geq 4k-2$. Thus we already have the cases $n = 3k$ and $n = 3k-1$ in the following

Claim 1.14. *$h(n) \geq \lfloor \frac{4}{3} n \rfloor$ for $n \geq 8$.*

Proof. We only need a construction for $n = 3k+1$, $k \geq 3$ to show $h(3k+1) \geq 4k+1$. It is enough to show $h(10) \geq 13$, $h(13) \geq 17$ and $h(16) \geq 21$, then the general case follows from $h(9) \geq 12$ using Claim 1.12.

Define the six triples of $\mathcal{F}_3$ as $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{1,4,7\}$, $\{2,5,8\}$ and $\{3,6,10\}$ and $\mathcal{H}_2$ as the 18 pairs covered by these triples and $\{9,10\}$. The graph $\mathcal{H}_2$ has only these 6 triangles, so (1.1)-(1.2) hold, and it is not difficult to check the four-tuples, too.

The other cases are similar: for $n = 13$ we can define $\mathcal{F}_3 := \{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{10,11,12\}$ and $\{1,4,10\}$, $\{2,5,7\}$, $\{6,8,11\}$, $\{3,9,13\}$ and $\mathcal{H}_2$ consists of these triangles and the pair $\{12,13\}$.

Finally, for $n = 16$ we define $\mathcal{F}_3$ as $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{10,11,12\}$, $\{13,14,15\}$ and $\{1,4,13\}$, $\{2,5,7\}$, $\{6,8,10\}$, $\{9,11,14\}$, and $\{3,12,16\}$. Again $\mathcal{H}_2$ consists of the triangles obtained from $\mathcal{F}_3$ and the edge $\{15,16\}$. □

2. Bases with deficiency at least 2

The aim of this paper is to prove Theorem 1.1 so suppose that $\mathcal{F}$ is a minimal 2-base of $\mathcal{B}(n, \leq 4)$ and that $\mathcal{F}_2 \cup \mathcal{F}_3$ satisfies (1.1)-(1.4).

Lemma 2.1. *If $abc \in \mathcal{F}_3$, then either $\{ab, bc, ca\} \subset \mathcal{F}_2$ or $\{ab, bc, ca\} \subset \mathcal{H}_2$.*

Proof. Suppose, on the contrary, that $ab \in \mathcal{F}_2$, $ac \notin \mathcal{F}_2$. Replace abc by ac in $\mathcal{F}$. Since $\sum_{F \in \mathcal{F}} |F|$ is minimal the family $\mathcal{F}' := \mathcal{F} \setminus \{abc\} \cup \{ac\}$ is not a 2-base. What can go wrong? Since we added a new pair, conditions (1.1) and (1.2) still hold. The only condition we can violate is (1.3)-(1.4). We removed abc, so there exists an $Q = abcd$ not a union of two members of $\mathcal{F}'$. So $abcd$ does not contain any triple from $\mathcal{F}'$ and also bd, $cd \notin \mathcal{F}'$. Consider bcd. We have $bcd \notin \mathcal{F}$ so (1.1) implies that $bc \in \mathcal{F}_2$. Consider acd. Since ac, cd, and $acd \notin \mathcal{F}$ again (1.1) implies that $ad \in \mathcal{F}_2$. However, then $Q = ad \cup bc$, a contradiction. □

Use the notation $\deg_2^-(x)$ for the degree of the vertex x in the graph $\mathcal{H}_2$ and $\deg_3(x)$ for the degree of x in $\mathcal{F}_3$. The difference $\deg_2^-(x) - \deg_3(x)$ is called the **deficiency** of the vertex $x \in V$. ¿From now on in this Section we suppose that

$$\deg_2^-(x) - \deg_3(x) \geq 2 \text{ for every } x \in [n]. \tag{2.1}$$

Let $N(x)$ denote the neighborhood of x in $\mathcal{H}_2$, $N(x) := \{y : xy \in \mathcal{H}_2\}$, $\deg_2^-(x) = |N(x)|$. Let $\mathcal{T}(x)$ denote the set of triples T from $\mathcal{F}_3$ with $x \in T \subset N(x) \cup \{x\}$, and let $t(x) := |\mathcal{T}(x)|$. Suppose that $D = \max_{x\in[n]} \deg_2^-(x)$, and a has maximum degree in $\mathcal{H}_2$. Consider $A = \{a\} \cup N(a)$, $|A| = D + 1$, let $t := t(a)$. Then (2.1) implies $t, t(x) \leq D - 2$.

