SEPARATUM

ACTA MATHEMATICA
ACADEMIAE SCIENTIARUM HUNGARICAE

TOMUS XV FASCICULI 3—4

GY. KATONA

INTERSECTION THEOREMS FOR SYSTEMS OF FINITE SETS

1964

INTERSECTION THEOREMS FOR SYSTEMS OF FINITE SETS

By

GY. KATONA (Budapest)

(*Presented by* A. RÉNYI)

Let $k \leqq m$, and M be a finite set of cardinal number m. Determine the largest number n such that there exists a system of n sets a_ν satisfying the conditions

$$a_\nu \subset M, \quad a_\mu \neq a_\nu; \quad |a_\mu a_\nu| \geqq k \quad (\mu < \nu < n),$$

where $|a|$ is the cardinal number of a.

If $m+k$ is even, then the system consisting of the sets a such that

$$a \subset M \quad \text{and} \quad |a| \geqq \tfrac{1}{2}(m+k)$$

has the required properties. P. ERDŐS, CHAO KO and R. RADO have guessed, that this system contains the maximum possible number of sets [1].

In this note I prove this conjecture, and determine the extremal system also in the case when $m+k$ is odd. For the proof I use a theorem (Theorem 2) which is also interesting in itself.

Notations:

The letters a, b, c, d, e denote finite sets of non-negative integers, all other lower-case letters denote non-negative integers. If $k \leqq l$, then $[k, l)$ denotes the set

$$\{k, k+1, \ldots, l-1\} = \{t : k \leqq t < l\}.$$

The obliteration operator $\widehat{\ }$ serves to remove from any system of elements the element above which it is placed. Thus $[k, l) = \{k, k+1, \ldots, \hat{l}\}$. The cardinal number of the set a is denoted by $|a|$; inclusion, union, difference and intersection of sets are denoted by $a \subset b$, $a+b$, $a-b$, ab.

If $k \leqq l \leqq m$, $S(k, l, m)$ denotes the set of all systems $\{a_0, a_1, \ldots, \hat{a}_n\}$ such that

$$a_\nu \subset [0, m), \quad |a_\nu| = l \quad (\nu < n),$$
$$a_\mu \neq a_\nu, \quad |a_\mu a_\nu| \geqq k \quad (\mu < \nu < n).$$

Put $A = \{a_0, \ldots, \hat{a}_n\}$, where $|a_\nu| = l$ $(\nu < n)$. A^g or $\{a_0, \ldots, \hat{a}_n\}^g$ denotes the system of sets b_ν such that $|b_\nu| = g$, $b_\mu \neq b_\nu$ $(\mu < \nu < |A^g|)$, and for some μ $b_\nu \subset a_\mu$.

Let us consider $A = \{a_0, \ldots, \hat{a}_n\}$, where a_ν are arbitrary sets. Denote by A_l the subsystem of sets a_ν satisfying the conditions $a_\nu \in A$ and $|a_\nu| = l$.

THEOREM 1. *If $1 \leqq g \leqq l$, $1 \leqq k \leqq l$ and $g+k < l$, further $\varepsilon > 0$, then there exists a system $A = \{a_0, \ldots, \hat{a}_n\} \in S(k, l, m)$ for which*

$$\frac{|A^g|}{n} < \varepsilon.$$

PROOF. Let $m \geqq k$ be a non-negative integer. If $a_0, a_1, \dots, a_n$ are distinct sets such that

$$[0, k) \subset a \subset [0, m) \quad \text{and} \quad |a| = l,$$

then $A = \{a_0, \dots, \hat{a}_n\} \in S(k, l, m)$ and $n = \binom{m-k}{l-k}$. Clearly $|A^g| \leqq \binom{m}{g}$ $\Big($in fact it is easy to see that $|A^g| = \binom{m}{g}$ $\Big)$, and

$$\frac{\binom{m}{g}}{\binom{m-k}{l-k}}$$

can be arbitrarily small, if m is sufficiently large, because $g < l-k$.

