Constructions via Hamiltonian Theorems☆

Gyula O.H. Katona

Alfréd Rényi Institute of Mathematics, HAS, P.O.B. 127, H-1364, Budapest, Hungary

Received 2 April 2003; received in revised form 11 August 2004; accepted 7 March 2005
Available online 10 October 2005

Abstract

Demetrovics et al [Design type problems motivated by database theory, J. Statist. Plann. Inference 72 (1998) 149–164] constructed a decomposition of the family of all k-element subsets of an n-element set into disjoint pairs $(A, B)(A \cap B = \emptyset, |A| = |B| = k)$ where two such pairs are relatively far from each other in some sense. The paper invented a proof method using a Hamiltonian-type theorem. The present paper gives a generalization of this tool, hopefully extending the power of the method. Problems where the method could be also used are shown. Moreover, open problems are listed which are related to the Hamiltonian theory. In these problems a cyclic permutation is to be found when certain restrictions are given by a family of k-element subsets.

Keywords: Families of subsets; Design; Hamiltonian cycle; Baranyai's theorem

1. Introduction

$\binom{X}{k}$ denotes the family of all k-element subsets of X. Let $[n] = \{1, 2, \ldots, n\}$. The following problem was raised in [2]. Can we decompose $\binom{[n]}{k}$ into (unordered) disjoint pairs (with one exception if $\binom{n}{k}$ is odd) in such a way that these pairs are not too close to each other in a certain sense? The answer was given in the following way.

☆ The work was supported by the Hungarian National Foundation for Scientific Research Grant numbers T037846 and AT048826.

E-mail address: ohkatona@renyi.hu.

Theorem 1.1. *Suppose that* $1 \leqslant k$ *is an integer and* n *is greater than some* $n(k)$. *Then there are* $\left\lfloor \frac{1}{2}\binom{n}{k} \right\rfloor$ *unordered pairs* (A_i, B_i) *of disjoint k-element subsets* $(A_i \cap B_i = \emptyset, |A_i| = |B_i| = k)$ *of* $[n]$ *such that*

$$\min\{|A_i \cap A_j|, |B_i \cap B_j|\} \leqslant \frac{k}{2}, \tag{1.1}$$

which implies

$$\min\{|A_i \cap B_j|, |B_i \cap A_j|\} \leqslant \frac{k}{2} \tag{1.2}$$

by the unorderedness.

The proof of this theorem in [2] is the starting point of the present paper. Let us show the main idea of this proof.

Sketch of the proof. Define two graphs $G_0=(V, E_0)$ and $G_1=(V, E_1)$ on the same vertex set $V = \binom{[n]}{k}$. Two k-element subsets are adjacent in G_0 if their intersection is empty, while they are adjacent in G_1 if they intersect in at least $\lceil (k+1)/2 \rceil$ elements. Let us formulate the statement of the theorem in terms of these graphs. A pair of disjoint subsets is an edge in G_0. Therefore we want to find a matching in G_0 containing all vertices with one possible exception. Conditions (1.1) and (1.2) can be expressed by G_1 in the following way: two edges of the matching do not form a cycle with two edges of G_1. Such a cycle will be called an *alternating four-cycle*.

We will show the existence of a Hamiltonian cycle (cycle containing every vertex of the graph exactly once) in G_0 with the same property. Choosing every second edge in this cycle gives us the desired matching. If we forget about the conditions (1.1) and (1.2) (i.e., the exclusion of an alternating four-cycle) then it is easy to construct a Hamiltonian cycle in G_0 using the following theorem of Dirac [3].

Theorem 1.2. *If* G *is a simple graph on* N *vertices and every degree in* G *is at least* $N/2$, *then* G *has a Hamiltonian cycle.*

In our case $N = \binom{n}{k}$, while the degree of the regular G_0 is $\binom{n-k}{k}$. The condition of the Dirac theorem holds for large n.

However, if we want to find a Hamiltonian cycle in which no two edges form an alternating four-cycle with the edges of the other graph then we need a Hamiltonian theorem for two graphs, proved in [2].

Theorem 1.3. *Let* $G_0 = (V, E_0)$ *and* $G_1 = (V, E_1)$ *be simple graphs on the same vertex set* $|V| = N$, *such that* $E_0 \cap E_1 = \emptyset$. *Let* r *be the minimum degree of* G_0 *and let* s *be the maximum degree of* G_1. *Suppose, that*

$$2r - 8s^2 - s - 1 > N \tag{1.3}$$

holds, then there is a Hamiltonian cycle in G_0 such that if (a, b) and (c, d) are two vertex-disjoint edges of the cycle, then they do not form an alternating cycle with two edges of G_1.

Here the graph G_1 is also regular, its degree is $s = \sum_{\lceil (k+1)/2 \rceil \leqslant j} \binom{k}{j}\binom{n-k}{k-j}$. It is easy to see that the largest exponent of n in this expression is $\lfloor (k-1)/2 \rfloor$, therefore the exponent of n in s^2 is $2\lfloor (k-1)/2 \rfloor < k$. Hence (1.3) holds for large n. □

Theorem 1.1 was sharpened in [4]. The proof was a modification of the proof sketched above. In Section 2 we extend the method proving a much more general Hamiltonian theorem than Theorem 1.3. It implies the result of [4], but many more applications are expected. The limits of the method are also illustrated.

Section 3 is of survey type. It is a mixture of related open problems, known results and easy remarks. Its main goal is to bridge the gap between the results like Theorem 1.3 and other known results, at least in a form of open questions. We also try to show what are the most natural conditions under which the existence of Hamiltonian cycles can be investigated and what are the most applicable conditions.

Theorem 1.1 has two motivations. One of them is coming from database theory (see [2]), where the construction of certain matrices was needed. Theorem 1.1 is actually this construction in another form. The other motivation comes from the following celebrated theorem of Baranyai [1].

Theorem 1.4 (*Baranyai [1]*). *If k divides n then there is a set of partitions of $[n]$ into k-element classes such that each element of $\binom{[n]}{k}$ is contained in exactly one such partition.*

This theorem has many applications. Observe, however, that the theorem imposes no condition on the relationship between two partitions. If something would be known how the classes of two distinct partitions pairwise intersect, it would enhance the applicability of the theorem. However it seems to be hopeless to obtain such a stronger Baranyai theorem. Theorem 1.1 can be considered as a very modest step toward this direction. We consider pairs of disjoint k-element subsets rather than partitions into k-element classes. We divide $\binom{[n]}{k}$ into the union of such pairs. Here we are able to prescribe some conditions on the intersections.