2.1. Eliminating the case $D \geq 5$

Claim 2.2. (2.1) *implies that* $D \leq 4$.

Proof. Consider the $\binom{D}{3}$ four-tuples of A containing x, let $\mathcal{B} := \{Q : a \in Q \subset A, |Q| = 4\}$. Note that none of these can satisfy (1.4) so each of them contains a member of $\mathcal{F}_3$. Classify them into two groups as follows:

$\mathcal{B}_1 := \{abcd : b, c, d \in A$ and there exits a $T \in \mathcal{F}_3$ with $a \in T \subset \{a, b, c, d\}\}$,

$\mathcal{B}_2 := \{abcd : abcd \subset A,\ abc, abd, acd \notin \mathcal{F}_3\}$.

Each $Q \in \mathcal{B}_2$ contains a member of $\mathcal{F}_3|N(a)$, hence

$$|\mathcal{B}_2| \leq |\mathcal{F}_3|N(a)|.$$

Each member of $\mathcal{T}(a)$ is contained in $D - 2$ four-tuples from $\mathcal{B}_1$, hence

$$|\mathcal{B}_1| \leq t(D-2). \tag{2.2}$$

Here the sum of the left hand sides is $\binom{D}{3}$. The sum of right hand sides can be estimated by the degrees of $\mathcal{F}_3$ on A. Using $\deg_3(x) \leq D - 2$ we obtain

$$\begin{aligned}\binom{D}{3} &= |\mathcal{B}_1| + |\mathcal{B}_2| \leq t(D-2) + |\mathcal{F}_3|N(a)| = t(D-3) + |\mathcal{F}_3|A| \\ &\leq t(D-3) + \frac{1}{3}\sum_{x\in A} \deg_3(x) \leq t(D-3) + \frac{1}{3}(t + D(D-2)).\end{aligned} \tag{2.3}$$

Hence

$$\frac{1}{6}D(D-2)(D-3) \leq t\frac{3D-8}{3}. \tag{2.4}$$

Since $t \leq D - 2$ we get $D \leq 6$. In case of $t \leq D - 3$ (2.4) implies $D \leq 4$. So two cases left in the proof of the Claim, namely $(D, t) = (6, 4)$ and $(5, 3)$.

In case of $D = 6$, $t = 4$ the right hand side of (2.2) can be improved by 2, since there are at least 2 coincidences when we estimated the cardinality of $\mathcal{B}_1$. So $|\mathcal{B}_1| \leq 14$, and we can decrease the right hand sides of (2.3) and (2.4) by 2, and that leads to a contradiction $12 \leq 4 \times \frac{10}{3} - 2$.

In case of $D = 5$, $t = 3$ we use two things. The first one is implied by Lemma 2.1 and (1.1):

(C1) If $abc \in \mathcal{T}(a)$ then $bc \in \mathcal{H}_2$; if $abc \notin \mathcal{T}(a)$ and $b, c \in N(a)$ then $bc \in \mathcal{F}_2$. Thus $\mathcal{F}_2|N(a)$ has exactly $\binom{D}{2} - t$ edges.

(C2) If $\deg_3(x) \geq 3$, then $t(x) = 3$. Indeed, (2.1) implies $\deg_2^-(x) \geq \deg_3(x) + 2 \geq 5$. Consequently $\deg_2^-(x) = 5 = D$, x has maximum degree, D, and then the previous considerations for a are valid for x, too, i.e., (2.4) implies that $t(x) = 3$ is the only possibility.

Now we are ready to show that, in fact, $(D, t) = (5, 3)$ is impossible. Suppose, on the contrary, that there is such a construction and let $N(a) = \{b, c, d, e, f\}$. Consider the 3-edge graph $G := \{xy : axy \in \mathcal{F}_3\}$. There are 4 non-isomorphic possibilities for G.