THEOREM 2. *If* $1 \leqq g \leqq l$, $1 \leqq k \leqq l$ *and* $g+k \geqq l$, *further* $A = \{a_0, \dots, \hat{a}_n\} \in S(k, l, m)$ *then*

$$n \frac{\binom{2l-k}{g}}{\binom{2l-k}{l}} \leqq |A^g|. \tag{1}$$

REMARK. From $g \geqq l-k$ and $l \geqq g$ it easily follows that

$$\frac{\binom{2l-k}{g}}{\binom{2l-k}{l}} \geqq 1 \tag{2}$$

and equality holds if and only if $g+k = l$ or $g = l$.

PROOF of Theorem 2. If $g = l$, the theorem is trivial (moreover always equality holds). In what follows we consider the case $g < l$.

We distinguish three cases.

Case 1: $2l-k \geqq m$.

By counting in two different ways the number of pairs (a_ν, c) where $c \in A^g$ and $c \subset a_\nu$, we obtain

$$n\binom{l}{g} \leqq |A^g| \binom{m-g}{l-g}. \tag{3}$$

We have to prove that

$$\frac{\binom{l}{g}}{\binom{m-g}{l-g}} \geqq \frac{\binom{2l-k}{g}}{\binom{2l-k}{l}}.$$

This is trivial, if $2l-k \geqq m$, moreover equality holds only in case $2l-k = m$. Hence we obtain that equality holds in (3) only if $m = 2l-k$ and every c is included in $\binom{m-g}{l-g}$ distinct sets a_ν, that is A is the system of all subsets of $[0, m)$. Thus equality in Case 1 can hold only in this way.

Case 2: $g=1$.

Since $g+k \geqq l$, we have $k \geqq l-1$. There are two cases: $k=l$ and $k = l-1$. If $k=l$, then $n=1$ and we can choose $m=k$. Here $2l-k = k = m$, therefore we have Case 1. Assume next $k = l-1$. If we have a system $A = \{a_0, \ldots, \hat{a}_n\} \in S(l-1, l, m)$ such that every set of $l-1$ is included at most in two a_ν, then consider the set a_0a_1. Clearly $|(a_0a_1)a_\nu| < l-1$, on the other hand $|(a_0a_1)a_\nu| < l-2$ is impossible, because in this case we should have

$$|a_0a_\nu| \leqq |(a_0a_1)a_\nu| + 1 < (l-2)+1 = l-1.$$

Thus $|a_0a_1a_\nu| = l-2$.

We have $a_0 - a_1 \subset a_\nu$ for every ν, because of $|a_0a_1| = l-1$ and $|(a_0a_1)a_\nu| = l-2$. Similarly $a_1 - a_0 \subset a_\nu$. Here a_0a_1, a_0-a_1, a_1-a_0 are disjoint sets, therefore

$$a_\nu = (a_0-a_1)+(a_1-a_0)+a_0a_1-\lambda_\nu, \tag{4}$$

where λ_ν is an element of a_0a_1. From this results $n \leqq l+1$, and every element is contained at most in l sets a_ν. Thus $n\frac{l}{l} \leqq |A^1|$, since every a_ν has exactly l elements.

If the system A is such that there is a set c satisfying $|c| = l-1$, which is included at least in 3 sets a_ν (for example a_0, a_1 and a_2) then for arbitrary $\nu < n$ $c \subset a_\nu$. Namely, $|ca_\nu| < l-2$ can not be true, because in this case $|a_0a_\nu| < l-1$, similarly $|ca_\nu| = l-2$ can not hold, since its consequence would be $a_\nu \supset a_0 - c$, $a_\nu \supset a_1 - c$ and $a_\nu \supset a_2 - c$ because of $|a_0a_\nu| = |a_1a_\nu| = |a_2a_\nu| = l-1$, that is $|a_\nu| \geqq l+1$, which is impossible. This completes the proof in Case 2, since here $|A^1| = n+l-1 > n$.

In Case 2 equality can hold if and only if every set of $l-1$ is included at most by two a_ν, and A consists of all sets a_ν satisfying (4). This falls under Case 1, where equality holds.