Section 4 is dealing with conjectures related to Baranyai's theorem. The results of this section are very modest: we pose a new, closely related conjecture and partially prove it, using another old conjecture concerning the existence of the rth *power of a Hamiltonian cycle* in a graph, that is the existence of a cyclic permutation $x_1, \ldots, x_n$ of the vertices such that $\{x_i, x_j\}$ is an edge of the graph for each pair satisfying $|i-j| \leqslant r$ or $|i-j| \geqslant n-r$. The method is a novel application of the method illustrated above.

2. A new Hamiltonian type theorem

In the following theorem $2k+1$ $(1 \leqslant k)$ simple graphs will be given on the same vertex set V: $G = (V, E)$, $H_i = (V, L_i)$, $J_i = (V, M_i)$ $(1 \leqslant i \leqslant k)$. Suppose that E is disjoint to all other edge sets. Let $|V| = N$, denote the minimum degree in G by r and the maximum degrees in H_i (J_i) by l_i (m_i). We say that the edges $\{x, y\}, \{y, z\}, \{z, u\}, \{u, x\}$ form *an alternating four-cycle* if $\{x, y\}, \{z, u\} \in E$ and $\{y, z\} \in L_i$, $\{u, x\} \in M_i$ hold for some i $(1 \leqslant i \leqslant k)$. A pair of distinct edges of G are called *acceptable* (*with respect to* $H_1, \ldots, H_k, J_1 \ldots, J_k$) if one cannot find two edges in L_i and M_i, resp., for some i $(1 \leqslant i \leqslant k)$ such that these four edges form an alternating four-cycle. A set of edges of G is said to be *acceptable* (*with respect to* $H_1, \ldots, H_k, J_1 \ldots, J_k$) if any pair of its vertex-disjoint distinct elements are acceptable.

Theorem 2.1. *If*

$$N \leqslant 2r - 8 \sum_{i=1}^{k} l_i m_i - \sum_{i=1}^{k} (l_i + m_i) \tag{2.1}$$

holds for the degrees of the graphs above, then there is an acceptable Hamiltonian cycle in G.

The proof follows the proof of Theorem 3.17 in [2]. We start with a lemma.

Lemma 2.2. *Suppose that there is an acceptable Hamiltonian path P in G from* $\alpha \in V$ *to* $\beta \in V$ *in G and that* (2.1) *holds. Then one can find vertices* γ *and* δ *neighboring, and located in this order along P which satisfy the following conditions*: (i) $\{\alpha, \delta\}, \{\beta, \gamma\} \in E$, (ii) *the set of three edges* $\{\alpha, \delta\}, \{\beta, \gamma\}, e \in P - \{\gamma, \delta\}$ *is acceptable for any choice of e.* ($\alpha = \gamma$ *and* $\beta = \delta$ *are allowed.*) *Therefore adding* $\{\alpha, \delta\}$ *and* $\{\beta, \gamma\}$, *and deleting* $\{\gamma, \delta\}$ *from P results in an acceptable Hamiltonian cycle.*

Proof. *Left* (*right*) *neighbor* means the neighbor towards α (β) along P, respectively. Let us first give a lower estimate on the number of pairs (γ, δ) satisfying only (i) at the moment. There are at least r vertices δ (among the $N - 1$ candidates) such that $\{\alpha, \delta\} \in E$. β has also at least r neighbors in G. Therefore there are at least r vertices whose right neighbor can be δ satisfying (i). Hence at least $2r - N +$ vertices δ satisfy both conditions: they are joined with α and their left neighbor is joined by β in G.

In what follows, we will subtract the number of cases, when the pair (γ, δ) does not satisfy (ii). This can happen in three different ways.

The pair of edges $\{\alpha, \delta\}, \{\beta, \gamma\}$ is not acceptable. Since $\{\gamma, \delta\}$ is not an edge of H_i or J_i $(1 \leqslant i \leqslant k)$, this case implies that $\{\beta, \delta\}$ belongs to $L_i \cup M_i$ for some $(1 \leqslant i \leqslant k)$. Therefore, the number of such δ's can be upperbounded by $\sum_{i=1}^{k} (l_i + m_i)$.

The pair $\{\alpha, \delta\}$ and some $\{x, y\} = e \in P$ is not acceptable. Suppose that the edges $\{\alpha, x\}\{x, y\}, \{y, \delta\}, \{\delta, \alpha\}$ form an alternating cycle and $\{\alpha, x\} \in L_i$ holds. Then $\{y, \delta\} \in M_i$ must also hold. There are at most l_i such choices for x, two choices for y (since it is a neighbor of x in P) and m_i choices for δ. Therefore, there are at most $2l_i m_i$ paths (α, x, y, δ) making an alternating cycle in this way. Since $\{\alpha, x\}$ can also lie in M_i and then $\{y, \delta\} \in L_i$ should hold, the total number of these cases is at most $4 \sum_{i=1}^{k} l_i m_i$.

The same upper bound is valid for the number of cases when the pair $\{\beta, \gamma\}$, $e \in P$ is not acceptable. Therefore, there is an appropriate pair γ, δ if $2r - N + 1 - \sum_{i=1}^{k}(l_i + m_i) - 8\sum_{i=1}^{k} l_i m_i$ is positive, proving the lemma. □

Proof of Theorem 2.1. If $\bigcup_{i=1}^{k} L_i \cup \bigcup_{i=1}^{k} M_i = \emptyset$, then the application of Dirac's theorem ensures the existence of a Hamiltonian cycle. The proof will use an indirect way. Suppose that this set of edges is not empty and G does not contain an acceptable Hamiltonian cycle. Delete edges from $\bigcup_{i=1}^{k} L_i \cup \bigcup_{i=1}^{k} M_i$ until such a Hamiltonian cycle appears. Without loss of generality it can be supposed that the last deleted edge $\{u, v\}$ is in L_1. That is, there is no acceptable Hamiltonian cycle with respect to $L_1'' = L_1' \cup \{\{u, v\}\} \subseteq L_1$, $L_i' \subseteq L_i$ $(2 \leqslant i \leqslant k)$, $M_i' \subseteq M_i$ $(1 \leqslant i \leqslant k)$, but there is one with respect to $L_i'(1 \leqslant i \leqslant k)$, M_i' $(1 \leqslant i \leqslant k)$. This cycle C must contain two edges of G which form an alternating four-cycle F with $\{u, v\}$ and an edge from M_1. (There can be more such F's!)