(α) G is a triangle, $\{bc, cd, bd\}$,

(β) G is a path of length 3, $\{bc, cd, de\}$,

(γ) G is a star, $\{bc, bd, be\}$,

(δ) G has 2 components, $\{bc, cd, ef\}$.

In each case we will find one or more $x \in N(a)$ with $t(x) = 3$. Then the triples containing x cover no pair from $\mathcal{F}_2$ and this will lead to a contradiction.

In case of (α) by (1.3) we have $bef, cef, def \in \mathcal{F}_3$. Hence $\deg_3(f) \geq 3$. Then (C2) implies that $t(f) = 3$ and then Lemma 2.1 gives that $\{b, c, d, e\} \subset N(f)$, $ef \notin \mathcal{F}_2$. However, $ef \in FF_2$ by (C1), a contradiction.

The other cases can be handled in the same way. In case of (β) we have $bdf, bef, cef \in \mathcal{F}_3$, hence $\deg_3(f) \geq 3$. Then $t(f) = 3$ and $\{b, c, d, e\} \subset N(f)$, $ef \notin \mathcal{F}_2$. In case of (γ) we have $cdf, cef, def \in \mathcal{F}_3$, hence $\deg_3(f) \geq 3$. Then $t(f) = 3$ and $\{c, d, e\} \subset N(f)$, $ef \notin \mathcal{F}_2$. In case of (δ) we have $bde, bdf \in \mathcal{F}_3$, hence $\deg_3(b) \geq 3$. Then $t(b) = 3$ and $\{c, d, e, f\} \subset N(b)$, $bf \notin \mathcal{F}_2$. This final contradiction completes the proof of the case $(D, t) = (5, 3)$ and Claim 2.2. □

2.2. The case $D \leq 4$

¿From now on in this Chapter we suppose that $D \leq 4$.

Claim 2.3. *(2.1) and $\deg_2^-(a) = 4$ imply that $t(a) = \deg_3(a) = 2$ and the two triples containing the element a meet only in a, e.g., $N(a) = bcde$ and $\mathcal{T}(a) = \{abc, ade\}$.*

Proof. Suppose, first, that $t(a) = 0$. Then all the four triples of the form xyz, $x, y, z \in N(a)$ belong to $\mathcal{F}_3$. Hence $\deg_3(b) \geq 3$, contradicting $D \geq \deg_2(x) \geq 2 + \deg_3(x)$. If $t(a) = 1$, say $abc \in \mathcal{T}(a)$, then $bde, cde \in \mathcal{F}_3$ is implied by (1.3). Hence $\deg_3(e) \geq 2$, so $\deg_2^-(e) = 4$. Since (1.1) implies that be, ce, $de \in \mathcal{F}_2$ we get that $N(e) \cap \{b, c, d\} = \emptyset$, so $t(e) = 0$. However, we have seen that $\deg_2^-(e) = D = 4$ implies $t(e) > 0$.

So we get $t(a) \geq 2$, i.e., by $t(x) \leq D - 2$ we have $t(a) = 2$. The only case left to exclude is when the triples in $\mathcal{T}(a)$ meet in two elements, say $\mathcal{T}(a) = \{abc, acd\}$. Then $bde \in \mathcal{F}_3$, so $\deg_3(b) \geq 2$. Hence we get $\deg_2^-(b) = 4$, this implies $t(b) = 2$ and $\{c, d, e\} \subset N(b)$. We get $ab, ae, be \in \mathcal{H}_2$, $abe \notin \mathcal{F}_3$, contradicting (1.1). □

Claim 2.4. *(2.1) and $\deg_2^-(x) = 3$ imply that $\deg_3(x) = 1$.*

Proof. Suppose, on the contrary, that $\deg_3(x) = 0$. Consider $N(x) = abc$, we have $ab, bc, ca \in \mathcal{F}_2$ by (1.1) and $abc \in \mathcal{F}_3$ by (1.3). Then $ab \in \mathcal{F}_2$ implies that $abc \notin \mathcal{T}(a)$

Therefore $t(a)$ cannot be $D-2=2$. So Claim 2.3 gives that $\deg_2^-(a)\neq 4$. Since $\deg_3(a)\geq 1$ we get that $\deg_2^-(a)=3$. Consider $N(a)=xyz$. Note that $y,z\notin\{x,a,b,c\}$. Then $xyz\in\mathcal{F}_3$ by (1.3). This contradicts $\deg_3(x)=0$, so we have $\deg_3(x)\geq 1$. On the other hand, (2.1) implies $\deg_3(x)\leq 1$. □

Claim 2.5. (2.1) *implies that* $h(\mathcal{F})\leq \frac{4}{3}n$.

