

# Formalizing set theory in diagonal-free cylindric algebras, searching for the weakest logic with Gödel's incompleteness property.

Draft version!\*

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## Abstract

We show that first-order logic can be translated into a very simple and weak logic, and thus set theory can be formalized in this weak logic. This weak logical system is equivalent to the equational theory of Boolean algebras with three commuting complemented closure operators, i.e., diagonal-free 3-dimensional cylindric algebras. Equivalently, set theory can be formulated in propositional logic with 3 commuting S5 modalities (i.e., in the multimodal logic [S5,S5,S5]). There are many consequences, e.g., free finitely generated  $Df_3$ 's are not atomic and [S5,S5,S5] has Gödel's incompleteness property.

## 1 Introduction

Tarski in 1953 [13, 14] formalized set theory in the theory of relation algebras. Why did he do this? Because the equational theory of relation algebras (RA) corresponds to a logic without individual variables, in other words, to a propositional logic. Tarski got the surprising result that a propositional logic can be strong enough to “express all of mathematics”, to be the arena for mathematics. The classical view before this result was that propositional logics in general were weak in expressive power, decidable, uninteresting in a sense. By using the fact that set theory can be built up in it, Tarski proved that the equational theory of RA is undecidable. This was the first propositional logic shown to be undecidable. This is why the title of the book [15] is “Formalizing set theory without variables”.

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\*This contains full proofs for the two main theorems. We make this preprint available in this draft version because so many people expressed strong interest in the proofs.

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From the above it is clear that replacing RA in Tarski’s result with a “weaker” class of algebras is an improvement of the result and it is worth doing. For more on this see Tarski-Givant [15, pp.89<sub>2</sub> – 90<sup>4</sup> and footnote 17 on p.90], especially the open problem formulated therein.

A result of J. D. Monk says that for every finite  $n$  there is a 3-variable FOL formula which is valid but which can be proved (in FOL) with more than  $n$  variables only. Intuitively this means that during any proof of this formula there are steps when we have to use  $n$  independent data (stored in the  $n$  variables as in  $n$  machine registers). For example, the associativity of relation composition of binary relations can be expressed with 3 variables but 4 variables are needed for any of its proofs.

Tarski’s main idea in [15] is to use pairing functions to form pairs, and so to store two pieces of data in one register. He used this technique to translate usual infinite-variable first-order logic FOL into the three-variable fragment of it. From then on, he used that any three-variable FOL-formula about binary relations can be expressed by an RA-equation, [5, sec 5.3]. He needed two registers for storing the data belonging to a binary relation and he had one more register available for making computations belonging to a proof.

The finite-variable fragment hierarchy of FOL corresponds to cylindric algebras (CA’s). The  $n$ -variable fragment  $\mathcal{L}_n$  of FOL consists of all FOL-formulas which use only the first  $n$  variables. By Monk’s result,  $\mathcal{L}_n$  is essentially incomplete for all  $n \geq 3$ , it cannot have a finite Hilbert-style complete and sound inference system. We get a finite Hilbert style inference system  $\vdash_n$  for  $\mathcal{L}_n$  by restricting a usual complete one for infinite-variable FOL to the first  $n$  variables (see [5, sec. 4.3]). This inference system  $\vdash_n$  belonging to  $\mathcal{L}_n$  expresses  $CA_n$ , it is sound but not complete:  $\vdash_n$  is much weaker than validity  $\models$ .

Relation algebras are halfway between  $CA_3$  and  $CA_4$ , the classes of 3-dimensional and 4-dimensional cylindric algebras, respectively. We sometimes jokingly say that RA is  $CA_{3.8}$ . Why is RA stronger than  $CA_3$ ? Because, the so-called relation algebra reduct of a  $CA_3$  is not necessarily an RA, e.g., associativity of relation composition can fail in the reduct. See [5, sec 5.3], and for more in this line see Németi-Simon [9]. Why is  $CA_4$  stronger than RA? Because not every RA can be obtained, up to isomorphism, as the relation algebra reduct of a  $CA_4$ . However, the same equations are true in RA and in the class of all relation algebra reducts of  $CA_4$ ’s (Maddux’s result, see [5, sec 5.3]). Thus Tarski formulated Set Theory, roughly, in  $CA_4$ , i.e., in  $\mathcal{L}_4$  with  $\vdash_4$ , or in  $\mathcal{L}_3$  with validity  $\models$ .

Németi [6], [7] improved this result by formalizing set theory in  $CA_3$ , i.e.,  $\mathcal{L}_3$  with  $\vdash_3$  in place of validity  $\models$ . The main idea for this improvement was using the pairing functions to store all data always, during every step of a proof, in one register only and so one got two registers to work with in the proofs. In this approach one represents binary relations as unary ones (of pairs). For the “execution” of this idea see sections 3-5 of the present paper.

First-order logic has equality as a built-in relation. One of the uses of equality in FOL is that it can be used to express (simulate) substitutions of variables,

thus to “transfer” content of one variable to the other. The reduct  $SCA_3$  of  $CA_3$  “forgets” equality  $d_{ij}$  but retains substitution in the form of the term-definable operations  $s_j^i$ . The logic belonging to  $SCA_3$  is weaker than 3-variable fragment of FOL. Zsolt Gyenis [4] improved parts of Némethi’s result by using  $SCA_3$  in place of  $CA_3$ .

We get a much weaker logic by forgetting substitutions, too, this is the logic corresponding to  $Df_3$  in which we formalize Set Theory in the present paper. Without equality or substitutions, if one has only binary relations, one cannot really use the third variable for anything; and it is known that the two-variable fragment of FOL is decidable, so it is already too weak for formalizing set theory. Therefore we need at least one ternary relation symbol (or atomic formula) in order to use the third variable, while in the language of set theory we only have one binary relation symbol, the elementhood-relation  $\epsilon$ . Therefore, while in formalizing set theory in the three-variable fragment of FOL (in  $CA_3$ ) we could do with one binary relation symbol, we did not have to change vocabulary during the formalization, in the present equality- and substitution-free case we have to change vocabulary, and we have to pay attention to this new feature of the translation mapping. A key device of our proofs will be a “translation mapping” translating FOL into the equational language of  $Df_3$ , or equivalently into the logic  $\mathcal{L}d_3$  defined in section 2 below.

$Df_3$  is nothing more than Boolean algebras with three commuting complemented closure operators. The only connection between these operators is commutativity. We know that without commutativity the class is too weak for supporting set theory because its equational theory is decidable [7]. We know that two such operators do not suffice, for the same reason. We do not know how much complemented-ness of the closure operators is important for supporting set theory. Several forms of this simple logic, revealing its propositional logic character, are presented in section 2.

What does it mean to formalize set theory? Why set theory? Have we learnt anything about set theory?

## 2 A simple logical system: three-variable logic without equality or substitutions

In this section we define the “target logic”  $\mathcal{L}d_3$  of our translation. We give several different forms for it to give a feeling of its expressive power. After this, we formulate three of our main theorems, all stating unexpected properties of this logic.

The language of our system contains three variable symbols,  $x, y, z$ , one ternary relational symbol  $P$ , and there is one atomic formula, namely  $P(x, y, z)$ . (We note that, e.g., the formula  $P(y, x, z)$  is not available in this language.) The logical connectives are  $\vee, \neg, \exists x, \exists y, \exists z$ . We denote the set of formulas by  $Fmd_3$ . We will use the derived connectives  $\forall, \wedge, \rightarrow, \leftrightarrow$ , too, as abbreviations:  $\forall v\varphi \stackrel{d}{=} \neg\exists v\neg\varphi$ ,  $\varphi \wedge \psi \stackrel{d}{=} \neg(\neg\varphi \vee \neg\psi)$ ,  $\varphi \rightarrow \psi \stackrel{d}{=} \neg\varphi \vee \psi$ ,  $\varphi \leftrightarrow \psi \stackrel{d}{=} (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ .

The proof system  $\vdash^d$  is a Hilbert style one with the following logical axioms and rules.

The logical axioms are the following. Let  $\varphi, \psi \in Fmd_3$  and  $v, w \in \{x, y, z\}$ .

- ((1))  $\varphi$ , if  $\varphi$  is a propositional tautology.
- ((2))  $\forall v(\varphi \rightarrow \psi) \rightarrow (\exists v\varphi \rightarrow \exists v\psi)$ .
- ((3))  $\varphi \rightarrow \exists v\varphi$ .
- ((4))  $\exists v\exists v\varphi \rightarrow \exists v\varphi$ .
- ((5))  $\exists v(\varphi \vee \psi) \leftrightarrow (\exists v\varphi \vee \exists v\psi)$ .
- ((6))  $\exists v\neg\exists v\varphi \rightarrow \neg\exists v\varphi$ .
- ((7))  $\exists v\exists w\varphi \rightarrow \exists w\exists v\varphi$ .

The inference rules are Modus Ponens ((MP), or detachment), and Generalization ((G)).