The vertices u and v are not neighbors in C, since $E \cap L_1 = \emptyset$. The edges in $F \cap E = F \cap C$ and $\{u, v\}$ must have common vertices. Let the neighbors of v in C be w and z. Then either $\{v, w\} \in F \cap C$ or $\{v, z\} \in F \cap C$ must hold. Thus the path P_1 obtained from C by deletion of the two edges at v is acceptable with respect to $H_1'', H_2', \dots, H_k', J_1', \dots, J_k'$.

Lemma 2.2 will be applied twice. In these steps "acceptable" will mean acceptable with respect to $H_1'', H_2', \dots, H_k', J_1', \dots, J_k'$. First apply Lemma 2.2 for the path $P_2 = C - \{v, w\}$. The obtained new cycle C_1 can be non-acceptable only because of $\{v, z\}$, since P_1 is acceptable. Apply Lemma 2.2 now for the path $P_3 = C_1 - \{v, z\}$. The resulting cycle is acceptable with respect to $H_1'', H_2', \dots, H_k', J_1 \dots, J_k'$, in contradiction with our assumption. □

Enomoto and Katona [4] contains the following sharpening of Theorem 1.1.

Theorem 2.3. *Suppose that* $1 \leqslant k$ *is an integer and* $n \geqslant n(k)$. *Then there are* $\lfloor \frac{1}{2}\binom{n}{k} \rfloor$ *unordered pairs* (A_i, B_i) *of disjoint k-element subsets* ($A_i \cap B_i = \emptyset$, $|A_i| = |B_i| = k$) *of* $[n]$ *such that*

$$|A_i \cap A_j| + |B_i \cap B_j| \leqslant k, \tag{2.2}$$

which implies

$$|A_i \cap B_j| + |B_i \cap A_j| \leqslant k \tag{2.3}$$

by the unorderedness.

The proof used a theorem analogous to Theorem 1.3, but it was developed specially for this purpose, its applicability was very limited. Here we sketch how to prove Theorem 2.3 by our new Theorem 2.1

Sketch of the proof of Theorem 2.3. As before, let $V = \binom{[n]}{k}$ and say two vertices are adjacent in G if they are disjoint subsets of X. Two vertices are adjacent in H_i if the intersection of the subsets has size i $(2 \leqslant i \leqslant k-1)$, they are adjacent in J_i if the intersection is of size at least $k - i + 1$ $(2 \leqslant i \leqslant k)$. Then the order of magnitude of l_i is n^{k-i} while the order of magnitude of m_i is n^{i-1}. Their product is asymptotically constant times n^{k-1}. Therefore (2.1) holds, Theorem 2.1 can be applied. □

Of course, the present proof of Theorem 2.3 is neither shorter nor easier than the original proof in [4], since it is based on Theorem 2.1 which is a generalization of the theorem used in the original proof. We have shown the present proof to illustrate the usage of our Theorem 2.1. Let us explain the main difference between Theorems 1.3 and 2.1. In both theorems the existence of a Hamiltonian cycle in the graph G (G_0 in Theorem 1.3) is ensured when certain pairs of edges are not allowed to be simultaneously in the cycle. These pairs are defined by another graph (G_1) in Theorem 1.3 and by a sequence of other graphs in Theorem 2.1. Translating this into the terms of applications, Theorem 1.3 can be used when the there is one forbidden situation for the pairs of disjoint pairs of k-element subsets while Theorem 2.1 can be used when there are more forbidden situations. Namely, in Theorem 1.1 it is excluded that both parts of the two pairs have an intersection of size more than $k/2$. On the other hand, in Theorem 2.3 more cases are excluded: if the intersection of the "first parts" has 0 elements then the intersection of the "second parts" cannot be larger than k, if the intersection of the "first parts" has one element then the intersection of the "second parts" cannot be larger than $k-1$, if the intersection of the "first parts" has 2 elements then the intersection of the "second parts" cannot be larger than $k-2$, and so on....Observe that Theorem 2.1 can also be applied in cases when other pairs of conditions are imposed on the sizes of the intersections of the "first parts" and the "second parts".

In the present paper the Hamiltonian-type theorems are applied only for subsets of an underlying set. However they could be applied for other combinatorial structures, as well. One could, for instance, obtain theorems analogous to Theorems 2.3 for pairs of k-dimensional subspaces of a finite affine geometry.

3. Cycles with edges of forbidden compositions

One further step in the generalization is the following model. Given a graph $G=(V,E)$ and a family $\mathscr{F}\subset\binom{V}{4}$. Find a Hamiltonian cycle in G such that no two edges of the cycle form a four-element set in $\mathscr{F}$. In Theorem 2.1 the members of $\mathscr{F}$ are defined as disjoint unions of edges of H_i and J_i (one from each, with fixed i). In the present section we will consider the generalization of this model, when a family $\mathscr{F}$ of k-element subsets is given and a Hamiltonian cycle is sought under some conditions determined by $\mathscr{F}$. First we generalize our previous result for these terms. Then we show some known results in this area and pose some open problems which lie inbetween.

The previous results suggest that the existence of a Hamiltonian cycle can be ensured by bounds on the degrees. This is why we define here the degree of $D\subset V$ in the family $\mathscr{F}$ by $d(D,\mathscr{F})=|\{F\in\mathscr{F}: D\subset F\}|$. The *maximum t-degree* $d_t(\mathscr{F})$ of $\mathscr{F}$ is the maximum of $d(D,\mathscr{F})$ for all t-element subsets $D\subset V$. Here $d_0(\mathscr{F})=|\mathscr{F}|$. If $\mathscr{F}\subset\binom{V}{k}$ then the inequality

$$d_{l-1}(\mathscr{F})\leqslant\frac{N-l+1}{k-l+1}d_l(\mathscr{F})\quad(1<l<k)\tag{3.1}$$

is obvious. This implies that restrictions for d_t imply restrictions on smaller values $s < t$. (Like in the case of simple graphs, an upper bound on the degree gives an upper bound on the number of edges.)

However, if we analyze the proofs of Lemma 2.2 and Theorem 2.1, it turns out that not the degrees above, but another notion plays an important role. Let $P = (v_1, \ldots, v_N)$ be a permutation of the elements of V and define

$$\overline{d}(v_1, P, \mathcal{F}) = |\{F \in \mathcal{F} : v_1 \in F,\ \ F \text{ contains two neighboring elements of } P\}|.$$

The *linear 1-degree of* v in $\mathcal{F}$

$$\overline{d}_1(v, \mathcal{F}) = \max_P \overline{d}(v, P, \mathcal{F}),$$

where the maximum is taken for all permutations P with $v_1 = v$. Finally the maximum linear 1-degree of $\mathcal{F}$ is

$$\overline{d}_1(\mathcal{F}) = \max_v \overline{d}_1(v, \mathcal{F}).$$

The *maximum linear 2-degree* $\overline{d}_2(\mathcal{F})$ is defined analogously.

$$\overline{d}(\{v_1, v_N\}, P, \mathcal{F}) = \\ |\{F \in \mathcal{F} : v_1, v_N \in F,\ \ F \text{ contains two other neighboring elements of } P\}|,$$

$$\overline{d}_2(D, \mathcal{F}) = \max_P \overline{d}(D, P, \mathcal{F}),$$

where $D = \{v_1, v_N\}$ is a two-element set and the maximum is taken for all permutations $P = (v_1, \ldots, v_N)$. The maximum linear 2-degree is

$$\overline{d}_2(\mathcal{F}) = \max_D \overline{d}_2(D, \mathcal{F}),$$

where the maximum runs for all two-element sets $D \subset V$.