Proof. For $x\in[n]$ define $\varphi(x):=\frac{1}{2}\deg_2^-(x)-\frac{1}{3}\deg_3(x)$. We are going to prove that $\varphi(x)\leq 4/3$ for every x. This implies the Claim as follows

$$h(\mathcal{F})=|\mathcal{H}_2|-|\mathcal{F}_3|=\sum_{x\in[n]}\varphi(x)\leq\frac{4}{3}n. \tag{2.5}$$

Using the previous three Claims one can split $[n]$ into three parts, $[n]=P\cup Q\cup R$, where $P:=\{x:\deg_2^-(x)=4,\ \deg_3(x)=2\}$, $Q:=\{x:\deg_2^-(x)=3,\ \deg_3(x)=1\}$, and $R:=\{x:\deg_2^-(x)=2,\ \deg_3(x)=0\}$. For each cases we have $\varphi\leq 4/3$. □

Note that $h(\mathcal{F})=\frac{4}{3}n$ in Claim 2.5 is only possible for Construction 1.13, especially

$$P=[n]\ \text{ and }\ Q=R=\emptyset. \tag{2.6}$$

3. Proof of the main result

Let $\mathcal{F}$ be a minimal 2-base for $\mathcal{B}(n,\leq 4)$. Then

$$\begin{aligned} 1+n+\binom{n}{2}-h(n) &= |\mathcal{F}|=|\mathcal{F}|([n]\setminus\{x\})|+1+(n-1-\deg_2^-(x))+\deg_3(x)\\ &\geq 1+n+\binom{n}{2}-h(n-1)-(\deg_2^-(x)-\deg_3(x)) \end{aligned} \tag{3.1}$$

gives that the deficiency of every vertex is at least $h(n)-h(n-1)$.

Proof of Theorem 1.1. We use induction on n to show that $h(n)\leq\frac{4}{3}n$. This is certainly true for $n\leq 2$. Suppose that $h(n-1)\leq\frac{4}{3}(n-1)$ and consider $h(n)$. If $h(n)\leq h(n-1)+1$, then we are done. If $h(n)\geq h(n-1)+2$, then, as we have seen in (3.1) there exists a minimal 2-base $\mathcal{F}$ on $[n]$ with deficiency at least 2. Then Claim 2.5 gives $h(n)=h(\mathcal{F})\leq\frac{4}{3}n$. □

Proofs of Claims 1.6, 1.8 and 1.10. The case $n\leq 4$ is trivial. Suppose that $5\leq n\leq 7$ and let $\mathcal{F}$ be a minimal 2-base on n vertices.

The case $n=5$ is easy. $h(\mathcal{F})\geq 6$ implies $|\mathcal{F}_2|+|\mathcal{F}_3|\leq 4$. If $|\mathcal{F}_2|=4$, then there is a unique way to satisfy (1.1) (namely, $\mathcal{F}_2$ is a union of an edge and a triangle) and then (1.4) is violated. If $|\mathcal{F}_2|=3$, then there are at least 2 triples not containing any member of $\mathcal{F}_2$, so (1.2) gives $|\mathcal{F}_3|\geq 2$. If $|\mathcal{F}_2|\leq 2$, then they satisfy (1.1) with at most $3|\mathcal{F}_2|$ triples. Hence, (1.2) gives $|\mathcal{F}_3|\geq 10-3|\mathcal{F}_2|$. Then $|\mathcal{F}_2|+|\mathcal{F}_3|$ exceeds 4, a final contradiction.

If the minimum deficiency of $\mathcal{F}$ is (at most) 1 then (3.1) gives $h(n)\leq h(n-1)+1$ and we are done. ¿From now on suppose that the deficiency of $\mathcal{F}$ is at least 2, i.e. (2.1) holds.