Case 3: $2l-k < m$ and $g > 1$.

We use induction over m, and we apply Cases 1 and 2.

Here $1 < g < l \leqq m$, thus $m \geqq 3$. First we consider $m=3$. Here $l=3$, thus $n=1$, $g=2$, $k=1$ or 2 ($k=3$ is impossible, since we should then have $2l-k = m$, and this is Case 1). Since $|A^2| = 3$ and $\frac{\binom{5}{2}}{\binom{5}{3}} = 1$, $\frac{\binom{4}{2}}{\binom{4}{3}} = \frac{3}{2}$, in both cases strict inequality holds.

Suppose that $m > 3$ and for $m-1$ Theorem 2 is true. We prove the theorem for m. Denote by s_ν the sum of the elements of a_ν. We can clearly assume that our system is such that $|A^g|$ is minimal and amongst all such systems $\sum_{\nu=0}^{n-1} s_\nu$ is minimal. Denote now by A the system $A = \{a_\nu : \nu < n\}$.

We separate in Case 3 two subcases.

Case 3a. Suppose that whenever

$$m-1\in a_\nu\in A \quad\text{and}\quad \lambda\in[0,m)-a_\nu$$

then

$$a_\nu-\{m-1\}+\{\lambda\}\in A.$$

We may assume that for some $n_0\leqq n$, $m-1\in a_\nu$ $(\nu<n_0)$ and $m-1\notin a_\nu$ $(n_0\leqq\nu<n)$. If $n=1$, this is Case 1, because we can choose $m=l$ and thus $2l-k\geqq m$ holds. Let be $n>1$. If $n_0=0$, then the theorem holds by our induction hypothesis. Suppose that $n_0\geqq 2$. Let be $\mu<\nu<n_0$. Then $|a_\mu+a_\nu|\leqq 2l-k<m$, and there exists an element $\lambda\in[0,m)-a_\mu-a_\nu$. Put $b_\mu=a_\mu-\{m-1\}$ $(\mu<n_0)$. Here $b_\mu+\{\lambda\}\in A$, $|b_\mu b_\nu|=|(b_\mu+\{\lambda\})b_\nu|=|(b_\mu+\{\lambda\})a_\nu|\geqq k$, and therefore $l-1\geqq k$ and $B=\{b_0,\dots,\hat b_{n_0}\}\in S(k,l-1,m-1)$. If $n_0=1$, since $n>1$, then $m-1\notin a_1$ and $|a_0a_1|\leqq l-1$. Thus also $l-1\geqq k$ and $B=\{b_0\}\in S(k,l-1,m-1)$. We can use our induction hypothesis, if $n_0\geqq 1$ and $g-1>1$, since both $g-1+k\geqq l-1$ (because of $g+k\geqq l$) and $g-1<l-1$ (because of $g<l$) hold, and $l-1\geqq k\geqq 1$. Therefore we have in this case

$$n_0\frac{\binom{2(l-1)-k}{g-1}}{\binom{2(l-1)-k}{l-1}}\leqq|B^{g-1}|=p. \tag{5}$$

We can not use the induction hypothesis, when $g-1=1$ that is $g=2$. However, (5) holds, because we can apply Theorem 2 for k, $l-1$ and $g-1=1$ (Case 2).

On the other hand $C=\{a_{n_0},\dots,\hat a_n\}\in S(k,l,m-1)$. We can use the induction hypothesis, if $l\leqq m-1$:

$$(n-n_0)\frac{\binom{2l-k}{g}}{\binom{2l-k}{l}}\leqq|C^g|=r. \tag{6}$$

If $l=m$, then this is Case 1, because $2l-k\geqq m$. Trivially

$$\frac{\binom{2l-k}{g}}{\binom{2l-k}{l}}\leqq\frac{\binom{2(l-1)-k}{g-1}}{\binom{2(l-1)-k}{l-1}} \tag{7}$$

since $l>g$ and $g+k-l\geqq 0$.

