1 VOA with unitary structure

1.1 Grading and stress-energy tensor

Let V be a VA with vacuum vector Ω and infinitesimal generator of translation T. A diagonalizable operator H on V satisfying

$$[H, Y(a,z)] = zY'(a,z) + Y(Ha,z) \tag{1}$$

for all $a \in V$ is called a **conformal Hamiltonian**. (Note that as a direct consequence of the above we have $H\Omega = 0$ and $[H,T] = T$.) A Vertex Algebra with a (fixed) conformal Hamiltonian H is usually referred as an **"H-graded" VA**.

An element $a \in V$ is called **homogeneous**, if it is an eigenvector of H. In this case the corresponding eigenvalue d_a is called the **conformal weight** of a and we shall set $a_n \equiv a_{(n+d_a-1)}$ ($n \in \mathbb{Z} - d_a$). With this convention, the field corresponding to a can be expressed as $Y(a,z) = \sum_{n\in\mathbb{Z}-d_a} z^{-n-d_a} a_n$. In general, $a \in V$ is a sum of homogeneous elements, and the operator a_n is defined by linearity. The commodity of using a_n rather than $a_{(n)}$ lies in its simple commutation relation with H; namely we have $[H, a_n] = -na_n$.

Motivated by physics, we shall be mainly interested by H-gradings satisfying the following two extra properties.

- **"Positivity of energy"**: All eigenvalues of H are nonnegative reals.
- **"Uniqueness of the vacuum"**: $\dim(\mathrm{Ker}(H)) = 1$; i.e. up to scalars the only eigenvector of H with eigenvalue 0 ("zero energy vector") is the vacuum.

It is a well-known phenomena that infinitesimal generators of geometrical symmetry transformations in a QFT model can be identified with such quantities as the (total) energy, (total) momentum, etc. In turn, these are integrals of local density quantities like the energy-density and so in general they can be expressed as moments of the stress-energy tensor.

Let us return to our H-graded VA. By the motivation that was indicated, we shall look for a homogenous element $\nu \in V$ of conformal weight 2 such that with the corresponding field denoted by $T(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$, we have

$$H = \oint_{S^1} T(z) z^2 \frac{dz}{2\pi i z} \equiv L_0 \quad \text{and} \quad T = \oint_{S^1} T(z) z \frac{dz}{2\pi i z} \equiv L_{-1}. \tag{2}$$

Having such a field, we shall say that it is **"stress-energy like"**. We shall now show that the Fourier modes of such a field automatically satisfy Virasoro relations. (In the framework of Wightmann axioms, this fact is called the Lüscher-Mack theorem; see e.g. [??].) For this reason, ν is often called a **Virasoro element**.

Proposition 1.1. *Let V be a VA with H-grading satisfying positivity of energy and uniqueness of vacuum, and assume that $T(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ is a stress-energy like field. Then for every $n, m \in \mathbb{Z}$*

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{-n,m}\mathbf{1}$$

where c is the scalar multiple with which $2L_2L_{-2}\Omega = c\Omega$.

Proof. As $HL_2\nu = 0$, by the uniqueness of the vacuum there exists a scalar multiple c such that $2L_2\nu = c\Omega$. Then by the *commutator formula* (see e.g. (?.?.??) in [??]) we get that $[L_n, L_m] = \sum_{j=0}^{\infty} \binom{n+j}{1} (L_{j-1}\nu)_{(n+m+2-j)} =$

$$(T\nu)_{(m+n+2)} + \binom{n+1}{1} 2\nu_{(m+n+1)} + \binom{n+1}{2} (L_1\nu)_{(m+n)} + \binom{n+1}{3} \frac{c}{2}\Omega_{(m+n-1)} \quad (3)$$

where we have used that by the positivity of energy $L_{j-1}\nu = 0$ for every $j > 3$, and that for $j = 0, 1, 2, 3$ we have the substitutions $L_{-1} = T$, $L_0\nu = H\nu = 2\nu$ and $L_2\nu = \frac{c}{2}\Omega$. As $(T\nu)_{(k)} = -k\nu_{(k-1)} = -kL_{k-2}$, and $\Omega_{(k)} = \delta_{-1,k}\mathbf{1}$, the above formula we may further simplify obtaining

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{-n,m}\mathbf{1} + \binom{n+1}{2} a_{(m+n)} \quad (4)$$

where we have set $a \equiv L_1\nu$. But using in the above equation that the commutator is anti-symmetric, one can easily deduce that a must be zero and thus finish the proof. □

In general — even after fixing an H-grading — there can be several stress-energy like fields. For example, if $T(z)$ is such a field, then so is $T(z) + Y(Ta, z)$ where $a \in V$ is any element of conformal weight 1 satisfying $a_0 \equiv a_{(0)} = 0$ (e.g. the current in the $U(1)$ model). In fact, we shall shortly see that this is the *only* kind of freedom.