For $n = 6$ Claim 2.5 gives that $h(\mathcal{F}) \le \frac{4}{3} \times 6 = 8$. By (2.6) $h(\mathcal{F}) = 8$ is only possible, if $P = [n]$, i.e., $\mathcal{H}_2$ is a 4-regular graph, and $\mathcal{F}_3$ consists of four triples. Then $\mathcal{F}_2$ is a matching, say, $\mathcal{F}_2 = \{a_1a_2, b_1b_2, c_1c_2\}$. Then (1.2) implies that all the eight triples of the form $a_ib_jc_k$ should belong to $\mathcal{F}_3$, a contradiction. We have obtained $h(\mathcal{F}) = h(6) \le 7$.

For $n = 7$ Theorem 1.1 implies $h(\mathcal{F}) \le \lfloor 7 \times \frac{4}{3} \rfloor = 9$. We claim that $h(7) = 8$. Suppose, on the contrary, that $h(\mathcal{F}) = 9$. Consider the partition of $[n] = P \cup Q \cup R$ defined in the proof of Claim 2.5. For $R \ne \emptyset$ (2.5) gives $|R| = 1$, $|P| = 6$, $Q = \emptyset$. Then $\mathcal{H}_2|P$ is a 4-regular graph, not joined to R so $\deg_2^-(R) = 2$ is impossible. Finally, if $R = \emptyset$, $|Q| = 2$ and $|P| = 5$ then we get $|\mathcal{F}_3| = 4$. The four members of $\mathcal{F}_3$ can pairwise meet in at most 1 vertex (by Claims 2.3 and 2.4) and have girth 4. But such an $\mathcal{F}_3$ does not exist on 7 vertices.

So we have obtained the exact value of $h(n)$ for every n. □

4. 2-bases for quadruples

Here we prove Theorem 1.2. Suppose that $\mathcal{F}$ is an extremal 2-base for $\mathcal{B}(n, 4)$, i.e., $|\mathcal{F}| = g_4(n)$, such that $|\mathcal{F}_1| + |\mathcal{F}_4|$ is minimal. The case $n = 5$ is a short finite process, $|\mathcal{F}| \le 4$ leads to a contradiction. So the pentagon gives $g_4(5) = 5$.

In the case $n = 6$ the 6 pairs of a hexagon and the 2 disjoint triples of second example in Construction 1.7 shows $g_4(6) \le 8$. Consider a minimal 2-base $\mathcal{F}$. If $\deg_{\mathcal{F}}(x) \ge 3$, then

$$|\mathcal{F}| = \deg_{\mathcal{F}}(x) + |\mathcal{F}|([n] \setminus \{x\})| \ge \deg_{\mathcal{F}}(x) + g_4(n-1) \tag{4.1}$$

implies $|\mathcal{F}| \ge 3 + 5$ and we are done. Moreover, it is easy to check that a hypergraph of 7 edges on 6 elements with maximum degree 2 cannot be a 2-base, so $g_4(6) \ge 8$. ¿From now on we may suppose that $n \ge 7$.

The upper bounds for $g_4(n)$ follows by leaving out the singletons and the empty set from Constructions 1.9 and 1.13 in Chapter 1. To prove a lower bound we proceed like in Chapter 2. The main idea of the proof is that first we investigate the minimal 2-bases with a maximum degree condition

$$\deg_{\mathcal{F}}(x) \le n - 3 \tag{4.2}$$

for all $x \in [n]$.

We claim that (4.2) implies that $\mathcal{F}_4 = \emptyset$. Indeed, suppose, on the contrary, that $Q \in \mathcal{F}_4$. If Q contains any proper subset $F \in \mathcal{F}$, $x \in F \subset Q$, $Q \ne F$, then one can replace Q by $Q \setminus \{x\}$ to obtain another 2-base with smaller $|\mathcal{F}_1| + |\mathcal{F}_4|$. So we may suppose that such a proper subset does not exist. Consider $Q \setminus \{x\} \cup \{y\}$ for some $x \in Q$, $y \in [n] \setminus Q$. This is a union of (at most) two sets $A, B \in \mathcal{F}$. Both of them contain y. We obtain that the sets $\{F : y \in F \subset Q \cup \{y, |F| > 1\}$ cover Q, and some vertex of Q is covered at least twice. Hence there exists an $x \in Q$ covered by these sets more than $n - 4$ times while y runs trough $[n] \setminus Q$. Take Q itself, too, we get that $\deg_{\mathcal{F}}(x) > n - 3$ contradicting (4.2).