Checking the proofs of Lemma 2.2 and Theorem 2.1 one can see that the following statement is true.

Theorem 3.1. *Let $G = (V, E)$ be a simple graph on $|V| = N$ vertices $\mathcal{F} \subset \binom{V}{4}$ and suppose that the following inequality holds:*

$$N \leqslant 2r - \overline{d}_2(\mathcal{F}) - 4\overline{d}_1(\mathcal{F}), \tag{3.2}$$

where r is the minimum degree of G. Then there is a Hamiltonian cycle in G such that no disjoint union of two edges of the cycle are in $\mathcal{F}$.

The proof is based on the following lemma, which is a generalization of Lemma 2.2. A set of edges of G is called *acceptable* if the union of no two edges is a member of $\mathcal{F}$.

Lemma 3.2. *Suppose that there is an acceptable Hamiltonian path P in G from $\alpha \in V$ to $\beta \in V$ in G and that* (3.2) *holds. Then one can find vertices γ and δ neighboring, and located in this order along P which satisfy the following conditions*: (i) $\{\alpha, \delta\}, \{\beta, \gamma\} \in E$,

(ii) *the set of three edges* $\{\alpha, \delta\}$, $\{\beta, \gamma\}$, $e \in P - \{\gamma, \delta\}$ *is acceptable for any choice of* e. ($\alpha = \gamma$ *and* $\beta = \delta$ *are allowed.) Therefore adding* $\{\alpha, \delta\}$ *and* $\{\beta, \gamma\}$, *and deleting* $\{\gamma, \delta\}$ *from* P *results in an acceptable Hamiltonian cycle.*

Proof. There are at least $2r - N + 1$ neighboring pairs $\{\gamma, \delta\}$ satisfying (i). The proof of this statement can be transferred from the proof of Lemma 2.2 without any change.

In what follows, we will subtract the number of cases, when the pair (γ, δ) does not satisfy (ii). This can happen in three different ways.

The pair of edges $\{\alpha, \delta\}$, $\{\beta, \gamma\}$ is not acceptable, i.e., $\{\alpha, \beta, \gamma, \delta\} \in \mathscr{F}$ holds. The number of such members of $\mathscr{F}$ cannot be more than $\overline{d}_2(\mathscr{F})$, by the definition of this parameter. Therefore the number of such pairs $\{\gamma, \delta\}$ can be upperbounded by $\overline{d}_2(\mathscr{F})$.

The pair $\{\alpha, \delta\}$ and some $\{x, y\} = e \in P$ is not acceptable, i.e., $F = \{\alpha, \delta, x, y\} \in \mathscr{F}$ holds. Observe that F contains the fixed α and two neighboring elements along P. Therefore the number of such F's cannot be more than $\overline{d}_1(\mathscr{F})$, by the definition of this parameter. In a given such F at most two element can play the role of δ (if the three elements different from α are three consecutive elements along P, otherwise there is only one "δ" in them). Consequently, the number of such δ's is at most $2\overline{d}_1(\mathscr{F})$.

The same upper bound is valid for the number of cases when the pair $\{\beta, \gamma\}$, $e \in P$ is not acceptable. Therefore there is an appropriate pair γ, δ if $2r - N + 1 - \overline{d}_2(\mathscr{F}) - 4\overline{d}_1(\mathscr{F})$ is positive, proving the lemma. □

Proof of Theorem 3.1. If $\mathscr{F}$ is empty then the application of Dirac's theorem ensures the existence of a Hamiltonian cycle. The proof will use an indirect way. Suppose that $\mathscr{F}$ is not empty and G does not contain an acceptable Hamiltonian cycle with respect to $\mathscr{F}$. Delete members from $\mathscr{F}$ until such a Hamiltonian cycle appears. Let A be the last deleted member. Then there is no acceptable Hamiltonian cycle with respect to some $\mathscr{F}' \subseteq \mathscr{F}$, but there is one with respect to $\mathscr{F}'' = \mathscr{F}' - \{A\}$. This cycle C must contain two edges of G whose union is A. Let one of these edges be $\{u, v\}$. If A consists of four consecutive element of C then choose $\{u, v\}$ not to be "in the middle". ((3.2) cannot hold when $N = 4$.)

Apply Lemma 3.2 for the path $P = C - \{u, v\}$, which is acceptable. The resulting cycle is acceptable with respect to $\mathscr{F}'$, in contradiction with our assumption. □

The most interesting case is when G is the complete graph. In the rest of the section only this case will be considered. Then $r = N - 1$ holds. If $\mathscr{F}$ is a family of 4-element subsets of V then we say that the cyclic permutation $(v_1, \ldots, v_N)$ of the elements of V is a (2, 2)-*Hamiltonian cycle* for $\mathscr{F}$ if $\{v_i, v_{i+1}, v_j, v_{j+1}\} \notin \mathscr{F}$ holds (the indices are considered mod m) for every pair $1 \leqslant i, j \leqslant N$. Theorem 3.1 implies the following corollary, if the obvious inequality $\overline{d}_2(\mathscr{F}) \leqslant \overline{d}_1(\mathscr{F})$ is substituted.

Corollary 3.3. *Let* V *be a set of* N *elements and suppose that the family* $\mathscr{F} \subset \binom{V}{4}$ *satisfies the inequality*

$$5\overline{d}_1(\mathscr{F}) \leqslant N - 2. \tag{3.3}$$

Then there is a (2, 2)-*Hamiltonian cycle for* $\mathscr{F}$.