We have not yet talked about invariant bilinear forms and unitary structure (we shall do so in a later subsection). However, let us remark in advance that they will be defined with respect to a *fixed* stress-energy field. In this respect, the next proposition will ensure that at most there can be one such field that can be physically interpreted as stress-energy tensor, i.e. such that it allows the existence of unitary structure or the existence of even just a nondegenerate invariant bilinear (Prop. 1.5). This field will be called *the* **stress-energy tensor**.

Proposition 1.2. *Let V be a VA with H-grading satisfying positivity of energy and uniqueness of vacuum, and assume that $T(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ and $\tilde{T}(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}\tilde{L}_n$ are stress-energy like fields. Then*

$$\tilde{T}(z) = T(z) + Y(Ta, z)$$

for some $a \in V$ of conformal weight 1.

Proof. Copying the argument of the last proposition one can easily find that

$$[L_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{\tilde{s}}{12}(n^3-n)\delta_{-n,m}\mathbf{1} + \binom{n+1}{2}\tilde{b}_{(m+n)} \quad (5)$$

where $\tilde{b} \equiv L_1 L_{-2}\Omega$ and $\tilde{s}$ is the scalar multiple so that $2L_2\tilde{L}_{-2}\Omega = \tilde{s}\Omega$. However, inverting the role of the two fields and introducing b and s similarly to $\tilde{b}$ and $\tilde{s}$ we also have that

$$\begin{aligned}[L_n, \tilde{L}_m] &= -[\tilde{L}_m, L_n] \\ &= (n-m)L_{n+m} + \frac{s}{12}(n^3-n)\delta_{-n,m}\mathbf{1} - \binom{m+1}{2} b_{(m+n)}. \end{aligned} \quad (6)$$

Substituting $n \equiv 1, m \equiv -3$ and confronting the two equations we get that $4\tilde{L}_{-2} + \tilde{b}_{(-2)} = 4L_{-2} - 3b_{(-2)}$ and hence that $\tilde{T}(z) = T(z) + Y(Ta, z)$ where $a \equiv (3b + \tilde{b})/4$. □

Let us fix a stress-energy field $T(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ in an H-graded VA satisfying positivity of energy and uniqueness of vacuum. An element $a \in V$ of conformal weight d_a and the corresponding field are called **(quasi) primary**, iff ($L_1 a = 0$) $L_n a = 0$ for all $n > 0$. This property is equivalent with the commutation relation $[L_k, a_n] = ((d_a - 1)k - n)a_{k+n}$ for all $k, n \in \mathbb{Z}$ (in the quasi primary case: for all $k = 0, \pm 1$ and $n \in \mathbb{Z}$).

If there exists a nondegenerate bilinear form $(\cdot,\cdot)$ on V such that $(L_n a, b) = (a, L_{-n}b)$ for all $a, b \in V$ and $n \in \mathbb{Z}$, then the representation of the Virasoro algebra given by the Fourier modes of $T(z)$ is a direct sum of lowest energy irreducible representations. It follows that in this case every field is a (finite) linear combination of quasi primary fields and derivatives of quasi primary fields. (In general, this is not true!)

1.2 VOA and invariant bilinear forms

One finds various (slightly) different definitions for the notion of Vertex Operator Algebra (VOA). It is important to fix the sense in which this notion is going to be used. Recall that we denote the vacuum by Ω.

An H-graded VA with a fixed stress-energy field $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ is called a **Vertex Operator Algebra** iff

(1) the eigenvalues of $H = L_0$ are integers,
(2) $V_n \equiv \mathrm{Ker}(H - n\mathbf{1})$ $(n \in \mathbb{Z})$ are finite dimensional,
(3) *positivity of energy*: $V_n = \{0\}$ for $n < 0$,
(4) *uniqueness of vacuum*: $V_0 = \mathbb{C}\Omega$.