This proof system is a direct translation of the equational axiom system of  $Df_3$ . Axiom ((2)) is needed for ensuring that the equivalence relation defined on the formula algebra by  $\varphi \equiv \psi \Leftrightarrow \vdash^d \varphi \leftrightarrow \psi$  be a congruence with respect to (w.r.t.) the operation  $\exists v$ . It is congruence w.r.t. the Boolean connectives  $\vee, \neg$  by axiom ((1)). Axiom ((1)) expresses that the formula algebra factorized with  $\equiv$  is a Boolean algebra, axiom ((5)) expresses that the quantifiers  $\exists v$  are operators on this Boolean algebra (i.e., they are additive w.r.t.  $\vee$ ), axioms ((3)),((4)) express that these quantifiers are closure operations, axiom ((6)) expresses that they are complemented closure operators (i.e., the negation of a closed element is closed again). Together with ((5)) they imply that the closed elements form a Boolean algebra, and hence the quantifiers are normal operators (i.e., the Boolean zero is a closed element). Finally, axiom ((7)) expresses that the quantifiers commute with each other.

We define  $\mathcal{L}d_3$  as the logic with formulas  $Fmd_3$  and with proof system  $\vdash^d$ . The logic  $\mathcal{L}d_3$  inherits a natural semantics from first-order logic (FOL). The proof system  $\vdash^d$  is sound with respect to this semantics, but it is not complete. Moreover, there is no finite Hilbert-style inference system which would be complete and sound at the same time w.r.t. this semantics (because the quasi-equational theory of  $RDf_3$  is not finitely axiomatizable, see [5] and [2]).

We note that in the above system, axiom ((6)) can be replaced with the following ((8)):

- ((8))  $\exists v(\varphi \wedge \exists v\psi) \leftrightarrow (\exists v\varphi \wedge \exists v\psi)$ .

In the present paper we will use our logic as introduced above. However, it has several different equivalent forms, each of which has advantages. We review below some of the different forms.

Restricted 3-variable FOL is introduced in [5, Part II, p.157], with proof system  $\vdash_{\overline{=}}$ . If we restrict this system  $\vdash_{\overline{=}}$  to formulas not containing the equality  $=$  then we get a system equivalent to our  $\mathcal{L}d_3$ . Lets call this system *restricted 3-variable FOL without equality*. That is, the formulas are those of restricted 3-variable FOL which contain no equality, and we leave out from the axioms of  $\vdash_{\overline{=}}$  the axioms which contain equality. This way we get a proof system with Modus Ponens and Generalization as deduction rules and with the following

axioms:

- ((V1))  $\varphi$ , if  $\varphi$  is a propositional tautology.
- ((V2))  $\forall v(\varphi \rightarrow \psi) \rightarrow (\forall v\varphi \rightarrow \forall v\psi)$ .
- ((V3))  $\forall v\varphi \rightarrow \varphi$ .
- ((V4))  $\varphi \rightarrow \forall v\varphi$ , if  $v$  does not occur free in  $\varphi$ .

Lets call this<sup>1</sup> *equality-free*  $\vdash_{\overline{r}}$ . Rule ((V4)) in this system essentially uses variables in its using the notion of free variables of a formula. On the other hand, no axiom in  $\vdash^{\underline{d}}$  needs to use the structure of a formula occurring in a rule, it is essentially variable-free. So, an advantage of  $\vdash^{\underline{d}}$  over equality-free  $\vdash_{\overline{r}}$  is that it is more “algebraic”, more like propositional logic. On the other hand, equality-free  $\vdash_{\overline{r}}$  contains fewer axioms (it contains only ((V1))-((V4)) as axioms).

The logic  $\mathcal{L}d_3$  has a neat *modal logic form*: three commuting S5 modalities. This is denoted as [S5,S5,S5], see [3, p.379, lines 15-20]. We recall this logic in a slightly simplified form. The language contains one propositional variable  $p$ , the connectives are  $\vee, \neg, \diamond_1, \diamond_2, \diamond_3$ . We use  $\Box_i \stackrel{d}{=} \neg \diamond_i \neg, \rightarrow, \leftrightarrow$  as derived connectives as before, and the axioms are the following (where  $\varphi, \psi$  are arbitrary formulas of the language and  $i, j \in \{1, 2, 3\}$ ):

- ((B))  $\varphi$ , if  $\varphi$  is a propositional tautology,
- ((K))  $\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$ ,
- ((S5))  $\diamond_i\varphi \rightarrow \Box_i\diamond_i\varphi$ ,
- ((C1))  $\diamond_i\diamond_j\varphi \rightarrow \diamond_j\diamond_i\varphi$ ,
- ((C2))  $\diamond_i\Box_j\varphi \rightarrow \Box_j\diamond_i\varphi$ .

The rules are Modus Ponens and Generalization (or, in other word, Necessitation, i.e.,  $\varphi \vdash \Box_i\varphi$ ). This modal logic is complete w.r.t. the frames consisting of three commuting equivalence relations as accessibility relations for the three modalities.

One can present this logic in yet one different form: *Equational logic* as the background logic, and the defining axioms of  $\mathbf{Df}_3$  as logical axioms. For completeness, we include this form here, too. The language consists of equations  $\tau = \sigma$  where  $\tau, \sigma$  are terms written up from (infinitely many) variables by the use of the function symbols  $+, -, f, g, h$  where  $+$  is binary and the rest are unary. The axioms are the following, where  $x, y, z$  are variables and  $F \in \{f, g, h\}$ :

- ((B1))  $x + y = y + x$ ,
- ((B2))  $x + (y + z) = (x + y) + z$ ,
- ((B3))  $-(-(x + y) + -(x + -y)) = x$ ,
- ((D1))  $x + Fx = Fx$ ,
- ((D2))  $FFx = Fx$ ,

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<sup>1</sup>We note that we also omitted ((V9)) of [5] because  $\forall v$  abbreviates  $\neg\exists v\neg$  in our approach, so ((V9)) is not needed.

- ((D3))  $F(x + y) = Fx + Fy$ ,  
 ((D4))  $F(-Fx) = -Fx$ ,  
 ((D5))  $f gx = g f x$ ,  $f h x = h f x$ ,  $g h x = h g x$ .

The rules are those of the equational logic:

Rules of equivalence:

$$\tau = \tau, \quad \tau = \sigma \vdash \sigma = \tau, \quad \tau = \sigma, \sigma = \rho \vdash \tau = \rho,$$

Rules of congruence:

$$\tau = \sigma, \rho = \delta \vdash -\tau = -\sigma, f\tau = f\sigma, g\tau = g\sigma, h\tau = h\sigma, \tau + \rho = \sigma + \delta,$$

Rule of invariance:

$\tau = \sigma \vdash \tau' = \sigma'$  where  $\tau', \sigma'$  are obtained from  $\tau, \sigma$  by replacing the variables simultaneously with arbitrary terms.

We note that the first three axioms are an axiom system for Boolean algebras, see [5, Problem 1.1, p.245] (this problem was solved affirmatively by a theorem prover).

Having formulated our logic  $\mathcal{L}d_3$  in several different ways, we now formulate some theorems. The first theorem says that this simple logic  $\mathcal{L}d_3$  is strong enough for doing all of mathematics in it. It says that we can do set theory in  $\mathcal{L}d_3$  as follows: in place of formulas  $\varphi$  of set theory we use their “translated” versions  $\text{Tr}(\varphi)$  in  $\mathcal{L}d_3$ , and then we use the proof system  $\vdash^d$  of  $\mathcal{L}d_3$  between the translated formulas in place of the proof system of FOL between the original formulas of set theory. Moreover, for sentences  $\varphi$  in the language of set theory,  $\varphi$  and  $\text{Tr}(\varphi)$  mean the same thing (are equivalent) modulo a “bridge”  $\Delta$  between the two languages. We need this bridge because the language  $\mathcal{L}_\omega$  of set theory contains only one binary relation symbol  $\epsilon$  and equality, and the language  $\mathcal{L}d_3$  contains only one ternary relation symbol  $P$ . When  $f : A \rightarrow B$  is a function and  $X \subseteq A$  then  $f(X) \stackrel{d}{=} \{f(a) : a \in X\}$  denotes the image of  $X$  under this function  $f$ .

**Theorem 2.1.** (Formalizability of set theory in  $\mathcal{L}d_3$ ) *There is a recursive translation function  $\text{Tr}$  from the language  $\mathcal{L}_\omega$  of set theory into  $\mathcal{L}d_3$  for which the following are true for all sentences  $\varphi$  in the language of set theory:*

- (i)  $ZF \models \varphi$  iff  $\text{Tr}(ZF) \vdash^d \text{Tr}(\varphi)$ .  
 (ii)  $ZF + \Delta \models \varphi \leftrightarrow \text{Tr}(\varphi)$ , where  
 $\Delta \stackrel{d}{=} \forall xyz [P(x, y, z) \leftrightarrow (x = y = z \vee \epsilon(x, y))]$ .

Theorem 2.1 is proved in section 5.

The next theorem is a partial completeness theorem for  $\mathcal{L}d_3$ .

**Theorem 2.2.** (Partial completeness theorem for  $\mathcal{L}d_3$ ) *There is a recursive subset  $K \subseteq \text{Fmd}_3$  and there is a recursive function  $\text{tr}$  mapping all FOL-formulas into  $K$  such that the following are true:*

- (i)  $\models \varphi$  iff  $\vdash^d \text{tr}(\varphi)$  for all  $\varphi \in K$ .

(ii)  $\models \varphi$  iff  $\models \text{tr}(\varphi)$  for all FOL sentences  $\varphi$ .

