FUNCTIONS DEFINED ON A DIRECTED GRAPH

by

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D. König has formulated the following theorem (see in [1] in a more general form):

Let G be a finite connected graph and g a real function defined on the vertices of G. Suppose that $g(x)$ is the arithmetic mean of the values attained by g on those vertices which are connected with x by an edge. Then the function g is constant on the set of vertices of G.

In this paper we try to generalize this problem for finite directed graphs. More precisely, we investigate for which graph holds the above property.

We will use the notation of C. Berge [2].

Finite directed graphs

Let $G = (X, \Gamma)$ a finite directed graph without loops and multiple edges, where $X = \{x_1, \ldots, x_n\}$ is the set of vertices and Γ is map of X into X $(x \notin \Gamma x)$. Suppose that

$$g(x_i) = \frac{1}{|\Gamma x_i|} \sum_{x \in \Gamma x_i} g(x) \qquad (\Gamma x_i \neq \varnothing, i = 1, 2, \ldots, n). \tag{1}$$

We say that *G has the König-property* if any function g satisfying (1) is constant on X. On the other hand a non-empty set A is called a *sink-set*, if $\Gamma A \subset A$.

We may formulate the following theorem.

THEOREM 1. *A finite directed graph $G = (X, \Gamma)$ has not the König-property if and only if it has two disjoint sink-sets.*

The PROOF OF THE NECESSITY is the same as the original proof of König. Let A be the set of vertices, where g attains its maximum, and similarly, let B be the set of vertices where g attains its minimum. Since, by the supposition, g is not constant, A and B are disjoint. Now we verify in an indirect way that A is a sink-set. Let x_i and x_j be two vertices such that $x_i \in A$, $x_j \notin A$, $x_j \in \Gamma x_i$. In this case

$$g(x_i) > \frac{1}{|\Gamma x_i|} \sum_{x \in \Gamma x_i} g(x)$$

holds, because the right side is the arithmetic mean of values smaller than or equal to $g(x_i)$, but at least one of them is strictly smaller. Similarly, B is a sink-set, too.

PROOF OF THE SUFFICIENCY. We have to show if there are two disjoint sink-sets A, B in X, then there exists a non-constant function g satisfying (1).

(α) We can suppose that there does not exist a sink-set C in X such that $C \subset X - A \cup B$. In the contrary case we can use $A \cup C$ instead of A. (Or finite times repeating this.)

Determine the indices so that $A = \{x_{k+1}, x_{k+2}, \ldots, x_l\}$ and $B = \{x_{l+1}, x_{l+2}, \ldots, x_n\}$ and rewrite (1) in the following form:

(2) $$-|\Gamma x_i|\, g(x_i) + \sum_{x \in \Gamma x_i} g(x) = 0 \qquad (1 \le i \le n).$$

(2) is a homogeneous linear equation system. We must show that it has a non-constant solution.

Let us consider the matrix **N** of coefficients of (2). Obviously, $a_{ii} = -|\Gamma x_i|$ and

$$a_{ij} = \begin{cases} 1 & \text{if } x_j \in \Gamma x_i \\ 0 & \text{otherwise.} \end{cases} \qquad (i \ne j).$$

Using the fact that A is a sink-set we obtain

(3) $$a_{ij} = 0 \begin{array}{l} \text{if} \quad k < i \le l \quad \text{and} \quad 1 \le j \le k; \\ \text{if} \quad k < i \le l \quad \text{and} \quad l < j \le n. \end{array}$$

Similarly

(4) $$a_{ij} = 0 \quad \text{if} \quad l < i \le n \quad \text{and} \quad 1 \le j \le l,$$

because B is a sink-set, too. Finally,

(5) $$\sum_{j=1}^{n} a_{ij} = 0 \qquad (1 \le i \le n)$$

is also obvious. The matrix **N** has the form

(6) $$\mathbf{N} = \left(\begin{array}{c|c|c} \mathbf{M} & & \\ \hline 0 & & 0 \\ \hline 0 & 0 & \end{array} \right).$$

Put

$$g(x_i) = \begin{cases} 0 & \text{if } k < i \le l, \\ 1 & \text{if } l < i \le n. \end{cases}$$

These values satisfy the last $n - k$ equations because of (3), (4) and (5). It remains to solve the equation system

$$\sum_{j=1}^{k} a_{ij}\, g(x_j) + \sum_{j=l+1}^{n} a_{ij} = 0 \qquad (1 \le i \le k).$$

We will show in an indirect way that $|\mathbf{M}| \neq 0$, where $\mathbf{M}$ denotes the matrix of coefficients of the variables $g(x_1), \ldots, g(x_k)$. Before the proof let us note a property of the sum of a row in $\mathbf{M}$. Instead of (5)

$$\sum_{j=1}^{k} a_{ij} \leq 0 \qquad (1 \leq i \leq k) \tag{7}$$

holds because non-negative elements are omitted from each row. If $|\mathbf{M}| = 0$, then the column vectors $\mathbf{v}_j$ of $\mathbf{M}$ are linearly dependent, that is, there are $c_1, \ldots, c_k$ real numbers such that

$$\sum_{j=1}^{k} c_j \mathbf{v}_j = 0, \qquad \sum_{j=1}^{k} c_j^2 > 0. \tag{8}$$