Note that to require (4) is somewhat less frequent in the literature[1]. Note also, that as we have mentioned, here we only work with Vertex Algebras satisfying the usual locality axiom rather than *super*-locality, and thus in agreement with the *spin-statistic theorem*, we have required the eigenvalues of L_0 to be integers rather than half-integers. Finally, let us remark that in our setting, as we have seen in the last subsection, it is a *consequence* (and not a requirement) that the operators L_n $(n \in \mathbb{Z})$ follow Virasoro commutation relation with a certain central charge $c \in \mathbb{C}$.

To introduce the notion of invariant bilinear forms, first we shall talk about the (restricted) dual V' of V. As a graded vector space it is defined as

$$V' = \bigoplus_{n=0}^{\infty} V_n^* \tag{7}$$

i.e. it is the algebraic direct sum of the duals V_n^*. The point is that V' can be naturally endowed with a V-module structure. Namely for any $n \in \mathbb{Z}$, we define the n-product (without parenthesis) of $a \in V_k$ and $v' \in V'$ by the formula

$$a_n \cdot v' \equiv (-1)^k \sum_{j=0}^{\infty} \frac{1}{j!} v' \circ (L_1^j a)_{-n} \tag{8}$$

where both sides are viewed as a functions from V to $\mathbb{C}$.

It is easy to see that the above formula is "good" in the sense that the right-hand side is still in the restricted dual. What is less evident, that this product rule is indeed an *action* of V on V'. However, it is true; see e.g. [**?**, Section 5.2]. In some sense, this is the "explanation" of the factors appearing in the formula of definition: they are needed in order to get an action. Note that it is "natural" or "canonical" only if V is considered with a *fixed* stress-energy field $T(z)$. To put it in another way: the action depends on the choice of $T(z)$ (in particular, on L_1).

A ($\mathbb{C}$-valued) bilinear form $(\cdot,\cdot)$ on V is called **invariant**, iff $(a,\cdot)$ is in the *restricted* dual V' for every $a \in V$, and the map given by the formula

$$a \mapsto (a,\cdot) \in V' \tag{9}$$

[1] Here we have done so to have more similarity with the axioms of local conformal nets. However, read the remark after Prop. 1.4 explaining the connection between simplicity, uniqueness of vacuum and existence of nondegenerate invariant bilinear forms.

is a module homomorphism. Note that "invariance" depends on the choice of the stress-energy field (as so does the module structure on V').

Looking at equation (8), it is an easy exercise to show that if $(\cdot,\cdot)$ is a bilinear form on V, then for every $b, c \in V$, $n \in \mathbb{Z}$ and homogenous element $a \in V_k$ we have

$$(a_n b, c) = (-1)^k \sum_{j=0}^{\infty} \frac{1}{j!} (b, (L_1^j a)_{-n} c). \tag{10}$$

This is also true vice versa. Suppose we have a bilinear form satisfying the above equality. Then, as $L_1 \nu = 0$ (where $\nu \equiv L_{-2}\Omega$), we have that

$$(L_n a, b) = (a, L_{-n} b) \tag{11}$$

for all $a, b \in V$ and $n \in \mathbb{Z}$. In particular, L_0 is $(\cdot,\cdot)$-symmetric and it follows that if $a \in V_k$ then $(a, \cdot) \in V_k^* \subset V'$ and so $(a, \cdot)$ is in the restricted dual for every $a \in V$. Then by a trivial check $a \mapsto (a, \cdot)$ is a module isomorphism and hence this form is invariant. We remark that from (10) also follows that such forms are always symmetric; see [**?**].

Let now $a, b \in V_n$. Then by the uniqueness of vacuum $(L_1^j a)_k b$ is parallel to Ω, and thus it follows that

$$(a, b)\Omega = (a_{-k}\Omega, b)\Omega = (\Omega, \Omega)(-1)^k \sum_{j=0}^{\infty} (L_1^j a)_k b \tag{12}$$

showing that such a form, up to to normalization, is uniquely determined by the algebraic structure. Actually, taking account of the uniqueness of vacuum, by [**?**, Theorem 1], such a (nonzero) form exists iff $L_1 V_1 = \{0\}$.