This can be considered a Dirac-type theorem, analogous to Theorem 1.2. There is, however, a fundamental difference. The total degree in case of simple graphs is $N-1$, Theorem 1.2 gives a linear bound in terms of N. (To make the analogy closer, we have to speak about the forbidden edges, the degree of the graph of forbidden edges is bounded from above.) Here $\overline{d}_1$ can be quadratic in N while our bound (3.3) is linear. By the trivial inequality $\overline{d}_1 \leqslant d_1$ (3.3) can be rewritten as $5d_1(\mathscr{F}) \leqslant N-2$. In this case the situation is worse than before, since the possible order of magnitude here is N^3. Can we expect that the statement of the corollary holds under a condition like $d_1 \leqslant cN^3$? Not really, since the analogous problem for graphs would sound in the following way. Give an upper bound on the degrees in a graph which ensures the existence of a cycle where not only the edges of the cycle are not edges in the graph, but no two vertices of the cycle are adjacent in it. The trivial answer shows that when the best upper bound for the degree d_t $(0 \leqslant t \leqslant 3)$ is searched which implies the existence of a (2,2)-Hamiltonian cycle then not even the exponent of N is trivial.

The following example given by Katona and Kierstead [7] is useful towards this aim since it seems to be nearly optimal. Suppose that N is odd and partition the set of edges of K_N into $r = (N-1)/2$ Hamiltonian cycles $H_1, \ldots, H_r$. The family $\mathscr{K}_N$ is defined as the set of all unions of two vertex-disjoint edges from the same Hamiltonian cycle H_i. If C is a Hamiltonian cycle then at least three of its $N = 2r+1$ edges are in one H_i. Two of them are disjoint. This contradiction shows that there is no (2,2)-Hamiltonian cycle for this family $\mathscr{F}$. It is easy to see that $d_3(\mathscr{K}_N) \leqslant 6$. Their construction for the case when N is even is similar. Then $d_3(\mathscr{K}_N) \leqslant 9$ holds. Using (3.1) we obtain

$$d_2(\mathscr{K}_N) \leqslant \frac{9}{2}N,\ d_1(\mathscr{K}_N) \leqslant \frac{3}{2}N^2,\ d_0(\mathscr{K}_N) = |\mathscr{K}_N| \leqslant \frac{3}{8}N^3. \tag{3.4}$$

This shows that a naive Dirac-type theorem is not true in these cases: the condition $d_t(\mathscr{F}) \leqslant cN^{4-t}$ does not ensure the existence of a (2,2)-Hamiltonian cycle for $\mathscr{F}$.

One could think that the condition $d_3(\mathscr{F}) \leqslant 1$ would be sufficient. We found the following construction which shows that this is not true for all N.

Let $N = 2^n$ and V be the set of all 0,1 vectors of n coordinates. $\mathscr{F}$ is defined as the family of all 4-element subsets $\{v_1, v_2, v_3, v_4\}$ of V satisfying $v_1 + v_2 + v_3 + v_4 = 0 \pmod 2$. These are the points and planes of the n-dimensional affine space over $GF(2)$. (For Steiner quadruple systems see, e.g. [9]). Suppose that $(v_1, v_2, \ldots, v_N)$ is a (2,2)-Hamiltonian cycle for $\mathscr{F}$. It is easy to see that $v_{i-1} + v_i$ and $v_i + v_{i+1}$ are different vectors. Since this is a (2,2)-Hamiltonian cycle, $v_i + v_{i+1}$ and $v_j + v_{j+1}$ (the indices are considered mod N) must be different in general, because they satisfy $v_i + v_{i+1} + v_j + v_{j+1} \neq 0$. Hence all the sums $v_i + v_{i+1} (1 \leqslant i \leqslant N)$ are different. Therefore, one of them is 0, i.e., $v_i = v_{i+1}$ for some i. This contradiction shows that there is no (2,2)-Hamiltonian cycle for this $\mathscr{F}$. On the other hand, every 3-element subset $\{v_1, v_2, v_3\}$ of V uniquely determines a v_4 satisfying $v_1 + v_2 + v_3 + v_4 = 0 \pmod 2$. Therefore $d_3(\mathscr{F}) = 1$ holds.

Problem 1. *For what N's does $d_3(\mathscr{F}) \leqslant 1$ imply the existence of a (2, 2)-Hamiltonian cycle for the family $\mathscr{F}$? (The condition is equivalent to $|F \cap G| < 3$ for every pair $F, G \in \mathscr{F}$.)*

To be more formal, introduce the notation

$$m_{2,2}(N,t) = \min\left\{ d_t(\mathscr{F}) : \mathscr{F} \subset \binom{V}{4} \text{ and there is no (2,2)-Hamiltonian cycle for } \mathscr{F} \right\}.$$

We have

$$m_{2,2}(N,3) \leqslant 9$$

by the construction of [7] and

$$m_{2,2}(2^n,3) = 0$$

by the constructions with 0,1-vectors. Problem 1 asks which N's satisfy the inequality $1 \leqslant m_{2,2}(N,3)$.

Problem 2. *Give estimates on* $m_{2,2}(N,0)$, $m_{2,2}(N,1)$ *and* $m_{2,2}(N,2)$.

In general, if a family

$$\mathscr{F} \subset \binom{V}{i_1+i_2+\cdots+i_r}$$

is given, we say that a cyclic permutation $P = \{v_1, \dots, v_N\}$ of the elements of V is an $(i_1, \dots, i_r)$-*Hamiltonian cycle* for $\mathscr{F}$ if there are no i_1 consecutive, i_2 consecutive, $\dots$, i_r consecutive elements, resp. of P whose union is in $\mathscr{F}$, i.e.,

$$\bigcup_{l=1}^{r} \{v_{j_l}, v_{j_l+1}, \dots, v_{j_l+i_l-1}\} \notin \mathscr{F}$$

holds for any choice of $j_1, \dots, j_r \pmod m$. Moreover, let us define

$$m_{i_1,\dots,i_r}(N,t) = \min\{d_t(\mathscr{F}) : \mathscr{F} \subset \binom{V}{i_1+i_2+\cdots+i_r} \text{ and there is no } (i_1,\dots,i_r)\text{-Hamiltonian cycle for } \mathscr{F}\}.$$

The most natural one is the case of k-Hamiltonian cycles (where $r=1$, $i_1 = k$ in the previous notation, and (k) is replaced by k). Here there is a real Dirac-type theorem.

Theorem 3.4 (*Katona and Kierstead [7]*).

$$\frac{N}{2k} + \frac{2}{k} - 4 \leqslant m_k(N, k-1) \leqslant \left\lceil \frac{N-k}{2} \right\rceil$$

In a very recent, deep and difficult work Rödl, Ruciński and Szemerédi almost completely solved the case $k=3$.

Theorem 3.5 (*Rödl et al. [12]*). $\frac{N}{2} - 2 \leqslant m_3(N,2)$ *holds if N is a "huge" number.*

Rödl et al. [12] hope that they can generalize their result for arbitrary k and are able to lower the constraint on the number of vertices.