Adding (5) and (6), applying (7) we get

$$n\frac{\binom{2l-k}{g}}{\binom{2l-k}{l}}\leqq p+r. \tag{8}$$

Denote by d_ν $(\nu<p)$ elements of B^{g-1}, and by c_ν $(\nu<r)$ elements of C^g. Let $e_\nu = d_\nu+\{m-1\}$ $(\nu<p)$. Then obviously $|e_\nu|=g$, $e_\mu\neq e_\nu$ $(\mu<\nu<n)$. Moreover for every $\nu<p$ there exists an index $\mu<n_0$ such that $d_\nu\subset b_\mu$. Hence $e_\nu\subset a_\mu$, since $e_\nu = d_\nu+\{m-1\}$ and $a_\mu = b_\mu+\{m-1\}$. Thus $e_\nu\in A^g$, moreover trivially $e_\mu\neq c_\nu$ $(\mu<p, \nu<r)$, since $m-1\in e_\mu$ and $m-1\notin c_\nu$. Consequently $c_0, c_1, ..., c_r, e_0, ..., e_p$ are distinct elements of A^g, that is

$$p+r \leqq |A^g|,$$

which completes the proof of 3a.

It remains to prove that in Case 3a equality can not hold. If $m=3$, this is true. Suppose now that $m>3$, and use induction over m. Apply the same steps, as in the proof of the inequality. In those cases, where then induction could be used, it can be used here too, that is if $m>2l-k$, then $m-1>2(l-1)-k$. Thus it follows by induction hypothesis that in (5) (and in the theorem) strict inequality holds. Those cases where induction could not be used are settled by Cases 1 and 2. Thus in Case 3a strict inequality always holds.

Case 3b. Suppose that there are $a\in A$ and $\lambda\in[0, m)-a$ such that $m-1\in a$ and $a-\{m-1\}+\{\lambda\}\notin A$. Then $\lambda<m-1$.

We may assume that the sets are labelled in such a way, that the following relations hold:

$$\begin{aligned}
&m-1\in a_\nu, \quad \lambda\notin a_\nu, \quad b_\nu = a_\nu-\{m-1\}+\{\lambda\}\notin A \quad (\nu<n_0),\\
&m-1\in a_\nu, \quad \lambda\notin a_\nu, \quad c_\nu = a_\nu-\{m-1\}+\{\lambda\}\in A \quad (n_0\leqq\nu<n_1),\\
&m-1\in a_\nu, \quad \lambda\in a_\nu \quad (n_1\leqq\nu<n_2),\\
&m-1\notin a_\nu \quad (n_2\leqq\nu<n).
\end{aligned}$$

Here $1\leqq n_0\leqq n_1\leqq n_2\leqq n$. Put $b_\nu=a_\nu$ $(n_0\leqq\nu<n)$. We have now to prove that

$$B = \{b_0, ..., \hat{b}_n\}\in S(k, l, m).$$

Let be $\mu<\nu<n$. We must prove that

$$b_\mu\neq b_\nu \quad \text{and} \quad |b_\mu b_\nu|\geqq k.$$

For $\mu<\nu<n_0$ or $n_0\leqq\mu<\nu$ these are obvious. Now let be $\mu<n_0\leqq\nu$. Then $b_\mu\notin A$, $b_\nu=a_\nu\in A$, and hence $b_\mu\neq b_\nu$.

If $n_0\leqq\nu<n_1$, then $c_\nu\in A$; and there are k distinct common elements of a_μ and c_ν. λ and $m-1$ are not among these, therefore they are common elements also of b_μ and $b_\nu=a_\nu$.

If $n_1\leqq\nu<n_2$, then $|a_\mu a_\nu|\geqq k$, but $\lambda\notin a_\mu a_\nu$. If instead of a_μ we take b_μ, then out of the common elements at most one is lost: $|b_\mu a_\nu|=|b_\mu b_\nu|\geqq k-1$, but λ, which is common element, does not belong to these $k-1$ elements. Thus $|b_\mu b_\nu|\geqq k$.

Finally, if $n_2\leqq\nu<n$, then a_μ and a_ν have k common elements. $m-1$ does not belong to them, since $m-1\notin a_\nu$. Therefore the same k elements are also common elements of b_μ and b_ν. Thus $B\in S(k, l, m)$ is proved.