The invariant bilinear form $(\cdot,\cdot)$ is **normalized**, iff $(\Omega, \Omega) = 1$. Let us now recollect the explained properties of invariant bilinear forms (adding also one that was not yet mentioned but should be trivial by what was said.)

Proposition 1.3 ([?, ?]). *Let V be a VOA with corresponding stress-energy field $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$. Then a ($\mathbb{C}$-valued) bilinear form $(\cdot,\cdot)$ is invariant if and only if it satisfies equation (10). Moreover, with $(\cdot,\cdot)$ standing always for such a form, we have that*

- *it is* **symmetric***: $(a, b) = (b, a)$ for all $a, b \in V$,*
- *$(L_n a, b) = (a, L_{-n} b)$ for all $a, b \in V$ and $n \in \mathbb{Z}$,*
- *a nonzero $(\cdot,\cdot)$* **exists** *if and only if $L_1 V_1 = \{0\}$ and in this case with normalization $(\Omega, \Omega) = 1$ it is* **unique***,*
- *there exists a* **nondegenerate** *$(\cdot,\cdot)$ if and only if $V \simeq V'$ as V modules.*

We have not yet defined unitary structure, however, let us mention in advance, that — as scalar product are positive definite — it will require the existence of a nondegenerate invariant bilinear form, or equivalently, to have $V \simeq V'$. Therefore we shall now have a closer look at this case.

Proposition 1.4. *Let V be a VOA with stress-energy $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ and suppose that there exists a nonzero invariant bilinear form $(\cdot,\cdot)$. Then $(\cdot,\cdot)$ is nondegenerate if and only if V is simple.*

Proof. Set $I \equiv \{a \in V | (a, \cdot) = 0\}$; then clearly it is an ideal of V which does not contain Ω (otherwise by (12) the form would be zero). Hence if V is simple, then it follows that $I = \{0\}$ showing that $(\cdot,\cdot)$ is nondegenerate.

Vice versa, suppose that $(\cdot,\cdot)$ is nondegenerate, and let $I \subset V$ be an ideal. As $L_0 I \subset I$, one has that $I = \sum_{n=0}^{\infty} I_n$ where $I_n \equiv I \cap V_n \subset V_n$. Moreover, by (10) we have that $I^{\perp} \equiv \{a \in V | \forall b \in I : (a, b) = 0\} = \sum_{n=0}^{\infty} I^{\perp} \cap V_n$ is still an ideal of V. Of course, if $\Omega \in I$ then $I = V$. On the other hand, if $\Omega \notin I$ then by the uniqueness of the vacuum $I_0 = \{0\}$ and hence $\Omega \in I^{\perp}$ and thus $I^{\perp} = V$, which — as $(\cdot,\cdot)$ in nondegenerate — implies that $I = \{0\}$. Thus V is simple as it has no proper ideals. □

Remark. Invariant bilinear forms may be introduced without requiring uniqueness of vacuum. Then for example $\Omega \in I^{\perp}$ would not follow from $\Omega \notin I$, and instead of the above theorem, we should prove the following: assuming $(\cdot,\cdot)$ to be nondegenerate, the vacuum is unique iff V is simple. This would be in complete similarity with the treatment of [**?**], where local nets are considered without uniqueness of vacuum, and then it is shown that uniqueness of vacuum is equivalent with irreducibility.

Recall that in the definition of VOA structure, requirements (1,2,3) and (4) regard the H-grading only, and not the particular choice of the stress-energy field. The existence of a nondegenerate invariant bilinear form however, may depend exactly on such a choice.

Proposition 1.5. *Let V be an H-graded VA satisfying (1,2,3,4). Then there exists at most one stress-energy field that allows the existence of a nondegenerate invariant bilinear form.*

Proof. Suppose $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ is a stress-energy field that makes the nondegenerate bilinear form $(\cdot,\cdot)$ to be invariant on V. By Prop. 1.2, every other stress-energy is of the form $\tilde{T}(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-2}\tilde{L}_n = T(z) + Y(Ta, z)$ for some $a \in V_1$.

