

Nicholas Vlamis - MAP - Dec. 1, 2008

S1) Suppose $e_1, e_2, e_3 \in H$ is an ONB in H and P_1, P_2 and Q are the orthogonal projections onto $\mathbb{C}e_1, \mathbb{C}e_2$ and $\mathbb{C}((2+i)e_1 + 2e_2)$, respectively. Suppose $p : P_r(H) \rightarrow [0, 1]$ is a probability law for which $p(P_1) = p(P_2) = 1/2$. Find all possible values of $p(Q)$.

Solution Since $\dim(H) = 3$, by Gleason's Theorem we know that all probability laws are given by density operators, therefore let $p(\cdot) = \text{Tr}(\rho \cdot)$ for some density operator ρ . Also, let $|\psi_0\rangle = (2+i)e_1 + 2e_2$, then define $|\psi\rangle = \frac{|\psi_0\rangle}{\| |\psi_0\rangle \|} = \frac{1}{3} \begin{bmatrix} 2+i \\ 2 \\ 0 \end{bmatrix}$. Therefore, we can represent Q as follows:

$$Q = |\psi\rangle\langle\psi| = \frac{1}{9} \begin{bmatrix} 5 & 4+2i & 0 \\ 4-2i & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now let us investigate ρ . Notice that $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and since $p(P_1) = \text{Tr}(\rho P_1) = \rho_{11} = 1/2$ and $p(P_2) = \text{Tr}(\rho P_2) = \rho_{22} = 1/2$, we know that $\rho_{11} = \rho_{22} = 1/2$, by simple calculation. Also, since ρ is a density operator we know that $\text{Tr}(\rho) = \rho_{11} + \rho_{22} + \rho_{33} = 1$, but we have already concluded that $\rho_{11} + \rho_{22} = 1$, which implies that $\rho_{33} = 0$. Since ρ is a density operator, we know that it is positive. Notice that $\rho_{33} = \langle e_3, \rho e_3 \rangle = 0$, but this can only happen if $\rho e_3 = 0$, by the positivity of ρ . Therefore, both the column and row of ρ containing ρ_{33} must consist of only zeros, since each entry consists of the term ρe_3 , since ρ is self adjoint and the matrix representation of ρ considered is with respect to an ONB (i.e. $\rho_{ij} = \langle e_i, \rho e_j \rangle$).

Thus, at this point we can realize that $\rho = \begin{bmatrix} 1/2 & z & 0 \\ \bar{z} & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for some $z \in \mathbb{C}$. Now, since ρ is

positive, its eigenvalues must be ≥ 0 . Therefore, looking at $\det(\rho - \lambda I) = 0$, we get the equation $\lambda^2 - \lambda + (1/4 - |z|) = 0$. By looking at the solutions of this binomial we find $\lambda = 1/2 \pm |z|$, which implies $|z| \leq 1/2$. Denoting $z = a + ib$ with $a, b \in \mathbb{R}$, we see that $a^2 + b^2 \leq 1/4$. Now, $p(Q) = \text{Tr}(\rho Q) = \frac{1}{9}(5/2 + (4+i2)\bar{z} + (4-i2)z + 2) = \frac{1}{9}(9/2 + 4(z + \bar{z}) + i2(\bar{z} - z)) = \frac{1}{2} + \frac{4}{9}(2a + b)$. It is clear that the maximum and minimum value then occurs at $z = 1/2$ and $z = -1/2$, respectively. Therefore, $p(Q) = \text{Tr}(\rho Q) \in [\frac{1}{18}, \frac{17}{18}]$.