Now we must show, that $|A^g|\geqq|B^g|$. Let c be such a set that $|c|=g$, $c\in B^g$ but $c\notin A^g$. Then $c\subset b_\nu$ for some $\nu<n$, because of $c\in B^g$. Obviously $\nu<n_0$, because if $n_0\leqq\nu<n$, then $b_\nu=a_\nu$ and $c\in A^g$.

$\lambda\in c$, because $c\subset b_\nu$ for some $\nu<n_0$, and $c\not\subset a_\nu = b_\nu+\{m-1\}-\{\lambda\}$.

On the other hand $m-1\notin c$, because of $m-1\notin b_\nu$ $(\nu<n_0)$.

Let be $d = c-\{\lambda\}+\{m-1\}$. Here $d \subset a_\nu$, that is $d \in A^g$, since $c \subset b_\nu$ and $b_\nu = a_\nu - \{\lambda\}+\{m-1\}$. However, $d \notin B^g$. If $d \subset b_\nu$ would hold for some $\nu < n$, then obviously $n_0 \leqq \nu < n_2$ because in the cases $\nu < n_0$ and $n_2 \leqq \nu < n$, $m-1 \notin b_\nu$ holds. If $n_0 \leqq \nu < n_1$, then $c \subset c_\nu = a_\nu - \{m-1\}+\{\lambda\}$ holds (for such ν, for which $d \subset b_\nu$) and since $c_\nu \in A$, follows $c \in A^g$, which contradicts our supposition. However, if $d \subset b_\nu$ holds for $n_1 \leqq \nu < n_2$, then $c \subset a_\nu$ because of $\lambda \in b_\nu = a_\nu$, $m-1 \in b_\nu = a_\nu$, and this also is a contradiction.

Hereby we associated a set d to every set c, which is an element of B^g, but is not one of A^g (to distinct sets c correspond distinct sets d) in such a way, that set d is an element of A^g, but is not one of B^g. From this follows

$$|A^g| \geqq |B^g|. \tag{9}$$

Since for fixed n we supposed A to be the system, for which $|A^g|$ is minimal, in (9) only equality can hold. However we have

$$f(b_0, \ldots, \hat{b}_n) - f(a_0, \ldots, \hat{a}_n) = n_0[-(m-1)+\lambda] < 0,$$

which contradicts the maximum property of A. This shows that Case 3b can not occur.

REMARKS.

1. In this proof I used the sequence of ideas contained in the proof of ERDŐS—CHAO KO—RADO's Theorem 1 ([1]).

2. We showed also, that equality can hold in Case 1, and here only if $m = 2l-k$, and A contains every subset of cardinal number l, or in the trivial case $g = l$.

The following are all consequences of Theorem 2.

3. THEOREM 1 OF ERDŐS—CHAO KO—RADO [1]. *If* $1 \leqq l \leqq \frac{1}{2}m$ *and*

$$A = \{a_0, \ldots, \hat{a}_n\} \in S(1, l, m), \text{ then } n \leqq \binom{m-1}{l-1}.$$

PROOF. Let $b_\nu = [0, m) - a_\nu$ and $B = \{b_0, \ldots, \hat{b}_n\}$. Then $|b_\nu| = m-l \geqq l$, $|b_\mu b_\nu| = |[0, m)-(a_\mu + a_\nu)| \geqq m-2l+1$, because $|a_\mu + a_\nu| \leqq 2l-1$ $(\mu < \nu < n)$. We use now Theorem 2 for $m-2l+1$, $m-l$ and l in place of k, l and g. We can apply the theorem, since $1 \leqq l \leqq m-l$, $1 \leqq m-2l+1 \leqq m-l$, and $l+(m-2l+1) \geqq m-l$. Thus

$$n \frac{\binom{m-1}{l}}{\binom{m-1}{m-l}} \leqq |B^l|. \tag{10}$$

Let be $c \in B^l$. Then there exists a number $\mu < n$ such that $c \subset b_\mu$. For this $c([0, m) - b_\mu) = ca_\mu = \emptyset$. Thus $c \notin A$. Consequently

$$|B^l| + |A| = |B^l| + n \leqq \binom{m}{l}.$$

From this, applying (10),

$$n \leqq \binom{m-1}{l-1}.$$

REMARKS.