As $(\cdot,\cdot)$ is nonzero (recall the theorem cited by us in Prop. 1.3), we have that $L_1 V_1 = \{0\}$. On the other hand, as we have assumed nondegeneracy,

$\exists b \in V_1 : (a, b) \neq 0$. Then

$$(\Omega, \tilde{L}_1 b) = (\Omega, (L_1 - 2a_1)b) = 2(a, b) \neq 0 \tag{13}$$

and thus $\tilde{L}_1 V_1 \neq \{0\}$. So $T(z)$ is in fact the only stress-energy field with which there exists a nonzero invariant bilinear form. $\square$

This uniqueness is important to us, as in particular it implies that it is preserved by automorphisms. An **automorphism** (**anti-automorphism**) of a Vertex Algebra V is an $\alpha : V \to V$ linear (conjugate-linear) bijection preserving the (n)-product; i.e. such that $\alpha(a_{(n)}b) = \alpha(a)_{(n)}\alpha(b)$ for every $a, b \in V$ and $n \in \mathbb{Z}$. It follows automatically that it preserves the vacuum: $\alpha(\Omega) = \Omega$, and it commutes with the generator of infinitesimal translations: $\alpha(Ta) = T\alpha(a)$ for every $a \in V$. If V is given an H-grading, we shall say that α is **preserves the H-grading** (or: that it is an **automorphism / anti-automorphism of the H-graded VA**) iff $\alpha(Ha) = H\alpha(a)$ for every $a \in V$, or equivalently, iff $\alpha(V_n) = V_n$ for every eigenvalue $n \in \mathrm{Sp}(H)$ (where of course $V_n \equiv \mathrm{Ker}(H - n\mathbf{1})$).

Corollary 1.6. *Let V be a VOA with sterss-energy $T(z) \equiv Y(\nu, z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ having a nondegenerate invariant bilinear form $(\cdot,\cdot)$. Then for an automorphism (anti-automorphism) α of V, the following three affirmations are equivalent.*

(i) *α preserves the H-grading.*

(ii) *α preserves $(\cdot,\cdot)$. That is: $(\alpha(a), \alpha(b)) = (a, b)$ (or: $(\alpha(a), \alpha(b)) = \overline{(a, b)}$ in the anti-automorphism case) for all $a, b \in V$*

(iii) *$\alpha(\nu) = \nu$ and thus $\alpha(L_n a) = L_n \alpha(a)$ for every $a \in V$ and $n \in \mathbb{Z}$.*

Proof. (i) $\Rightarrow$ (iii) follows from the previous uniqueness result. (iii) $\Rightarrow$ (ii): as α commutes with L_1, it is easy to see that $(\alpha(\cdot), \alpha(\cdot))$ (or: $\overline{(\alpha(a), \alpha(b))}$ in the anti-automorphism case) is an invariant bilinear form. But as $\alpha(\Omega) = \Omega$, this new invariant form is normalized in the same way as $(\cdot,\cdot)$, and hence it coincides with it. (ii) $\Rightarrow$ (i): if α preserves $(\cdot,\cdot)$ then also its inverse does so, and as α^{-1} also commutes with $T \equiv L_{-1}$, we have

$$(a, L_1\alpha(b)) = (Ta, \alpha(b)) = (T\alpha^{-1}(a), b) = (\alpha^{-1}(a), L_1 b) = (a, \alpha(L_1 b)) \tag{14}$$

for all $a, b \in V$. Thus by the nondegeneracy of $(\cdot,\cdot)$ it follows that $L_1\alpha(b) = \alpha(L_1 b)$; i.e. that α commutes with L_1. But then it also commutes with $H \equiv L_0 = \frac{1}{2}[L_1, L_{-1}]$. $\square$

1.3 Two equivalent definitions of unitary structure

In this subsection we shall consider two definitions for unitary structure and then show their equivalence. One of them is more natural from the mathematical point of view, as it will be based on the concept of invariant bilinear forms. The other one may be considered more natural from the physical point of view and it has a direct counterpart in the Wightmann framework of Quantum Fields.

Let V be a VOA with stress-energy $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$ having a normalized invariant bilinear form $(\cdot,\cdot)$. A scalar product (i.e. a positive definite $\mathbb{C}$-valued form, which is antilinear in its first, linear in its second variable) $\langle\cdot,\cdot\rangle$ is said to be an **invariant (normalized) scalar product** iff there exists a grading preserving anti-automorphism Θ of V such that

$$\langle\Theta\cdot,\cdot\rangle = (\cdot,\cdot). \tag{15}$$

We may reformulate this in terms of the map $a \mapsto \langle a,\cdot\rangle \in V'$. Of course, unlike with invariant bilinear forms, it cannot be a homeomorphism (as it is not linear but conjugate-linear), but it may be an anti-homomorphism of V modules, and in fact it is not hard to see, that our requirement is exactly this.