According to the above theorem, the proof system  $\vdash^d$  is complete within  $K$ . But is  $K$  big enough? Yes, we can prove any valid FOL-formula  $\varphi$  by translating it into  $K$  and then proving the translated formula by  $\vdash^d$ . We know that  $\vdash^d$  is not strong enough to prove all valid  $Fmd_3$  formulas (i.e.,  $K$  is necessarily a proper subset of  $Fmd_3$ ), because as stated in the introduction, no finite Hilbert-style axiom system can be sound and complete at the same time for  $\mathcal{L}d_3$ . However, we can formulate each sentence in a slightly different form, namely as  $\text{tr}(\varphi)$  so that this “version” of  $\varphi$  can now be proved by  $\vdash^d$  iff it is valid.

**Theorem 2.3.** (Gödel style incompleteness theorem for  $\mathcal{L}d_3$ ) *There is a formula  $\varphi \in Fmd_3$  such that no consistent recursive extension  $T$  of  $\varphi$  is complete.*

**Discussion 2.1.** In Theorems 2.1-2.3, at least one at least ternary relation symbol  $P$  is needed in the “target-language”  $\mathcal{L}d_3$ , the axiom of commutativity ((7)) is needed in the proof system  $\vdash^d$  (because omitting ((7)) from  $\vdash^d$  results decidability of the so obtained proof system, see [7]). We do not know whether complementedness of the closure operators ((6)) is needed or not. Also, two variables do not suffice because the satisfiability problem of the two-variable fragment of FOL is decidable.  $\square$

### 3 Finding QRA-reducts in $Df_3$

In this section we begin the proof of Theorem 2.1. We show that every  $Df_3$  contains lots of quasi-projective relation algebras (QRA’s) in them. We do this by defining relation algebra type operations in the term language of  $Df_3$  and proving that these operations form QRAs in appropriate relativizations. Since QRAs are representable, this will amount to a “partial” representation theorem for  $Df_3$ ’s, and to “partial” completeness theorem for  $\mathcal{L}d_3$ . We will work in  $\mathcal{L}d_3$  in place of  $Df_3$ .

There will be parameters in the definitions to come. These will be formulas in  $Fmd_3$ , namely  $\delta_{xy}, \delta_{xz}$  with free variables  $\{x, y\}$  and  $\{x, z\}$  respectively, together with two other formulas  $p_0, p_1$  with free variables  $\{x, y\}$ . Thus, if you choose  $\delta_{xy}, \delta_{xz}, p_0, p_1$  with the above specified free variables then you will arrive at a QRA-reduct of any  $Df_3$  corresponding to these. We get the QRA-reduct by assuming some properties of the meanings of these formulas, this will be expressed by a formula  $Ax$ . In section 5 then we will choose these parameters so that they fit set theory, which means that the formula  $Ax$  built up from them is provable in set theory. Intuitively, the formulas  $\delta_{xy}, \delta_{xz}$  stand for equality  $x = y, x = z$  and  $p_0, p_1$  will be arbitrary pairing functions.

So, choose formulas  $\delta_{xy}, \delta_{xz}, p_0, p_1$  with the above specified free variables arbitrarily, they will be parameters of the definitions to come. To simplify notation, we will *not* indicate these parameters.

We now set ourselves to defining the above relation algebra type operations on  $Fmd_3$ . To help readability, we often write just comma in place of conjunction

in formulas, especially when they begin with a quantifier. E.g., we write  $\exists x(\varphi, \psi)$  in place of  $\exists x(\varphi \wedge \psi)$ . Further, **True** denotes a provably true formula, say  $\mathbf{True} \stackrel{d}{=} \delta_{xy} \vee \neg\delta_{xy}$ . First we introduce notation to support the intuitive meaning of the parameters  $\delta_{xy}, \delta_{xz}$  as equality.

**Definition 3.1.** (Simulating equality between variables)

$$\begin{aligned}
x \doteq y &\stackrel{d}{=} \delta_{xy}, \\
x \doteq z &\stackrel{d}{=} \delta_{xz}, \\
y \doteq z &\stackrel{d}{=} \exists x(x \doteq y \wedge x \doteq z), \\
y \doteq x &\stackrel{d}{=} x \doteq y, \\
z \doteq x &\stackrel{d}{=} x \doteq z, \\
z \doteq y &\stackrel{d}{=} y \doteq z, \\
x \doteq x &\stackrel{d}{=} \mathbf{True}, \\
y \doteq y &\stackrel{d}{=} \mathbf{True}, \\
z \doteq z &\stackrel{d}{=} \mathbf{True}.
\end{aligned}$$

**Definition 3.2.** (Simulating substitution with (simulated) equality)

$$\begin{aligned}
\varphi(x, y) &\stackrel{d}{=} \varphi, \\
\varphi(x, z) &\stackrel{d}{=} \exists y(y \doteq z \wedge \varphi), \\
\varphi(y, z) &\stackrel{d}{=} \exists x(x \doteq y \wedge \varphi(x, z)), \\
\varphi(y, x) &\stackrel{d}{=} \exists z(x \doteq z \wedge \varphi(y, z)), \\
\varphi(z, x) &\stackrel{d}{=} \exists y(y \doteq z \wedge \varphi(y, x)), \\
\varphi(z, y) &\stackrel{d}{=} \exists x(x \doteq z \wedge \varphi), \\
\varphi(x, x) &\stackrel{d}{=} \exists y(x \doteq y \wedge \varphi), \\
\varphi(y, y) &\stackrel{d}{=} \exists x(x \doteq y \wedge \varphi), \\
\varphi(z, z) &\stackrel{d}{=} \exists x(x \doteq z \wedge \varphi(x, x)).
\end{aligned}$$

**Remark 3.3.** In FOL,  $\varphi(u, v)$  is semantically equivalent with the formula we get from  $\varphi$  by replacing  $x, y$  with  $u, v$  everywhere simultaneously, when  $\delta_{xy}, \delta_{xz}$  are  $x = y, x = z$  respectively. This is Tarski’s fabulous trick to simulate substitutions.  $\square$

Next we introduce notation supporting intuition about the pairing functions  $p_0, p_1$ . First we define some auxiliary formulas. We will use the notation  $2 = \{0, 1\}$ , to make the text shorter. Let  $2^*$  denote the set of all finite sequences of 0, 1 including the empty sequence  $\langle \rangle$  as well. If  $i, j \in 2^*$  then  $ij$  denotes their “concatenation” usually denoted by  $i \frown j$ , and  $|i|$  denotes the “length” of  $i$ . Further, if  $k \in 2$ , then we write  $k$  instead of  $\langle k \rangle$  for the sequence  $\langle k \rangle$  of length 1. Accordingly,  $00$  denotes the sequence  $\langle 0, 0 \rangle$ .

We are going to define  $Fmd_3$ -formulas  $u_i \doteq v_j$  for  $u, v \in \{x, y, z\}$  and  $i, j \in 2^*$ . The intuitive meaning of  $u_{i_0 \dots i_n} \doteq v_{j_0 \dots j_k}$  is that if  $p_0, p_1$  are partial functions



then  $p_{i_n} \dots p_{i_0} u = p_{j_k} \dots p_{j_0} v$ . As usual in the partial algebra literature, the equality holds if both sides are defined and are equal. E.g., the intuitive meaning of  $x_0 \doteq y_{01}$  is that all of  $p_0x, p_0y, p_1p_0y$  exist and  $p_0x = p_1p_0y$ .

**Definition 3.4.** (Simulating projections)

Let  $\{u, v, w\} = \{x, y, z\}$ ,  $i, j \in 2^*$  and  $k \in 2$ .

$$\begin{aligned} (u_{\langle \rangle} \doteq v_{\langle \rangle}) &\stackrel{d}{=} u \doteq v, \\ (u_k \doteq v_{\langle \rangle}) &\stackrel{d}{=} p_k(u, v), \\ (u_{ik} \doteq v_{\langle \rangle}) &\stackrel{d}{=} \exists w(u_i \doteq w_{\langle \rangle}, p_k(w, v)) \quad \text{if } i \neq \langle \rangle, \\ (u_i \doteq v_j) &\stackrel{d}{=} \exists w(u_i \doteq w_{\langle \rangle}, v_j \doteq w_{\langle \rangle}) \quad \text{if } j \neq \langle \rangle, \\ (x_i \doteq x_j) &\stackrel{d}{=} \exists y(x \doteq y, x_i \doteq y_j), \\ (y_i \doteq y_j) &\stackrel{d}{=} \exists x(x \doteq y, x_i \doteq y_j), \\ (z_i \doteq z_j) &\stackrel{d}{=} \exists x(z_i \doteq x_{\langle \rangle}, z_j \doteq x_{\langle \rangle}). \end{aligned}$$

We will omit the index  $\langle \rangle$  in formulas  $u_i \doteq v_j$ , i.e., we write  $u_i \doteq v$  and  $u \doteq v_i$  for  $u_i \doteq v_{\langle \rangle}$  and  $u_{\langle \rangle} \doteq v_i$  resp. if  $i \in 2^*$ .

So far we did nothing but introduced notation supporting the intuitive meanings of the parameters  $\delta_{xy}, \delta_{xz}, p_0, p_1$  as equality and partial pairing functions. Almost any of the concrete formulas supporting this would do, we only had to fix one of them since our proof system  $\vdash^d$  is very weak, it would not prove equivalence of most of the semantically equivalent forms. Now we write up a statement **Ax** about the parameters using the just introduced notation. Let  $H \stackrel{d}{=} \{i \in 2^*, |i| \leq 3\}$ . Notice that  $H$  is finite.