1. [1] contains this theorem in a more general form which follows from the form proved here by a simple step, shown in [1].

2. If $2l-k \geqq m$, $|a_\nu| = l$ $(\nu < n)$, and $\{a_0, \ldots, \hat{a}_n\} \in S(k, l, m)$, then trivially $n \leqq \binom{m}{l}$, and this estimate is the best possible. If $2l-k < m$, then according to Theorems 1 and 2 of ERDŐS—CHAO KO—RADO [1], the estimate $n \leqq \binom{m-k}{l-k}$ holds in most cases. The estimate is however not true for every case: In [1] an interesting example is cited. Further simple example:

2a. Let $k = l-1$, $|a_\nu| = l$. Then either $n \leqq l+1$ or $n > l+1$.

Consider in the latter case the subsets of a_0 having $l-1$ elements. The number of these is l, and one of these is included in a_ν $(1 \leqq \nu < n)$. Thus there exists a set $c \subset a_0$ such that $|c| = l-1$, and there exist two sets, for example a_1 and a_2 for which $c \subset a_1$, $c \subset a_2$. We showed if there is a set c for which $|c| = l-1$ and which is included at least in 3 sets a_ν, then for every $\nu < n$ $c \subset a_\nu$. As a consequence $n \leqq m-l+1$, because there can exist at most as many sets a_ν as the number of distinct elements which are not contained in set c. That is $n \leqq \max(l+1, m-l+1)$, and there is always a system satisfying the equality.

2b. Let $m = 2l-k+1$, $|a_\nu| = l$ $(\nu < n)$ and $k > 1$. Use Theorem 2 for $g = l-k+1$:

$$n\frac{l}{l-k+1} \leqq |\{a_0, \ldots, \hat{a}_n\}^{l-k+1}| = p. \tag{11}$$

If $c \in \{a_0, \ldots, \hat{a}_n\}^{l-k+1}$ then $|[0, m)-c| = l$. Moreover, since $c \subset a_\nu$ for some $\nu < n$, $|a_\nu([0, m)-c)| = |a_\nu - c| = k-1$. Thus $[0, m)-c \notin A$ and the elements of A and the complementary sets of the elements of A^{l-k+1} are distinct, therefore

$$n+p \leqq \binom{2l-k+1}{l}.$$

Hence applying (11)

$$n \leqq \binom{2l-k}{l} = \binom{m-1}{l}.$$

Here equality holds if and only if A is the system of all subsets of $[0, m-1)$ having l elements. Namely, according to Remark 2 of Theorem 2 equality in (11) can hold only if the number of elements is $2l-k$ and A is as specified. In this case equality is trivial.

THEOREM 4. *Let* $2 \leqq k \leqq m$. *If* $A = \{a_0, \ldots, \hat{a}_n\}$ *is a system such that* $a_\mu \neq a_\nu$, $|a_\mu a_\nu| \geqq k$, $a_\nu \subset [0, m)$ $(\mu < \nu < n)$, *then either*

$$k+m = 2v \qquad n \leqq \sum_{i=v}^{m} \binom{m}{i}, \tag{a}$$

or

$$k+m = 2v-1 \qquad n \leqq \binom{m-1}{v-1} + \sum_{i=v}^{m} \binom{m}{i}. \tag{b}$$

Moreover there exists a unique maximal system of sets a such that $a \subset [0, m)$ and $|a| \geqq v$ in case (a), *and in case* (b) *a system of sets of the same property and additionally of all the sets satisfying the conditions*

$$a \subset [0, m-1) \quad \textit{and} \quad |a| = v-1.$$

PROOF. 1. If $1 \leqq k \leqq l \leqq m$ and $A = \{a_0, \ldots, \hat{a}_n\} \in S(k, l, m)$, then $n \leqq \binom{m}{l-k}$. From Theorem 2 for $l-k = g$ follows $n \leqq |A^{l-k}|$. However, $|A^{l-k}| \leqq \binom{m}{l-k}$, that is $n \leqq \binom{m}{l-k}$.