Note that for the existence of an invariant scalar product, $(\cdot,\cdot)$ must necessarily be nondegenerate, since a scalar product is always nondegenerate. Note also that the anti-automorphism Θ is in fact uniquely determined by the above equation. This (unique) anti-automorphism determined by an invariant scalar product $\langle\cdot,\cdot\rangle$, is called the **PCT operator** associated to $\langle\cdot,\cdot\rangle$.

Proposition 1.7. *Let V be a VOA with stress-energy $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$ having an invariant scalar product $\langle\cdot,\cdot\rangle$, and let Θ be the associated PCT operator. Then*

- Θ *is an involution:* $\Theta^2 = \mathbf{1}$,
- Θ *is anti-unitary:* $\overline{\langle\Theta\cdot,\Theta\cdot\rangle} = \langle\cdot,\cdot\rangle$.

Proof. A scalar product — by definition — is (conjugate) symmetric, and we also know that an invariant bilinear forms is always symmetric. So let $a, b \in V$, then using Corollary 1.6 we have that

$$(a,\Theta^2 b) = \langle\Theta a,\Theta^2 b\rangle = \overline{\langle\Theta^2 b,\Theta a\rangle} = \overline{(\Theta b,\Theta a)} = (b,a) = (a,b) \tag{16}$$

and hence by the nondegeneracy of $(\cdot,\cdot)$ it follows that $\Theta^2 = \mathbf{1}$. Moreover, now using also that Θ is an involution, we have

$$\overline{\langle\Theta a,\Theta b\rangle} = \overline{(\Theta^2 a,\Theta b)} = (\Theta^{-1} a, b) = \langle a, b\rangle \tag{17}$$

showing that Θ is indeed anti-unitary. $\square$

By Corollary 1.6 the H-preserving Θ automatically commutes with L_n, and these operators satisfy equation (11). Thus we can draw the following easy conclusion.

Corollary 1.8. *Let V be a VOA with stress-energy $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ having an invariant scalar product $\langle\cdot,\cdot\rangle$. Then $T(z)$ is* **hermitian***; i.e.*

$$\langle a, L_n b\rangle = \langle L_{-n}a, b\rangle \tag{18}$$

for all $a, b \in V$ and $n \in \mathbb{Z}$.

By definition, a **unitary VOA** is a VOA with a (fixed) invariant scalar product. Let us now see a different approach to unitary structure.

Having a scalar product permits us to define the adjoint of fields. In general, changing the scalar product changes the adjoint. In agreement with the Wightmann axioms, our basic requirement of unitarity should be the following: *the scalar product must be such that the adjoint of every local field remains local.* We shall now formulate this requirement and then show its equality with our former requirement on unitarity, thus receiving a second (equivalent) definition of unitary structure.

Let V be a VOA with stress-energy field $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ and let $\langle\cdot,\cdot\rangle$ be a scalar product on V making $T(z)$ hermitian in the sense of equation (18). Then in particular L_0 is symmetric and the finite dimensional subspaces $V_n \equiv \mathrm{Ker}(L_0 - n\mathbf{1})$, $(n \in \mathbb{Z})$ are pairwise orthogonal.

In general, if $A \in \mathrm{End}(W)$ where W is a finite dimensional space with a given scalar product, then there exists a unique operator $A^+ \in \mathrm{End}(W)$ such that $\langle u, Aw\rangle = \langle A^+u, w\rangle$ for all $u, w \in W$. If W is infinite dimensional, then such an operator A^+ does not necessarily exists, but if it exists, it is still unique. In this case we shall call A^+ the **formal adjoint** of A. The reader is reminded that the (true) adjoint — which we denote by a star rather than a cross — is defined in the von Neumann sense and in general its domain $\mathcal{D} \subset \overline{W}$ is different from that of A; i.e. from W.

Lemma 1.9. *For any $a \in V$ and $n \in \mathbb{Z}$ the formal adjoint $a_n^+ \in \mathrm{End}(V)$ exists. Moreover, for any $b \in V$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $a_{-n}^+ b = 0$ and if a is quasi primary of dimension d then*

$$\overline{a}(z) \equiv \sum_{n\in\mathbb{Z}} z^{-n-d} a_{-n}^+$$

is a translation covariant field.