**Definition 3.5** (pairing axiom **Ax**). We define  $\mathbf{Ax} \in Fmd_3$  to be the conjunction of the union of the following finite sets (A1),..., (A4) of formulas:

$$\begin{aligned} \text{(A1)} &\{u_i \doteq v_j, v_j \doteq w_k \rightarrow u_i \doteq w_k : u, v, w \in \{x, y, z\}, i, j, k \in H\} \\ \text{(A2)} &\{u_i \doteq v_j, u_{ik} \doteq u_{ik} \rightarrow u_{ik} \doteq v_{jk} : u, v \in \{x, y, z\}, ik, jk \in H, k \in 2\} \\ \text{(A3)} &\{u_i \doteq u_i, v_j \doteq v_j \rightarrow \exists w(w_0 \doteq u_i, w_1 \doteq v_j) : u, v, w \in \{x, y, z\}, w \notin \{u, v\}, i, j \in H\} \\ \text{(A4)} &\{\exists w u \doteq w : u, w \in \{x, y, z\}\}. \end{aligned}$$

We are ready to define our relation-algebra type operations on  $Fmd_3$ . They will have the intended meanings on formulas with one free variable  $x$ . Let  $Fmd_3^1$  denote the set of formulas in  $Fmd_3$  with at most one free variable  $x$ .

If we assume uniqueness of pairs (as in **SAx** later) then we can write  $\varphi \wedge \mathbf{pair}$  in place of  $\varphi \odot 1$  in the definition of **Dra**. Then we may use the simpler definition  $\mathbf{Dra} \stackrel{d}{=} \{\varphi \in Fmd_3^1 : \mathbf{Ax} \mid^d \varphi \rightarrow \mathbf{pair}\}$  for **Dra**, in Def.3.6 below.

**Definition 3.6** (relation algebra reduct  $\mathcal{Dra}$  of  $Fmd_3$ ). Let  $\varphi, \psi \in Fmd_3$ .

$$\begin{aligned}
\text{pair} &\stackrel{d}{=} \exists y p_0 \wedge \exists y p_1. \\
\varphi u_i &\stackrel{d}{=} \exists x (x \dot{=} u_i, \varphi) \quad \text{if } u \in \{y, z\} \text{ and } i \in 2^*. \\
\varphi \odot \psi &\stackrel{d}{=} \exists y (\varphi y_0, \psi y_1, x_0 \dot{=} y_{00}, y_{01} \dot{=} y_{10}, y_{11} \dot{=} x_1), \quad \text{see Figure 3,} \\
\varphi^\cup &\stackrel{d}{=} \exists y (\varphi y, y_0 \dot{=} x_1, y_1 \dot{=} x_0), \\
\dot{1} &\stackrel{d}{=} x_0 \dot{=} x_1, \\
\div \varphi &\stackrel{d}{=} \text{pair} \wedge \neg \varphi, \\
\varphi + \psi &\stackrel{d}{=} \varphi \vee \psi. \\
\text{Dra} &\stackrel{d}{=} \{\varphi \in Fmd_3 : \text{Ax} \vdash^d \varphi \leftrightarrow \psi \odot \dot{1} \text{ for some } \psi \in Fmd_3^1\}, \\
\mathfrak{Dra} &\stackrel{d}{=} \langle \text{Dra}, +, \div, \odot, \cup, \dot{1} \rangle.
\end{aligned}$$

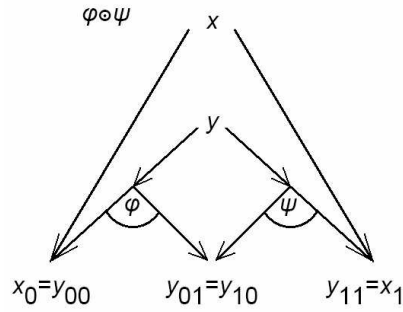


Figure 3. Illustration of  $\varphi \odot \psi$

In the definition of  $\text{Dra}$ , intuitively  $\varphi \odot \dot{1}$  means that we close the meaning of  $\varphi$  under the equivalence relation of “equivalent pairs”. I.e., let us define  $x \sim y \stackrel{d}{=} x_0 \dot{=} y_0, x_1 \dot{=} y_1$ , then  $\varphi \odot \dot{1}$  means the closure of  $\varphi$  under this equivalence relation.

The next theorem is the heart of formalizing set theory in  $Fmd_3$ . Let us define the equivalence relation  $\equiv_T$  on  $Fmd_3$  by

$$\varphi \equiv_T \psi \stackrel{d}{=} T \vdash^d \varphi \leftrightarrow \psi,$$

where  $T \subseteq Fmd_3$ .

**Theorem 3.7.** (i)  $\mathfrak{Dra}$  is an algebra, i.e., the operations  $+, \div, \odot, \cup, \dot{1}$  do not lead out of  $\text{Dra}$ .

(ii)  $\text{Dra} \supseteq \{\varphi \odot \psi : \varphi, \psi \in Fmd_3^1\}$ .

(iii)  $\mathfrak{Dra} / \equiv_{\text{Ax}}$  is a relation algebra.

(iv) The images of the formulas  $x_1 \dot{=} x_{00}$  and  $x_1 \dot{=} x_{01}$  form a quasi-projection pair in  $\mathfrak{Dra} / \equiv_{\text{Ax}}$ .

**Proof.** The proof of the analogous theorem in [7, Thm.9, p.43] goes through with some modifications. We indicate here these modifications.

The “proof explanations” (CA), . . . , (KV) introduced on [7, p.44] can be used in our proof, too, except that we always have to check whether the explanation uses only the Df-part of (CA). If it uses axiom C7 of CA, then we have to give an alternate proof. Let (Df) denote the part of (CA) which does not use C7. We can use (UV) because it can be derived from Ax and (Df). Of course, throughout we have to change  $=$  to  $\doteq$ .

We now go through the proof given in [7, pp.46-64] and indicate the changes needed for our proof. All the statements on pp.48-54 beginning with (A1) and ending with (S13) follow from Ax and (Df). In fact, we could just add these statements to Ax since they do not contain formula-schemes denoting arbitrary formulas (such as, e.g., (0) on p.54 does), and so they amount to finitely many formulas only. Because of this, we do not indicate the changes needed in the proofs of these items.

We have to avoid statement (0) at the end of p.54 by all means, because it uses axiom C7 of (CA) essentially. Fortunately, we do not really use (0) in the proof, changing  $\varphi$  to  $\varphi x$  in some steps will suffice for eliminating (0).

The proof of (2/a) on p.55 has to be modified, and that can be done as follows. Recall from [7] that  $\varphi$  has at most one free variable  $x$ , and  $\varphi u \stackrel{d}{=} \exists x(x \doteq u, \varphi)$  when  $u$  is different from  $x$  while  $\varphi x \stackrel{d}{=} \exists y(y \doteq x, \varphi y)$ . Assume  $x \notin \{u, v\}$ . The proof for  $\varphi u, u \doteq v \mid^d \varphi v$  is as on [7, p.55]. Then

$$\begin{aligned} &\varphi x, x \doteq z \mid^d && \text{by definition of } \varphi x \\ &\exists y(y \doteq x, \varphi y), x \doteq z \mid^d \\ &\exists y(y \doteq x, \varphi y, x \doteq z) \mid^d && \text{by Ax} \\ &\exists y(y \doteq z, \varphi y) \mid^d && \text{by the case when } x \notin \{u, v\} \\ &\exists y(\varphi z) \mid^d \\ &\varphi z. \end{aligned}$$

$$\begin{aligned} &\varphi x, x \doteq y \mid^d && \text{by Ax} \\ &\exists z(z \doteq x, \varphi x, x \doteq y) \mid^d && \text{by Ax} \\ &\exists z(z \doteq y, \varphi x, x \doteq z) \mid^d && \text{by the previous case} \\ &\exists z(z \doteq y, \varphi z) \mid^d && \text{by the case when } x \notin \{u, v\} \\ &\varphi y. \end{aligned}$$

$$\begin{aligned} &\varphi y, y \doteq x \mid^d \\ &\exists y(y \doteq x, \varphi y) \mid^d && \text{by definition of } \varphi x \\ &\varphi x. \end{aligned}$$

$$\begin{aligned} &\varphi z, z \doteq x \mid^d && \text{by Ax} \\ &\exists y(y \doteq z, z \doteq x, \varphi z) \mid^d && \text{by Ax} \\ &\exists y(y \doteq x, z \doteq y, \varphi z) \mid^d && \text{by the case when } x \notin \{u, v\} \\ &\exists y(y \doteq x, \varphi y) \mid^d \\ &\varphi x. \end{aligned}$$

On p.58, in the last line of the proof of (9) we have to write  $\gamma x$  in place of  $\gamma$ . Similarly, on p.59, in lines 7 and 8 we have to change  $\psi$  to  $\psi x$  and then we can cross out reference to (0). That's all! **QED**

## 4 Finding CA-reducts in $Df_3$

Simon [12] defines a  $CA_n$ -reduct in every QRA, for all  $n \in \omega$ , and also proves that these reducts are representable. We will use, in this paper, the  $CA_3$ -reduct of our QRA defined in the previous section,  $\mathcal{Dra}/\equiv_{SAx}$ .