If in addition $l-k < \frac{1}{2}(m-1)$ then $n \leqq \binom{m}{l-k} < \binom{m}{l-k+1}$.

2. If $l < \frac{1}{2}(m+k-1)$ and A is an arbitrary system satisfying the conditions of Theorem 4, then

$$|A_l| + |A_{m-l+k-1}| \leqq \binom{m}{m-l+k-1} = \binom{m}{l-k+1}, \tag{12}$$

and equality can hold only if $|A_l| = 0$ and $A_{m-l+k-1}$ consists of all the sets a such that $|a| = m-l+k-1$.

Proof of (12): If $l < \frac{1}{2}(m+k-1)$, then $l-k < \frac{1}{2}(m-1)$ thus by 1:

$$|A_l| < \binom{m}{l-k+1}.$$

If $|A_l| = 0$, (12) is true. If $0 < |A_l| < \binom{m}{l-k+1}$, we shall show, that

$$|A_{m-l+k-1}| < \binom{m}{m-l+k-1} - |A_l|.$$

Let $c \in (A_l)^{l-k+1}$. Then there exists a number v such that $c \subset a_v$, $|a_v| = l$ and $a_v \in A$, thus $|a_v([0, m) - c)| = |a_v - c| = k-1 < k$, and hence $[0, m) - c \notin A$. Since $|[0, m) - c| = m-l+k-1$, there are $|(A_l)^{l-k+1}|$ sets of cardinal number $m-l+k-1$, which can not be elements of A and $A_{m-l+k-1}$ respectively. We have

$$|A_{m-l+k-1}| \leqq \binom{m}{m-l+k-1} - |(A_l)^{l-k+1}|.$$

To complete our proof we must show, that $|(A_l)^{l-k+1}| > |A_l|$. This trivially follows from Theorem 2. We can use the theorem because of $k \geqq 2$, $(l-k+1)+k \geqq l$ and $l > l-k+1$ and thus the coefficient (2) is larger than 1. Equality can hold only in the case $|A_l| = 0$.

3. $|A_\mu| = 0$ $(\mu < k)$, thus we have to determine the maximum of $|A| = |A_k| + \ldots$ $\ldots + |A_m|$. By 2 the pairs $|A_k| + |A_{m-1}|$, $|A_{k+1}| + |A_{m-2}|$, ... are maximal, if the first term is 0. The last pair is $|A_{\frac{1}{2}(m+k-2)}| + |A_{\frac{1}{2}(m+k)}|$ and here also $l = \frac{1}{2}(m+k-2) < \frac{1}{2}(m+k-1)$. The maximum of $|A_m|$ is 1. This completes the proof in the case (a).

Similarly in case (b) for the maximal system $|A_\mu| = 0$ $\left(\mu < \frac{1}{2}(m+k-1)\right)$ and $|A_m| = 1$. Only the term $|A_{\frac{1}{2}(m+k-1)}|$ remains. In the Remark 2b we have shown that

$|A_{\frac{1}{2}(m+k-1)}| \leqq \binom{m-1}{\frac{1}{2}(m+k-1)} = \binom{m-1}{v-1}$, and equality can hold only if $A_{\frac{1}{2}(m+k-1)}$ is the system of all the sets satisfying the conditions

$$a \subset [0, m) \quad |a| = \tfrac{1}{2}(m+k-1).$$

This system trivially satisfies the conditions. Therefore this is the maximal system as stated.

MATHEMATICAL INSTITUTE,
EÖTVÖS LORÁND UNIVERSITY,
BUDAPEST

(Received 1 August 1963)

Bibliography

[1] P. Erdős, Chao Ko (Szechuan), R. Rado (Reading), Intersection theorems for systems of finite sets. *The Quarterly Journal of Mathematics*, **12** (48), (1961, Oxford).