Another result for k-Hamiltonian cycles is the following one.

Theorem 3.6 *(Katona and Kierstead [7] and Frankl and Katona [5]).*

$$\frac{4k}{4k-1}\frac{1}{N}\binom{N}{k}\leqslant m_k(N,0)\leqslant\binom{N-2}{k-1}. \tag{3.5}$$

The upper estimate is improved by the following pretty construction of Tuza [14] under the assumption that an $(N-1, 2k-3, k-2)$ *Steiner system* exists, i.e., a family $\mathscr{S}$ of $2k-3$-element subsets of an $N-1$-element set such that every $k-2$-element subset is contained in exactly one member of $\mathscr{S}$. Let v be a fixed element of V and take a Steiner system $\mathscr{S}\subset 2^{V-\{v\}}$ satisfying the above conditions. Then define the family $\mathscr{F}$ in the following way using the notation $\mathscr{A}+a=\{A\cup\{a\}: A\in\mathscr{A}\}$:

$$\mathscr{F}=\left(\binom{V-v}{k-1}+v\right)-\bigcup_{S\in\mathscr{S}}\left(\binom{S}{k-1}+v\right) \tag{3.6}$$

Let us see that there is no k-Hamiltonian cycle for this $\mathscr{F}$. An indirect way will be used: suppose that there is a cyclic ordering of the elements of V so that no consecutive k elements form a member of $\mathscr{F}$. Investigate especially the elements around the distinguished v: $(v_{-k+1},\ldots,v_{-2},v_{-1},v,v_1,v_2,\ldots,v_{k-1})$. There are k intervals of length k in this part of the cyclic ordering. None of them is a member of $\mathscr{F}$, i.e., each of them is a subset of $S\cup\{v\}$ for some $S\in\mathscr{S}$. In other words, any of the k intervals of length $k-1$ in the sequence $(v_{-k+1},\ldots,v_{-2},v_{-1},v_1,v_2,\ldots,v_{k-1})$ is a subset of some member of $\mathscr{S}$. Suppose $\{v_{-k+1},\ldots,v_{-2},v_{-1}\}\subset S_1$ and $\{v_1,v_2,\ldots,v_{k-1}\}\subset S_2$ where $S_1,S_2\in\mathscr{S}$ holds. Since $|\{v_{-k+1},\ldots,v_{-2},v_{-1},v_1,v_2,\ldots,v_{k-1}\}|=2k-2$ and the members of $\mathscr{S}$ have only $2k-3$ elements, S_1 and S_2 must be different. Let i be the smallest integer for which $\{v_{-i},v_{-i+1},\ldots,v_{-1},v_1,v_2,\ldots,v_{k-i-1}\}\subset S_1$ still holds. Then $\{v_{-i+1},v_{-i+2},\ldots,v_{-1},v_1,v_2,\ldots,v_{k-i}\}$ is a subset of some $S_3\neq S_1$ (but $S_3=S_2$ might be true). Taking the intersections we obtain $\{v_{-i+1},\ldots,v_{-1},v_1,v_2,\ldots,v_{k-i-1}\}\subset S_1\cap S_3$ with a contradiction, since the left-hand side has $k-2$ elements, but $|S_1\cap S_3|<k-2$ holds by the property of $\mathscr{S}$.

Determine now $|\mathscr{F}|$. It is easy to see that

$$|\mathscr{S}|=\frac{\binom{N-1}{k-2}}{\binom{2k-3}{k-2}},$$

since every $k-2$-element subset is a subset of exactly one member of $\mathscr{S}$. Hence we have

$$|\mathscr{F}|=\binom{N-1}{k-1}-\frac{\binom{N-1}{k-2}}{\binom{2k-3}{k-2}}\binom{2k-3}{k-1}=\binom{N-1}{k-1}-\binom{N-1}{k-2}.$$

by (3.6). This implies the following improvement of (3.5) if the Steiner system in question exists.

$$m_k(N,0) \leqslant \binom{N-1}{k-1} - \binom{N-1}{k-2}.$$

It is easy to see that

$$\binom{N-1}{k-1} - \binom{N-1}{k-2} < \binom{N-2}{k-1}$$

and the difference of the two sides is

$$\binom{N-2}{k-3}.$$

Summarising, the order of magnitude (for large N, fixed k) of the upper estimate in (3.5) is correct and (3.6) improves the second term only, if the Steiner system in question exists. This is definitely true for $k = 3$ (trivial) and $k = 4$ (see [15,16]) with infinitely many N's (with positive density), but is unknown for $k \geqslant 5$.

Problem 3. *Give estimates on* $m_k(N, t)$ $(1 \leqslant t < k-1)$.

The problem of $(1, 1, \ldots, 1, 2)$-Hamiltonian cycles is in fact a traditional Hamiltonian problem for graphs.

Problem 4. *Give estimates on* $m_{2,\ldots,2}(N, t)$.

Our following construction shows that the example of [7] can be replaced by a family $\mathscr{F}$ of quadratic size if we are looking for cyclic permutations which are not only (2,2)-Hamiltonian cycles, but also (3,1)-Hamiltonian cycles. In other words, the cycle to be found cannot contain two distinct "edges" whose union is either a member of the given family or it is contained in one of the members. Let us define the problem more formally. Given a family $\mathscr{F}$ of 4-element subsets of V, we say that the cyclic permutation $(v_1, \ldots, v_N)$ of the elements of V is a 4_+-*Hamiltonian cycle* for $\mathscr{F}$ if $\{v_i, v_{i+1}, v_j, v_{j+1}\} \notin \mathscr{F}$ and $\{v_i, v_{i+1}, v_{i+2}, v_j\} \notin \mathscr{F}$ hold (the indices are considered mod m) for every pair $1 \leqslant i, j \leqslant N$. Define

$$m_{4_+}(N,0) = \min\left\{ |\mathscr{F}| : \mathscr{F} \subset \binom{V}{4} \text{ and there is no } 4_+\text{-Hamiltonian cycle for } \mathscr{F} \right\}.$$

Let $4 < N$, $V = \{1, \ldots, N\}$ and define $\mathscr{T}_N$ by

$$\mathscr{T}_N = \left\{ \{1, 2, i, j\} : \text{ either } \left(3 \leqslant i < j \leqslant \frac{N+1}{2}\right) \text{ or } \left(\frac{N+3}{2} \leqslant i < j \leqslant N\right) \right\}$$

for odd N and by

$$\mathcal{T}_N = \left\{\{1,2,i,j\} : \text{ either } \left(3 \leqslant i < j \leqslant \frac{N}{2}+1\right) \text{ or } \left(\frac{N}{2}+2 \leqslant i < j \leqslant N\right)\right\}$$
$$\bigcup \left\{\{1,2,3,j\} : \frac{N}{2}+2 \leqslant j \leqslant N-1\right\}$$

for even N. Deleting the set $\{1,2\}$ from each member of $\mathcal{T}_N$ a simple graph is obtained on the vertex set $\{3,\ldots,N\}$. Let us call this graph the *reduced graph*.