We will use the following stronger form of  $SAx$ , just for convenience:

**Definition 4.1.** [strong paring axioms  $SAx$ ] We define  $SAx \in Fmd_3$  to be the conjunction of the union of the finite sets (A1),..., (A4) of Def.3.5 together with the following:

$$(A5) \{ x_0 \dot{=} y_0, x_1 \dot{=} y_1 \rightarrow x \dot{=} y, \quad x_0 \dot{=} x_0 \leftrightarrow x_1 \dot{=} x_1 \}.$$

The formulas in (A5) express that pairs are unique, and that  $p_0$  is defined exactly when  $p_1$  is defined. We use (A5) for convenience only, this way formulas will be shorter. We could omit (A5) on the expense that formulas will be longer and more complicated. If we assume  $SAx$  then  $\varphi \wedge \text{pair} \in \text{Dra}$  for all  $\varphi \in Fmd_3^1$ .

Every QRA has a  $CA_3$ -reduct, which is representable, see Simon [12]. The following definition is recalling this  $CA_3$ -reduct from [12] in our special case of  $\mathcal{Dra}/\equiv_{SAx}$ . The definition below is simpler than in [12] because we will assume uniqueness of pairs in  $SAx$ , which is not assumed in [12]. We will use the abbreviation

$$\varphi x_i \stackrel{d}{=} \exists y(y \dot{=} x_i, \varphi y), \text{ when } \varphi \in Fmd_3^1 \text{ and } i \in 2^*.$$

Beware:  $\vdash^d \varphi \leftrightarrow \varphi x$  usually is not true for  $\varphi \in Fmd_3^1$ . (For the definition of  $\varphi y_j$  see Def.3.6.)

**Definition 4.2.** [cylindric reduct  $\mathcal{Dca}$  of  $Fmd_3$ ]

Triplet  $\stackrel{d}{=} x_{11} \dot{=} x_{11}$ , see Figure,

$$(0) \stackrel{d}{=} 0, \quad (1) \stackrel{d}{=} 10, \quad (2) \stackrel{d}{=} 11, \quad \text{and for all } i, j < 3 \text{ and } \varphi, \psi \in Fmd_3^1$$

$$T_i \stackrel{d}{=} \text{Triplet}x_0 \wedge \text{Triplet}x_1 \wedge \bigwedge \{x_{0(j)} \dot{=} x_{1(j)} : i \neq j < 3\},$$

$$c_i \varphi \stackrel{d}{=} \varphi \odot T_i,$$

$$d_{ij} \stackrel{d}{=} \text{Triplet}x_1 \wedge x_{1(i)} \dot{=} x_{1(j)},$$

$$- \varphi \stackrel{d}{=} \text{Triplet}x_1 \wedge \neg \varphi,$$

$$\varphi + \psi \stackrel{d}{=} \varphi \vee \psi,$$

$$\text{Dca} \stackrel{d}{=} \{ \varphi \in Fmd_3 : SAx \vdash^d \varphi \leftrightarrow \psi x_1 \wedge \text{Triplet}x_1 \text{ for some } \psi \in Fmd_3^1 \},$$

$$\mathcal{Dca} \stackrel{d}{=} \langle \text{Dca}, +, -, c_i, d_{ij} : i, j < 3 \rangle.$$

**Theorem 4.3.** *The set  $\text{Dca}$  is closed under the operations defined in Def.4.2 and the algebra  $\mathcal{Dca}/\equiv_{SAx}$  is a representable  $CA_3$ .*

**Proof.** We show that the algebra  $\mathfrak{Dca}/\equiv_{\mathbf{SA}_X}$  is the  $\mathbf{CA}_3$ -reduct of the quasi-projective relation algebra  $\mathfrak{Dra}/\equiv_{\mathbf{SA}_X}$ , as defined in [12]. Let  $\mathfrak{p} \stackrel{d}{=} x_1 \dot{=} x_{00}$ ,  $\mathfrak{q} \stackrel{d}{=} x_1 \dot{=} x_{01}$ . In the following we will omit referring to  $\equiv_{\mathbf{SA}_X}$ , so we will look at  $\mathfrak{p}$  as an element of  $\mathfrak{Dra}/\equiv_{\mathbf{SA}_X}$ , while only  $\mathfrak{p}/\equiv_{\mathbf{SA}_X}$  is such. Let

$$\begin{aligned} \mathfrak{e} &\stackrel{d}{=} \text{Triplet}x_0 \wedge x_0 \dot{=} x_1, \\ \pi_{(i)} &\stackrel{d}{=} \text{Triplet}x_0 \wedge x_1 \dot{=} x_{0(i)}, \\ \chi_{(i)} &\stackrel{d}{=} \text{Triplet}x_0 \wedge \text{Triplet}x_1 \wedge x_{0(i)} \dot{=} x_{1(i)}. \end{aligned}$$

Now, one can show that  $\mathfrak{e}$  is the same as  $\in^{(3)}$  in [12, Def.3.1], i.e.,

$$\mathbf{SA}_X \vdash^d \mathfrak{e} \leftrightarrow (\mathfrak{q} \odot \mathfrak{q} \odot \mathfrak{q}^{\cup} \odot \mathfrak{q}^{\cup} \wedge \mathfrak{1}).$$

Similarly, one can show that  $\pi_{(i)}, \chi_{(i)}$  are the same as the ones in [12], and  $\text{Triplet}$  is the same as  $1^{(3)}$  in [12, Def.3.1], e.g.,

$$\begin{aligned} \mathbf{SA}_X \vdash^d \pi_{(1)} &\leftrightarrow (\mathfrak{e} \odot \mathfrak{q} \odot \mathfrak{p}), \\ \mathbf{SA}_X \vdash^d \chi_{(1)} &\leftrightarrow (\pi_{(1)} \odot \pi_{(1)}^{\cup}), \\ \mathbf{SA}_X \vdash^d \text{Triplet} &\leftrightarrow (\text{pair} \odot \mathfrak{e}). \end{aligned}$$

From this one can show, similarly to the above, that [12]'s  $\mathfrak{t}_i, \mathfrak{d}_{ij}, \mathfrak{t}$  are the same as our  $T_i, \mathfrak{d}_{ij}$  and  $\mathfrak{e}$  in Def.4.2 and above. Thus, our reduct  $\mathfrak{Dca}$  is the same as the 3-reduct defined in [12], and then we can use [12, Thm.3.2, Thm.5.2]. **QED**

## 5 Formalizing set theory in $\mathcal{L}d_3$

The 3-variable restricted fragment  $\mathcal{L}_3$  of  $\mathcal{L}_\omega$  is defined as follows. Language: three variable symbols,  $x, y, z$ , one binary relational symbol  $\epsilon$ , so there is one atomic formula, namely  $\epsilon(x, y)$ . Logical connectives:  $\vee, \neg, \exists x, \exists y, \exists z$  and  $u = v$  for  $u, v \in \{x, y, z\}$ . We denote the set of formulas by  $Fm_3$ . Derived connectives are  $\forall, \forall, \rightarrow, \leftrightarrow$ . The proof system  $\vdash_{-3}$  of  $\mathcal{L}_3$  is a Hilbert style one defined in [5, Part II, p.157]. The word-algebra of  $\mathcal{L}_3$  is denoted as  $\mathfrak{Fm}_3$ , it is the absolutely free  $\mathbf{CA}_3$ -type algebra generated by the formula  $\epsilon(x, y)$ .

In this section we prove Theorem 2.1 stated in section 2. As a first step, we will define a translation function  $\mathfrak{h}$  from  $Fm_3$  to  $Fmd_3$ . This will be analogous to the one defined in [7], but here a novelty is that the vocabulary of  $Fm_3$  is different (disjoint) from that of  $Fmd_3$ , and we will have to pay attention to this difference. Namely,  $Fm_3$  contains one binary relation symbol  $\epsilon$  and equality  $u = v$  for  $u, v \in \{x, y, z\}$  while  $Fmd_3$  contains one ternary relation symbol  $P$  and does not contain equality. The formula  $\Delta$  introduced in section 2 bridges the difference between the two languages. Namely,  $\Delta$  is stated in a language which contains both  $Fm_3$  and  $Fmd_3$  and  $\Delta$  is a definition of  $P$  in  $Fm_3$ , so it

leads from  $Fm_3$  to  $Fmd_3$ . But  $\Delta$  is a two-way bridge, because of the following. Let

$$\Delta' \stackrel{d}{=} [\epsilon(x, y) \leftrightarrow \forall z P(x, y, z)] \wedge [x = y = z \leftrightarrow (P(x, y, z) \wedge \neg \forall z P(x, y, z))] .$$

Then  $\Delta'$  provides a definition of equality  $=$  and  $\epsilon$  in  $Fmd_3$ , and thus it is a bridge leading from  $Fmd_3$  to  $Fm_3$ . Now, the two definitions are equivalent in non-trivial models, namely

$$\exists xy(x \neq y) \models \Delta \leftrightarrow \Delta' .$$

As usual, we begin with definitions. To start, we work in  $Fm_3$ . Recall the concrete definitions of  $p_0, p_1$ , and  $\pi$  from [7, p.71, p.35]. Define

$$\begin{aligned} \pi^+ &\stackrel{d}{=} \pi \wedge \\ &\quad \forall xy[\exists z(\mathbf{p}_0((x, z)), \mathbf{p}_0((y, z))), \exists z(\mathbf{p}_1((x, z)), \mathbf{p}_1((y, z))) \rightarrow x = y] \wedge \\ &\quad \forall x(\exists y\mathbf{p}_0(x, y) \leftrightarrow \exists y\mathbf{p}_1(x, y)) \wedge \exists xy(x \neq y). \end{aligned}$$

Then  $p_0, p_1, \pi, \pi^+$  are formulas of  $Fm_3$ . We note that the notation  $\varphi((u, v))$  in [7] is the same as our  $\varphi(u, v)$  introduced in Def.3.2, except that instead of  $x \doteq y$  etc. the notation  $\varphi((u, v))$  uses real equality  $x = y$  etc. available in  $Fm_3$ .