We may assume there is a positive number among $c_1, \ldots, c_k$. Thus $\max_{1 \leq j \leq k} c_j = d > 0$. Let us choose the indices in such a manner that $c_1 = c_2 = \ldots = \; = c_r = d$ but $c_j < d$ for $r < j \leq k$. By condition (α) there are m $(1 \leq m \leq r)$and $t > r$ such that

$$a_{mt} = 1. \tag{9}$$

We separate two cases:

(a) $1 \leq t \leq k$;
(b) $k < t \leq n$.

In the case (a) $c_j < d$ and in the case (b) we obtain from (5) and (9) the inequality

$$\sum_{j=1}^{k} a_{mj} < 0 \tag{10}$$

instead of (7).

We will show that (8) cannot hold for the m-th co-ordinates. The following inequality is trivial:

$$\begin{aligned} \sum_{j=1}^{k} c_j a_{mj} &= - c_m |\Gamma x_m| + \sum_{\substack{j=1 \\ j \neq m}}^{k} c_j a_{mj} \leq - d |\Gamma x_m| + \\ &+ \sum_{\substack{j=1 \\ j \neq m}}^{k} d a_{mj} = d \sum_{j=1}^{k} a_{mj}; \end{aligned} \tag{11}$$

and in the case (a) strict inequality holds. Thus, in the case (a) (11) and (7) result

$$\sum_{j=1}^{k} c_j a_{mj} < 0. \tag{12}$$

In the case b) (12) follows from (11) and (10). The proof is completed.

Corollary. *Every tournament and strongly connected graph has the König-property.*

The proof is obvious.

Let G be an undirected graph. If we assign directions to the edges of G in an arbitrary manner, then the resulting graph G^* is called *an orientation* of G. If G is a complete graph, then any orientation of G is a tournament, thus, by Corollary, it has the König-property. However, for the other cases we have the following theorem.

THEOREM 2. *Let G be a finite undirected non-complete graph. Then G has an orientation G_1^* having the König-property and another orientation G_2^* having not this property.*

PROOF. Since G is non-complete, we have two distinct vertices x_i and x_j which are not connected by an edge. Assign directions to the edges incident with x_i or x_j in such a manner that x_i and x_j have only incoming edges. The directions of the other edges are arbitrary. In this case $\{x_i\}$ and $\{x_j\}$ will be sink-sets in the obtained graph G^*. By Theorem 1 G^* has not the König-property.

We prove the second part of the theorem first for trees. Let us consider a terminal vertex e of a tree T. We can assign directions to the edges of T such that there exists a directed path from any vertex to e. Obviously, any sink-set of the resulting T^* contains the vertex e, that is, there are no two disjoint sink-sets; T^* has the König-property.

Let now G be an arbitrary undirected graph and T a spanning tree of G. Assign directions to the edges of T in above manner and to other edges of G in an arbitrary manner. We know that T^* has not two disjoint sink-sets, but the adding new edges does not fail this property, that is, G^* has the König-property, indeed.

General solution of (1)

If a directed graph G has the König-property, every solution of the system (1) is constant. However, it is also interesting what is the general solution of (1) if G has not the König-property.

The following properties of a directed graph G are well-known (see e.g. [3], p. 149.). We say that two vertices x_i and x_j are *equivalent* if they are mutually connected (there is a directed path from x_i to x_j and another directed path from x_j to x_i) or $x_i = x_j$. This is an equivalency relation; the equivalency classes are called *leaf*, the section graphs defined by a leaf are the *leaf graphs*. All the edges connecting two leaves have the same orientation. We can construct a leaf composition graph G' whose vertices are the leaves of G. Two leaves L_1 and L_2 are connected by a directed edge in G' when there are directed edges from L_1 to L_2 in G. A leaf graph is strongly connected.

Let L be a leaf of G and let the corresponding vertex of L a sink. We call such a set L *minimal sink-set*. (It is obvious by definition of G' that L is a sink-set in G.) Let further g be a function defined on G satisfying (1). Since L has not outcoming edges, g satisfies (1) on L, too. However, L is strongly connected, thus, applying Corollary we obtain that g is constant on L.

Consider all the minimal sink-sets $L_1, L_2, \ldots, L_m$ *of* G. g *must be constant on each of* L_i's. *Let us given these constant values in an arbitrary manner on* $L_1, L_2, \ldots, L_m$. *The other values of* g *are already uniquely determined by* (1).

The proof of this statement is the same as the proof of sufficiency of Theorem 1. Only that we have more than 2 sink-sets.

REFERENCES

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[3] Ore, O.: *Theory of graphs,* Amer. Math. Soc. Colloq. Publ. Vol. 38. Providence, 1960.