Try to find a 4_+-Hamiltonian cycle for $\mathcal{T}_N$. First suppose that 1 and 2 are neighbors in the cyclic permutation $P = \{v_1,\ldots,v_N\}$, say $v_1 = 1$, $v_N = 2$. Observe that the complement of the reduced graph is not Hamiltonian. Therefore the cyclic permutation $(v_2, v_3, \ldots, v_{N-1})$ contains an edge of the reduced graph. If this is $\{v_i, v_{i+1}\}(2 \leqslant i \leqslant N-2)$ then both $\{1,2\}$ and $\{v_i, v_{i+1}\}$ are neighbors in P and their union is in $\mathcal{T}_N$, a contradiction. On the other hand, if $\{v_2, v_{N-1}\}$ is in the reduced graph, then both $\{1, v_2\}$ and $\{2, v_{N-1}\}$ are neighbors in P and their union is in $\mathcal{T}_N$, a contradiction, again.

Suppose now that 1 and 2 are not neighbors in the cyclic permutation P. Let $v_1 = 1$, then $v_2 \neq 2$, $v_N \neq 2$, $v_2 \neq v_N$. Since $4 < N$ then one of the neighbors of 2 in the cyclic permutation, say v_j is different from both v_2 and v_N. Observe that the reduced graph "contains no empty triangle" that is one of the pairs $\{v_2, v_N\}$, $\{v_2, v_j\}$, $\{v_j, v_N\}$ is an edge of the reduced graph. If $\{v_2, v_j\}$ is this edge, then its union with $\{1,2\}$ is in $\mathcal{T}_N$, but this is also a union of two pairs of neighbors in P : $\{1, v_2\} \cup \{2, v_j\}$. If the edge in the reduced graph is $\{v_j, v_N\}$ then the same argument works. Finally, if $\{v_2, v_N\}$ is in the reduced graph then $\{v_N, v_1, v_2, 2\} \in \mathcal{T}_N$. In all three cases, the cyclic permutation contains two distinct edges whose union is either a member of $\mathcal{T}_N$ or a subset of a member. This contradiction finishes the proof of the following statement.

Proposition 3.7. *There is no 4_+-Hamiltonian cycle for $\mathcal{T}_N (4 < N)$.*

Since

$$|\mathcal{T}_N| = \begin{cases} \dfrac{(N-3)^2}{4} & \text{if } N \text{ is odd,} \\ \dfrac{N(N-4)}{4} & \text{if } 4 < N \text{ is even} \end{cases} \tag{3.7}$$

is quadratic in N, $|\mathcal{T}_N|$ is much smaller than $|\mathcal{K}_N|$ although it "almost" satisfies the condition that it contains no (2,2)-Hamiltonian cycle. Our proposition proves that $m_{4_+}(N, 0)$ is at most (3.7). We believe that $\mathcal{T}_N$ is the best construction for this modified problem, that is, $m_{4_+}(N, 0)$ is actually equal to (3.7).

Problem 5. *Is it true that if $|\mathcal{F}|$ is less than* (3.7) *then there is a 4_+-Hamiltonian cycle for $\mathcal{F}$?*

Let us summarize the content of the present section. There are some nice results concerning the k-Hamiltonian cycles. They seem to be the most natural generalizations of Dirac's classical theorem. The author is hoping that they will have important applications. On the other hand, the existing applications need results on the existence of (2,2)-Hamiltonian cycles. Moreover, the conditions in these theorems (Theorem 3.1, Corollay 3.2) use some unusual concepts of ("linear") degrees. The results involving the most natural generalization of the regular degree are not strong enough for the known application. Since these two types of problems are rather far from each other, there are many open question inbetween which sound natural to ask after considering the existing results. These questions are asked in forms of Problems.

4. A Baranyai-type conjecture

The following conjecture of Baranyai and the author tries to give a result analogous to the Baranyai theorem for the case when k does not divide n. Let m be the lowest common multiple of k and n, use the notation $a = m/k$. Define

$$\mathscr{K} = \{\{1,\ldots,k\},\{k+1,k+2,\ldots,2k\},\ldots,\{(a-1)k+1,\\ (a-1)k+2,\ldots,ak\}\},$$

where the elements of the sets are considered mod n. The families obtained from $\mathscr{K}$ by permuting the elements of the underlying set $[n]$ are called *wreaths*. If k divides n then a wreath is just a partition.

Conjecture 4.1 (*see Katona [6]*). $\binom{[n]}{k}$ can be decomposed into disjoint wreaths.

The aim of the present section is to state the following conjecture.

Conjecture 4.2. The members of $\binom{[n]}{k}$ can be listed in such a way that any $\lfloor n/k \rfloor - 1$ consecutive ones are disjoint.

First we prove the latter conjecture in the easy case $k = 2$.

Theorem 4.3. *One can list the edges of the complete graph K_n in such a way that any $\lfloor n/2 \rfloor - 1$ consecutive ones are disjoint.*

Proof. It was proved in the 19th century [11] that K_n can be decomposed into perfect matchings if n is even. (That is, Baranyai's theorem for $k = 2$.) We use here Walecki's proof (see [10]) . Cases are distinguished according to the parity of n.

1. *Suppose that n is even.* The decomposition of Walecki starts with the perfect matching

$$P_1 = \left\{\{1, n-1\}, \{2, n-2\}, \ldots, \left\{\frac{n}{2}-1, \frac{n}{2}+1\right\}, \left\{\frac{n}{2}, n\right\}\right\}. \tag{4.1}$$

Let P_i $(1 \leqslant i \leqslant n-1)$ denote the set of edges obtained by replacing the vertices j $(1 \leqslant j \leqslant n-1)$ by $j+i-1 \pmod{n-1}$ while n remains unchanged. (Then it is easy to see that P_i's are pairwise disjoint and their union is K_n.)