Next, we work in  $Fmd_3$  and we get the versions of these formulas that the definition  $\Delta$  of  $P$  provides. We will write out the details. First we fix the parameters  $\delta_{xy}, \delta_{xz}, p_0, p_1$  occurring in the formula  $\text{SAx}$  of  $Fmd_3$ .

**Definition 5.1.** [fixing the parameters  $\delta_{xy}, \delta_{xz}$  of  $Fmd_3$ ]

$$\begin{aligned} E &\stackrel{d}{=} \forall z P(x, y, z), \\ D &\stackrel{d}{=} P(x, y, z) \wedge \neg E, \\ \delta_{xy} &\stackrel{d}{=} \exists z D, \\ \delta_{xz} &\stackrel{d}{=} \exists y D. \end{aligned}$$

The next definition fixes the parameters  $p_0, p_1$  of  $Fmd_3$ . To distinguish them from their  $Fm_3$ -versions, we will denote them by  $\mathbf{p}_0, \mathbf{p}_1$ . The definition below is a repetition of the definition of the corresponding formulas  $p_0, p_1$  on p.71 of [7] such that we write  $E$  and the above defined concrete  $\delta_{xy}, \delta_{xz}$  in place of  $\epsilon$  and  $x = y, x = z$ . We will use the notation introduced in section 2 and we will use notation to support set theoretic intuition. Thus, if we introduce a formula  $\varphi$  denoted as, say,  $x \doteq \{y\}$ , then  $u \doteq \{v\}$  denotes the formula  $\varphi(u, v)$  (see Definitions 3.2, 3.1). Below, “op” abbreviates “ordered pair” (to distinguish it from the formula pair defined earlier.).

**Definition 5.2.** [fixing the parameters  $\mathbf{p}_0, \mathbf{p}_1$  of  $Fmd_3$ ]

$$\begin{aligned} u\varepsilon v &\stackrel{d}{=} E(u, v), \quad \text{for } u, v \in \{x, y, z\}, \\ x \doteq \{y\} &\stackrel{d}{=} \forall z(z\varepsilon x \leftrightarrow z \doteq y), \\ \{x\}\varepsilon y &\stackrel{d}{=} \exists z(z \doteq \{x\}, z\varepsilon y), \end{aligned}$$

$$\begin{aligned}
x \dot{=} \{\{y\}\} &\stackrel{d}{=} \exists z(z \dot{=} \{y\}, x \dot{=} \{z\}), \\
x\varepsilon \cup y &\stackrel{d}{=} \exists z(x\varepsilon z, z\varepsilon y), \\
\text{op}(x) &\stackrel{d}{=} \exists y\forall z(\{z\}\varepsilon x \leftrightarrow y \dot{=} z) \wedge \\
&\quad \forall yz[(y\varepsilon \cup x, \neg\{y\}\varepsilon x, z\varepsilon \cup x, \neg\{z\}\varepsilon x) \rightarrow y \dot{=} z], \forall y\exists z(y\varepsilon x \rightarrow z\varepsilon y), \\
p_0 &\stackrel{d}{=} \text{op}(x) \wedge \{y\}\varepsilon x, \\
p_1 &\stackrel{d}{=} \text{op}(x) \wedge [x \dot{=} \{\{y\}\} \vee (y\varepsilon \cup x, \neg\{y\}\varepsilon x)].
\end{aligned}$$

It is not hard to check that

$$(1) \quad \Delta, \exists xy(x \neq y) \models \pi \leftrightarrow \text{Ax}, \pi^+ \leftrightarrow \text{SAx}.$$

We are ready to define our translation mapping  $h$  from  $Fm_3$  to  $Fmd_3$ . For a formula  $\varphi \in Fmd_3^2$  and  $i, j \in 2^*$  we define

$$\varphi(x_i, x_j) \stackrel{d}{=} \exists yz(y \dot{=} x_i, z \dot{=} x_j, \varphi(y, z)).$$

**Definition 5.3.** [translation mapping  $h$ ]

- (i)  $h' : Fm_3 \rightarrow Fmd_3^1$  is defined by the following:  
 $h'$  is a homomorphism from  $\mathfrak{Fm}_3$  into  $\mathfrak{Dca}$  such that  
 $h'(\varepsilon(x, y)) \stackrel{d}{=} E(x_{1(0)}, x_{1(1)})$ .
- (ii)  $\text{SAx}^* \stackrel{d}{=} \text{SAx} \wedge \forall x(\text{Triplet}x_1 \rightarrow h'(\pi^+))$ .
- (iii) We define the mapping  $h : Fm_3 \rightarrow Fmd_3^0$  as

$$h(\varphi) \stackrel{d}{=} \forall x(\text{SAx}^* \wedge \text{Triplet}x_1 \rightarrow h'(\varphi)).$$

We say that a translation function  $f$  is *Boolean preserving w.r.t.*  $\stackrel{d}{|}$  iff for all sentences  $\varphi, \psi \in Fmd_3$  we have that  $\stackrel{d}{|} f(\varphi \wedge \psi) \leftrightarrow (f(\varphi) \wedge f(\psi))$  and  $\stackrel{d}{|} f(\varphi \rightarrow \psi) \rightarrow (f(\varphi) \rightarrow f(\psi))$ .

**Theorem 5.4.** *Let  $\varphi$  be a sentence and  $T$  be a set of sentences of  $Fm_3$ . Then the following (i)-(iii) are true.*

- (i)  $T \cup \{\pi^+\} \models \varphi \Leftrightarrow h(T) \stackrel{d}{|} h\varphi$ .
- (ii)  $\pi^+ \wedge \Delta \models \varphi \leftrightarrow h(\varphi)$ .
- (iii)  $h$  is Boolean preserving and  $\text{SAx}^* \stackrel{d}{|} h(\neg\varphi) \rightarrow \neg h(\varphi)$ .

**Proof.** Proof of (ii): Let  $\mathfrak{M} = \langle M, \varepsilon^{\mathfrak{M}}, P^{\mathfrak{M}} \rangle$  be a model of  $\pi^+ \wedge \Delta$ . Let  $V \stackrel{d}{=} \{x, y, z\}$  and  $\text{Val} \stackrel{d}{=} {}^V M$ , the set of evaluations of the variables in  $\mathfrak{M}$ . Now, by  $\mathfrak{M} \models \pi^+ \wedge \Delta$  we have that  $p_i$  and  $p_i$  have the same meanings in  $\mathfrak{M}$ , and they form pairing functions. Then one can show by induction that

$$(2) \quad \mathfrak{M} \models (u_i = v_j)[k] \quad \text{iff} \quad \mathfrak{M} \models (u_i \dot{=} v_j)[k] \quad \text{iff} \quad k(u)_i = k(v)_j \text{ in } \mathfrak{M}.$$

We say that  $a \in M$  is a *triplet* iff  $a_{11}$  is defined. Then  $a_0, a_{10}$  are also defined by  $\mathfrak{M} \models \pi^+$ . If  $a$  is a triplet, then we assign an evaluation  $\text{val}(a) \in \text{Val}$  to  $a$  such that  $\text{val}(a)$  assigns to  $x, y, z$  the elements  $a_{(0)}, a_{(1)}, a_{(2)}$  respectively. We will prove by induction the following statement: For all  $\varphi \in Fm_3$  and  $k \in \text{Val}$  we have

$$(3) \quad \mathfrak{M} \models h'(\varphi)[k] \quad \text{iff} \quad (k(x)_1 \text{ is a triplet and } \mathfrak{M} \models \varphi[\text{val}(k(x)_1)]).$$

From (3) we get the following:

$$(4) \quad \mathfrak{M} \models \varphi \leftrightarrow \forall x(\text{Triplet}x_1 \rightarrow h'\varphi) \quad \text{for all} \quad \varphi \in Fm_3^0.$$

Indeed, assume  $\mathfrak{M} \models \varphi$ , then  $\mathfrak{M} \models \varphi[k]$  for all  $k \in \text{Val}$ . We show  $\mathfrak{M} \models \forall x(\text{Triplet}x_1 \rightarrow h'\varphi)$ . Indeed, let  $k \in \text{Val}$  be such that  $k(x)_1$  is a triplet. By  $\mathfrak{M} \models \varphi$  then  $\mathfrak{M} \models \varphi[\text{val}(k(x)_1)]$ , by (3) then  $\mathfrak{M} \models h'\varphi[k]$  and we are done. Assume next  $\mathfrak{M} \not\models \varphi$ , then  $\mathfrak{M} \not\models \varphi[k]$  for all  $k \in \text{Val}$  because  $\varphi$  is a sentence (i.e., does not contain free variables). We show  $\mathfrak{M} \not\models \forall x(\text{Triplet}x_1 \rightarrow h'\varphi)$ . Indeed, let  $a \in M$  be such that  $a_{111}$  is defined. Such an  $a \in M$  exists by  $\mathfrak{M} \models \pi$ . Let  $k \in \text{Val}$  be such that  $k(x) = a$ . Then  $k(x)_1$  is a triplet and  $\mathfrak{M} \not\models \varphi[\text{val}(k(x)_1)]$ , so  $\mathfrak{M} \not\models h'\varphi[k]$  by (3), so  $\mathfrak{M} \not\models (\text{Triplet}x_1 \rightarrow h'\varphi)[k]$ , so  $\mathfrak{M} \not\models \forall x(\text{Triplet}x_1 \rightarrow h'\varphi)$ . This shows that (4) indeed holds.