Use that notations of the previous section, like $D := \max \deg_2^-(x)$ and $\deg_2^-(a) = D$, etc. We claim that (4.2) implies that

$$D \le 4.$$

In the proof of this one cannot use Lemma 2.1 neither (1.1) nor (1.2), however (2.2)–(2.4) still hold, implying $D \leq 6$. Furthermore, $ab, ac, ad \notin \mathcal{F}_2$, and $abc, abd, acd \notin \mathcal{F}_3$ imply not only $bcd \in \mathcal{F}_3$ but $a \in \mathcal{F}_1$. Thus in the case $\mathcal{B}_2 \neq \emptyset$ (e.g., for $D > 4$) one gets $a \in \mathcal{F}_1$. Then (4.2) gives $t(a) \leq \deg_2^-(a) - 3 = D - 3$. So (2.4) gives $D \leq 4$.

Using the same idea one can see that Claim 2.3 remains true. The following analog of Claim 2.4 is obviously true: $\deg_2^-(x) = 3$ implies $\deg_1(x) + \deg_3(x) = 1$.

Like in Claim 2.5 we show that (4.2) implies

$$|\mathcal{F}| \geq \binom{n}{2} - \frac{4}{3}n. \tag{4.3}$$

Indeed, for $x \in [n]$ define $\varphi(x) := \frac{1}{2}\deg_2^-(x) - \frac{1}{3}\deg_3(x) - \deg_1(x)$. As before we have that (4.2) implies that $\varphi(x) \leq 4/3$ for every x, completing the proof of (4.3) for this case.

Finally, for hypergraphs with maximum degree at least $n-2$ one can use induction on n. The inequality(4.1) implies that (4.3) always holds.

The case $n = 7$ can be finished like in the proof of Claim 2.5 considering a partition of $[n]$ into three parts, $[n] = P \cup Q \cup R$, where now $Q := \{x : \deg_2^-(x) = 3,\ \deg_1(x) + \deg_3(x) = 1\}$. The details are omitted. □

5. More hypergraphs

Let $T(n, k, r)$ denote the minimum size of a hypergraph $\mathcal{F} \subseteq \mathcal{B}(n, r)$ such that every k-subset of $[n]$ contains a member of $\mathcal{F}$. The determination of $T(n, k, r)$ is proposed by Turán [8] who solved the case $r = 2$ (the case of graphs, see [7]) and has a longstanding conjecture $T(n, 4, 3) = (\frac{4}{9} + o(1))\binom{n}{3}$. For a survey on this see Sidorenko [6].

One can prove for every odd integer k that our $f_k(n)$ equals to $(1 + o(1))T(n, k, (k+1)/2)$, but the even case is more involved and apparently leads to a new Turán type problem. The authors intend to return to this topic in a future work.

References

[1] B. Bollobás, Extremal Graph Theory, Academic Press, London, (1978).

[2] Paul Erdős, personal communication.

[3] Z. Füredi, Turán type problems, pp. 253–300, in "Surveys in Combinatorics, 1991", edited by A. D. Keedwell, Cambridge University Press 1991.

[4] Katona, Nemetz, Simonovits A new proof of a theorem of P. Turán and some remarks on a generalization of it, (in Hungarian), *Mat. Lapok* **15** (1964), 228-238.

[5] D. Mubayi and V. Rödl, On the Turán number of triple systems, *J. Combin. Theory, Ser. B* **100** (2002), 135-152.

[6] A. F. Sidorenko, What we know and what we do not know about Turán numbers, *Graphs Combin.* **11** (1995), 179–199.

[7] P. Turán, On an extremal problem in graph theory, (in Hungarian), *Math. Fiz. Lapok* **48** (1941), 436–452.

[8] P. Turán, Research problems, *MTA Mat. Kutató Int. Közl.* **6** (1961), 417-423.