List the edges following the order in (4.1), first the edges in P_1 then in $P_2, \ldots, P_{n-1}$. Consider the beginning of the list obtained from P_1 and P_2:

$$\left\{\{1, n-1\}, \{2, n-2\}, \ldots, \left\{\frac{n}{2}-1, \frac{n}{2}+1\right\},\right.$$
$$\left.\left\{\frac{n}{2}, n\right\}, \{2, 1\}, \ldots, \left\{\frac{n}{2}, \frac{n}{2}+2\right\}, \left\{\frac{n}{2}+1, n\right\}\right\}. \tag{4.2}$$

It is easy to see that any $n/2-1$ consecutive edges are (vertex)-disjoint. By the construction, any $n/2-1$ consecutive edges are in $P_i \cup P_{i+1}$ for some i. However this part of the list is isomorphic to (4.2).

2. *Let n be odd.* The proof in this case is analogous, but n is not distinguished. Define

$$P_1 = \left\{\{1, n\}, \{2, n-1\}, \ldots, \left\{\frac{n-1}{2}, \frac{n+3}{2}\right\}\right\}.$$

Let P_i $(1 \leqslant i \leqslant n)$ denote the set of edges obtained by replacing the vertices j $(1 \leqslant j \leqslant n)$ by $j+i-1 \pmod{n}$. After giving these definitions, the proof follows the previous case. □

For general k we can only prove a much weaker statement under the assumption that another conjecture holds. We say that a graph *contains* H^r if the vertices of the graph can be listed in such a way that if the list is $x_1, \ldots, x_N$ then $\{x_i, x_j\}$ is an edge for each pair satisfying $|i-j| \leqslant r$ or $|i-j| \geqslant N-r$.

Conjecture 4.4 (*Seymour [13]*). If the minimum degree of a graph on N vertices is at least $[(r-1)/r]N$ then the graph contains H^r.

In the particular case $r=2$ "a graph contains H^r" becomes "a graph contains a Hamiltonian cycle". The conjecture becomes Dirac's theorem [3]. The conjecture is almost proved for every r, namely the following theorem holds.

Theorem 4.5 (*Komlos et al. [8]*). *If the minimum degree of a graph on N vertices is at least $[(r-1)/r]N$ and N is large enough ($N_0(r) \leqslant N$) then the graph contains H^r.*

Our very weak result is the following one.

Theorem 4.6. *If Conjecture* 4.4 *is true and $n_0(k) < n$ then the members of $\binom{[n]}{k}$ can be listed in such a way that any $\lfloor n/k^2 \rfloor$ consecutive ones are disjoint.*

Proof. Define a graph G whose vertex set is $\binom{[n]}{k}$ and say two vertices are adjacent iff the corresponding k-element sets are disjoint (Kneser graph). The degree of a vertex is $\binom{n-k}{k}$. Therefore the condition in Conjecture 4.4 is

$$\frac{r-1}{r}\binom{n}{k} \leqslant \binom{n-k}{k}.$$

This is equivalent to

$$r \leqslant \frac{\binom{n}{k}}{\binom{n}{k} - \binom{n-k}{k}}.$$

That is, we have to prove the inequality

$$\frac{n}{k^2} \leqslant \frac{n(n-1)\cdots(n-k+1)}{n(n-1)\cdots(n-k+1) - (n-k)\cdots(n-2k+1)}.$$

We will check the validity of the equivalent

$$(n-k^2)n(n-1)\cdots(n-k+1) < n(n-k)(n-k-1)\cdots(n-2k+1) \tag{4.3}$$

for large n and fixed k. It is easy to see that the coefficients of n^{k+1} and n^k, resp. of the two polynomials in (4.3) are the same. On the other hand, the coefficients of n^{k-1} on the left-hand side is

$$k^2\binom{k}{2} + \sum_{1 \leqslant i < j \leqslant k-1} ij. \tag{4.4}$$

The coefficient on the right-hand side is

$$\sum_{k \leqslant i < j \leqslant 2k-1} ij = \sum_{0 \leqslant i < j \leqslant k-1} (i+k)(j+k) = k^2\binom{k}{2} + k \sum_{0 \leqslant i < j \leqslant k-1} (i+j) + \sum_{1 \leqslant i < j \leqslant k-1} ij.$$

The latter one is obviously larger than (4.4) when $2 \leqslant k$. This proves (4.3) for large n. □

Acknowledgements

The author is deeply indebted to the anonymous referee for his/her careful work finding smaller and larger errors and for suggesting to add more illuminating comments.

References

[1] Zs. Baranyai, On the factorization of the complete uniform hypergraph, in: A. Hajnal, R. Rado, V.T. Sós, (Eds.), Infinite and Finite Sets, vol. I, Colloquium of Keszthely, 1973, North Holland, Amsterdam, Colloquium of Mathematical Society of János Bolyai, vol. 10, pp. 91–108.

[2] J. Demetrovics, G.O.H. Katona, A. Sali, Design type problems motivated by database theory, J. Statist. Plann. Inference 72 (1998) 149–164.

[3] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. Ser. 3 (2) (1952) 69–81.

[4] H. Enomoto, G.O.H. Katona, Pairs of disjoint q-element subsets far from each other, Electronic J. Combin. 8 (2001) R7.

[5] P. Frankl, G.Y. Katona, k-edge-hamiltonian hypergraphs, manuscript.

[6] G.O.H. Katona, Rényi and the combinatorial search problems, Studia Sci. Math. Hungar. 26 (1991) 363–378.
[7] G.Y. Katona, H. Kierstead, Hamiltonian chains in hypergraphs, J. Graph Theory 30 (3) (1999) 205–212.
[8] J. Komlós, G. Sárközy, E. Szemerédi, On a conjecture of Seymour, Ann. Combin. 2 (1998) 43–60.
[9] C.C. Lindner, A. Rosa, Steiner quadruple systems, a survey, Discrete Math. 22 (1978) 147–181.
[10] E. Lucas, Récréations Mathématiques I–IV, 1882–1884, vol. II, p. 176.
[11] M. Reiss, Über eine Steinersche kombinatorische Aufgabe, J. Reine Angew. Math. 56 (1859) 326.
[12] V. Rödl, A. Ruciński, E. Szemerédi, manuscript.
[13] P. Seymour, Problem section, in: T.P. McDonough, V.C. Mavron (Eds.), Combinatoric: Proceedings of the British Combinatorial Conference, 1973, Cambridge University Press, London/New York, 1974, pp. 201–202.
[14] Zs. Tuza, personal communication.
[15] R.M. Wilson, An existence theory for pairwise balanced designs I. Composition theorems and morphisms, J. Combin. Theory Ser. A 13 (1972) 220–245.
[16] R.M. Wilson, An existence theory for pairwise balanced designs I. The structure of PBD-closed sets and the existence conjectures, J. Combin. Theory Ser. A 13 (1972) 246–273.