From (4) and  $\mathfrak{M} \models \pi^+ \wedge \Delta$  then we have  $\mathfrak{M} \models \text{SAx}^*$ . This together with (4) and the definition of  $h$  finishes the proof of (ii).

Proof of (iii): The proofs of (4) and (6) in [7, p.73], which prove that  $\kappa$  is Boolean-preserving w.r.t.  $\vdash_3$ , work for showing that  $h$  is Boolean preserving w.r.t.  $\vdash^a$ , because  $h$  has the same “structure” as  $\kappa$ . Similarly, the proof of (5) in [7, p.73] is good for proving the second statement of the present (iii).

Proof of (i): First we prove (i) for the special case when  $T$  is the empty set. Let  $\varphi$  be a sentence of  $Fm_3$ , we want to prove  $\vdash^a h(\varphi)$  implies  $\pi^+ \not\models \varphi$ . So, assume that  $\vdash^a h\varphi$ , i.e.,  $\vdash^a \forall x(\text{SAx}^* \wedge \text{Triplet}x_1 \rightarrow h'\varphi)$ . Then  $\text{SAx} \vdash^a \forall x(\text{Triplet}x_1 \wedge h'(\pi^+) \rightarrow h'(\varphi))$ , i.e.,

$$(5) \quad \text{SAx} \vdash^a h'(\pi^+ \rightarrow \varphi).$$

Recall that  $h'$  is a homomorphism from  $\mathfrak{Fm}_3$  to  $\mathfrak{Dca}/\equiv_{\text{SAx}}$ , and the latter is a representable  $\text{CA}_3$ . Let  $\psi \stackrel{d}{=} \pi^+ \rightarrow \varphi$ . By (5) we have that the image of  $\psi$  is not 1 under  $h'$ , therefore there is a homomorphism  $g$  from  $\mathfrak{Dca}/\equiv_{\text{SAx}}$  to a cylindric set algebra  $\mathfrak{C}$  such that the image of  $h'\psi$  under  $g$  is not 1. Let  $f \stackrel{d}{=} gh'$ , then

$$(6) \quad f : \mathfrak{Fm}_3 \rightarrow \mathfrak{C} \quad \text{and} \quad f(\psi) \neq 1.$$

Let  $U$  be the base set of  $\mathfrak{C}$ , let  $R \stackrel{d}{=} \{\langle s_0, s_1 \rangle : s \in f(\epsilon(x, y))\}$  and define the model  $\mathfrak{M}$  as  $\langle U, R \rangle$ . Then for all  $\varphi \in Fm_3$  and  $s \in U^3$  we have that  $\mathfrak{M} \models \varphi[s]$  iff  $s \in f(\varphi)$ . Thus  $\mathfrak{M} \not\models \psi$  by (6), and we are done with showing  $\pi^+ \not\models \varphi$ .



In the other direction, we have to show that  $\vdash^d \mathbf{h}\varphi$  implies  $\pi^+ \models \varphi$ . By soundness of the proof system  $\vdash^d$  we have  $\models \mathbf{h}\varphi$ , then by (ii) we have  $\pi^+ \wedge \Delta \models \varphi$ . Since  $P$  does not occur in  $\varphi$  and in  $\pi^+$ , this means that  $\pi^+ \models \varphi$  and we are done.

Next, assume that  $T$  is a set of sentences of  $Fm_3$ . We want to show that  $T \cup \{\pi^+\} \models \varphi$  iff  $\mathbf{h}(T) \vdash^d \mathbf{h}(\varphi)$ . Now,

$T \cup \{\pi^+\} \models \varphi$  iff (by compactness of  $Fm_3$ )  
 $T_0 \cup \{\pi^+\} \models \varphi$  for some finite  $T_0 \subseteq T$ , iff  
 $\pi^+ \models \bigwedge T_0 \rightarrow \varphi$  for some finite  $T_0 \subseteq T$ , iff (by first part of (i))  
 $\vdash^d \mathbf{h}(\bigwedge T_0 \rightarrow \varphi)$  for some finite  $T_0 \subseteq T$ . Then, by Boolean preserving of  $\mathbf{h}$   
 $\vdash^d \bigwedge \mathbf{h}(T_0) \rightarrow \mathbf{h}(\varphi)$ , and so by Modus Ponens we get  
 $\mathbf{h}(T_0) \vdash^d \mathbf{h}(\varphi)$ . Conversely, from this we get by the soundness of  $\vdash^d$  that

$$(7) \quad \mathbf{h}(T_0) \models \mathbf{h}(\varphi).$$

From here on we have to deal with the difference between the two languages.

Let  $\mathfrak{M}$  be an arbitrary model of  $\mathcal{L}_3$ , so  $\mathfrak{M}$  contains one binary relation, say  $\mathfrak{M} = \langle M, \epsilon^{\mathfrak{M}} \rangle$ . Define  $P^{\mathfrak{M}}$  according to the definition  $\Delta$ , i.e.,  $P^{\mathfrak{M}} \stackrel{d}{=} \{ \langle a, a, a \rangle : a \in M \} \cup \{ \langle a, b, c \rangle \in M \times M \times M : \langle a, b \rangle \in \epsilon^{\mathfrak{M}} \}$ . Let  $\mathfrak{M}^+ \stackrel{d}{=} \langle M, \epsilon^{\mathfrak{M}}, P^{\mathfrak{M}} \rangle$  be the expansion of  $\mathfrak{M}$  with this new relation, and let  $\mathfrak{M}^- \stackrel{d}{=} \langle M, P^{\mathfrak{M}} \rangle$  be the reduct of this expansion to the language  $\mathcal{L}_3$ .

Assume now  $\mathfrak{M} \models T_0 \cup \{\pi^+\}$ . Then  
 $\mathfrak{M}^+ \models T_0 \cup \{\pi^+ \wedge \Delta\}$ . Thus by (ii) we have that  
 $\mathfrak{M}^+ \models \mathbf{h}(T_0)$ , then  
 $\mathfrak{M}^- \models \mathbf{h}(\varphi)$  by (7). Thus  
 $\mathfrak{M}^+ \models \mathbf{h}\varphi \wedge \pi^+ \wedge \Delta$ , so by (ii)  
 $\mathfrak{M}^+ \models \varphi$ , and so  
 $\mathfrak{M} \models \varphi$  and thus, since  $\mathfrak{M}$  was an arbitrary model of  $\mathcal{L}_3$   
 $T_0 \cup \{\pi^+\} \models \varphi$  and we are done. **QED**

We are almost done with proving Thm.2.1, all we have to do is to use a connection between  $\mathcal{L}_\omega$  and  $\mathcal{L}_3$  established in [7], [6].

**Proof of Thm.2.1:** By [7, Lemma 3.1, p.35], or by [6, Lemma 2.2, Remark 2.4, p.25, p.30] there is a recursive function  $f : Fm_\omega^2 \rightarrow Fm_3^2$  for which

$$(8) \quad \pi \models f(\varphi) \leftrightarrow \varphi, \quad f(\neg\varphi) = \neg f(\varphi), \quad f(\varphi \vee \psi) = f(\varphi) \vee f(\psi),$$

for all sentences  $\varphi, \psi$  of  $\mathcal{L}_\omega$ . Take such an  $f'$  and extend it to  $Fm_\omega$  by letting  $f(\varphi) \stackrel{d}{=} f'(\varphi')$  where  $\varphi'$  is the universal closure of  $\varphi$  (if  $n$  is the smallest number such that the free variables of  $\varphi$  are among  $v_0, \dots, v_n$ , then  $\varphi'$  is  $\forall v_0 \dots \forall v_n \varphi$ ). Then  $f$  has the same properties as  $f'$  and it is defined on the whole of  $Fm_\omega$ , not only on  $Fm_\omega^2$ . Define  $\text{Tr} : Fm_\omega \rightarrow Fm_3$  by

$$\text{Tr}(\varphi) \stackrel{d}{=} \mathbf{h}(f(\varphi))$$

for all  $\varphi \in Fm_\omega$ . We show that this translation function satisfies the requirements of Thm.2.1. First,  $\text{Tr}$  is recursive because both  $f$  and  $h$  are such. We defined the parameters  $p_0, p_1$  so that

$$(9) \quad ZF \models \pi^+$$

holds. Thus  $ZF \models ZF + \pi^+ \models \varphi \leftrightarrow f(\varphi)$  by the chosen properties of  $f$ , and then  $ZF + \Delta \models ZF + \pi^+ \wedge \Delta \models \varphi \leftrightarrow hf\varphi$  by Thm.5.4(ii), so  $ZF + \Delta \models \varphi \leftrightarrow \text{Tr}\varphi$  for all sentences  $\varphi$  of  $L_\omega$ . This is Thm.2.1(ii). To prove Thm.2.1(i), first we show

$$(10) \quad ZF \models \varphi \Leftrightarrow f(ZF) + \pi^+ \models f(\varphi).$$

Indeed, assume  $ZF \models \varphi$  and let  $\mathfrak{M} \models f(ZF) + \pi^+$ . Then  $\mathfrak{M} \models ZF$  by (8), so  $\mathfrak{M} \models \varphi + \pi^+$  by our assumption  $ZF \models \varphi$ , so  $\mathfrak{M} \models f(\varphi)$  by (8). This shows  $f(ZF) + \pi^+ \models f(\varphi)$ . Conversely, assume now the latter, and we want to prove  $ZF \models \varphi$ . Let  $\mathfrak{M} \models ZF$ , then  $\mathfrak{M} \models ZF + \pi^+$  by (9), thus  $\mathfrak{M} \models f(ZF) + \pi^+$  by (8), then  $\mathfrak{M} \models f(\varphi) + \pi^+$  by our assumption, and then  $\mathfrak{M} \models \varphi$  by (8) again. We have shown that (10) holds.

By combining this (10) with Thm.5.4(i) we get  $ZF \models \varphi$  iff  $\text{Tr}(ZF) \stackrel{d}{\models} \text{Tr}\varphi$  which is Thm.2.1(i).

Later, in section 6, we will also need the following:

$$(11) \quad \text{Tr is Boolean preserving} \quad \text{and} \quad \text{SAX}^* \stackrel{d}{\models} \text{Tr}(\neg\varphi) \rightarrow \neg\text{Tr}\varphi.$$

Indeed, this follows from Thm.5.4(iii) and from (8): Let  $\varphi, \psi \in Fm_\omega^0$ . Then  
 $\stackrel{d}{\models} \text{Tr}(\varphi \wedge \psi)$ , iff by the definition of  $\text{Tr}$   
 $\stackrel{d}{\models} hf(\varphi \wedge \psi)$ , iff by (8)  
 $\stackrel{d}{\models} h(f\varphi \wedge f\psi)$ , iff by Thm.5.4(iii)  
 $\stackrel{d}{\models} hf\varphi \wedge hf\psi$ , iff by the definition of  $\text{Tr}$   
 $\stackrel{d}{\models} \text{Tr}\varphi \wedge \text{Tr}\psi$ .

Similarly,

$\stackrel{d}{\models} \text{Tr}(\varphi \rightarrow \psi)$ , implies by the definition of  $\text{Tr}$   
 $\stackrel{d}{\models} hf(\varphi \rightarrow \psi)$ , implies by (8)  
 $\stackrel{d}{\models} h(f\varphi \rightarrow f\psi)$ , implies by Thm.5.4(iii)  
 $\stackrel{d}{\models} hf\varphi \rightarrow hf\psi$ , implies by the definition of  $\text{Tr}$   
 $\stackrel{d}{\models} \text{Tr}\varphi \rightarrow \text{Tr}\psi$ .

Finally,  $\text{SAX}^* \stackrel{d}{\models} h(\neg f\varphi) \rightarrow \neg h(f\varphi)$ , by Thm.5.4(iii), so  $\text{SAX}^* \stackrel{d}{\models} h(f(\neg\varphi)) \rightarrow \neg h(f\varphi)$  by (8), and then  $\text{SAX}^* \stackrel{d}{\models} \text{Tr}(\neg\varphi) \rightarrow \neg\text{Tr}(\varphi)$ , as was to be shown.

**QED(Thm.2.1)**

## 6 Free algebras

In this section we prove that the one-generated free  $\text{Df}_3$  is not atomic.

**Theorem 6.1.** *The one-generated free  $\text{Df}_3$  is not atomic.*

**Proof.** It is enough to show that the zero-dimensional part of the free  $Df_3$  is not atomic by [5, 1.10.3(i)]:

$$(12) \quad \exists \delta \exists \tau_1 Df_3 \text{ is not atomic implies that } \exists \tau_1 Df_3 \text{ is not atomic.}$$

Let us define  $\equiv$  as  $\equiv_{\emptyset}$ , i.e.

$$\varphi \equiv \psi \quad \text{iff} \quad \vdash^d \varphi \leftrightarrow \psi.$$

Let  $\mathfrak{Fm}d_3$  and  $\mathfrak{Fm}d_3^0$  denote the word-algebra of  $Fmd_3$  and the word-algebra of sentences of  $Fmd_3$ , respectively (the latter under the operations of  $\vee, \neg$ ). It is easy to show that

$$(13) \quad \mathfrak{F}\tau_1 Df_3 \text{ is isomorphic to } \mathfrak{Fm}d_3 / \equiv \text{ and}$$

$$(14) \quad \exists \delta \exists \tau_1 Df_3 \text{ is isomorphic to } \mathfrak{Fm}d_3^0 / \equiv .$$

There is a non-separable formula  $\lambda \in Fm_{\omega}^0$  which is consistent with  $\pi^+$  (with our concrete pairing formulas), by [7, pp.69-71] (or equivalently by [6, Lemma 2.7, p.34]). Define

$$\psi \stackrel{d}{=} SAx^* \wedge Tr\lambda.$$

Then  $\psi \in Fmd_3^0$ . We will show that there is no atom below  $\psi / \equiv$  and the latter is nonzero in  $Fmd_3^0 / \equiv$ . Assume the contrary, i.e., that

$$(15) \quad \delta / \equiv \text{ is an atom below } \psi / \equiv$$

and we will derive a contradiction. Let

$$T \stackrel{d}{=} \{\varphi \in Fm_{\omega}^0 : \vdash^d \delta \rightarrow Tr\varphi\}.$$

Then  $T \subseteq Fm_{\omega}^0$ . We will show that  $T$  is recursive and it separates the consequences of  $\lambda$  from the sentences refutable from  $\lambda$  which contradicts the choice of  $\lambda$ , namely that  $\lambda$  is inseparable. From now on let  $\varphi \in Fm_{\omega}^0$  be arbitrary. Now,  $\delta / \equiv$  being an atom implies

$$(16) \quad \vdash^d \delta \rightarrow Tr\varphi \Leftrightarrow \vdash^d \delta \rightarrow \neg Tr\varphi.$$

From (16) we get that both  $T$  and the complement of  $T$  are recursively enumerable, so  $T$  is recursive. Next we show

$$(17) \quad \lambda \models \varphi \text{ implies } \varphi \in T.$$

Indeed, assume that  $\lambda \models \varphi$ . Then  $\models \lambda \rightarrow \varphi$ . Then, in particular,  $\pi^+ \models \lambda \rightarrow \varphi$ , and so  $\vdash^d Tr(\lambda \rightarrow \varphi)$  by Thm.2.1(i). Then  $\vdash^d Tr(\lambda) \rightarrow Tr\varphi$  by (11). By Modus Ponens then  $SAx^* + Tr(\lambda) \vdash^d Tr\varphi$ , i.e.,  $\psi \vdash^d Tr\varphi$ . By (15) we have

$$(18) \quad \vdash^d \delta \rightarrow \psi$$

so we have  $\vdash^d \delta \rightarrow Tr\varphi$ , i.e.,  $\varphi \in T$  as was desired. Next we show

$$(19) \quad \lambda \models \neg\varphi \text{ implies } \varphi \notin T.$$

Indeed, assume  $\lambda \models \neg\varphi$ . Then  $(\neg\varphi) \in T$  by the previous case (17), and this means  $\vdash^a \delta \rightarrow \text{Tr}(\neg\varphi)$ . Now, by (18) and the definition of  $\psi$  we have  $\vdash^a \delta \rightarrow \text{SAx}^*$ , and thus by  $\vdash^a \delta \rightarrow \text{Tr}(\neg\varphi)$  and (11)  $\vdash^a \delta \rightarrow \neg\text{Tr}(\varphi)$ . Thus by (16) we get  $\vdash^{a/} \delta \rightarrow \text{Tr}\varphi$ , by the definition of  $T$  then  $\varphi \notin T$ .

By the above we have shown that there is no atom below  $\psi/\equiv$ . It remains to show that the latter is nonzero. This follows from the fact that  $\lambda \wedge \pi^+$  has a model. Let  $\mathfrak{M}$  be such that  $\mathfrak{M} \models \lambda \wedge \pi^+$ . Expand this model with  $P^{\mathfrak{M}}$  so that  $\mathfrak{M}^+ \models \Delta$  also. Then  $\mathfrak{M}^+ \models \text{Tr}(\lambda)$  by Thm.2.1(ii), and also  $\mathfrak{M}^+ \models \text{SAx}^*$  by  $\mathfrak{M}^+ \models \pi^+ \wedge \Delta$ . Thus  $\mathfrak{M}^+ \models \psi$ , and so  $\mathfrak{M}^- \models \psi$  where  $\mathfrak{M}^-$  is the reduct of  $\mathfrak{M}^+$  to the language of  $\psi$ . **QED**

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