Proof. Since $a_n(V_k) \subset V_{k-n}$, we may view $a_n|_{V_k}$ as an operator between two finite dimensional scalar product spaces, and so it has a well-defined adjoint $(a_n|_{V_k})^* \in \mathrm{Hom}(V_{k-n}, V_k)$. It is easy to check that

$$a_n^+ \equiv \bigoplus_{k\in\mathbb{N}} (a_n|_{V_k})^* \qquad (19)$$

is indeed the formal adjoint of a_n (and so it exists). From its actual form we also see that $a_{-n}^+(V_k) \subset V_{k-n}$ which shows that indeed for any $b \in V$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $a_{-n}^+ b = 0$. Finally, if a is (quasi) primary of dimension d, then by what was said $\overline{a}(z)$ is indeed a field, and since

$$\begin{aligned} [L_{-1}, a_{-n}^+] &= -[L_{-1}^+, a_{-n}]^+ = -[L_1, a_{-n}]^+ \\ &= -((d-1+n)a_{-n+1})^+ = -(d-1+n)a_{-n+1}^+, \end{aligned} \qquad (20)$$

it is actually translation covariant. □

We shall now give a general definition of the term "hermitian", although we have already used it in connection with the stress-energy tensor. A field given by a quasi primary element $a \in V$ is called **hermitian**, iff $\overline{a}(z) = a(z)$, or equivalently, iff $a_n^+ = a_{-n}$ for all $n \in \mathbb{Z}$.

We shall now give a second definition of unitary structure on V as follows. Given a scalar product $\langle\cdot,\cdot\rangle$ on V normalized so that $\|\Omega\| = 1$, we shall say that the pair $(V, \langle\cdot,\cdot\rangle)$ is a **unitary VOA** (in the Wightmann sense) iff

- $T(z)$ is hermitian
- $\overline{a}(z)$ is *local* for every quasi primary element $a \in V$.

(Naturally, in the above definition adjoints are considered with respect to the fixed scalar product $\langle\cdot,\cdot\rangle$). Note that by the *state-field correspondence*, the second requirement is equivalent with saying that $\overline{a}(z) = Y(a_d^+\Omega, z)$ where d is of course the conformal weight of a.

Proposition 1.10. *Let V be a VOA with stress-energy $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ and let $\langle\cdot,\cdot\rangle$ be a scalar product on V. Then $(V, \langle\cdot,\cdot\rangle)$ is a unitary VOA if and only if it is unitary in the Wightmann sense.*

Proof. The "only if" part is rather trivial. Indeed, suppose that $(V, \langle\cdot,\cdot\rangle)$ is a unitary VOA and denote by Θ the corresponding PCT operator. If $a \in V$ is a quasi-primary element of conformal weight d, then by (10) it is not hard to see, that $\overline{a}(z) = (-1)^d Y(\Theta a, z)$ and hence that $\overline{a}(z)$ is local.

Let us prove now the "if" part. Suppose $(V, \langle\cdot,\cdot\rangle)$ is unitary in the Wightmann sense. Then in particular the stress-energy field $T(z)$ is hermitian, and thus the representation of the Virasoro algebra given by its Fourier modes is a direct sum (recall the finite dimensionality of the conformal eigenspaces!) of irreducible representations, with each of them being isomorphic to one of the so-called "unitary lowest energy" representations. It follows that $L_1 V_1 = \{0\}$ and hence there exists a *normalized* invariant bilinear form $(\cdot,\cdot)$ (which, by the uniqueness of vacuum, is unique).

The finite dimensional subspaces V_n $(n \in \mathbb{N})$ are pairwise $\langle\cdot,\cdot\rangle$-orthogonal, but also pairwise $(\cdot,\cdot)$-orthogonal. Thus there exists a unique $\Theta : V \to V$ conjugate-linear, grading preserving map such that $(\cdot,\cdot) = \langle\Theta\cdot,\cdot\rangle$.

All we have to show is that the above introduced conjugate linear map Θ is actually an anti-automorphism of V. First of all, let us observe that by the fact that $T(z)$ is hermitian, it follows that Θ commutes with the operators L_n $(n \in \mathbb{Z})$. Second, that to prove simplicity at Prop. 1.4, all we used is that the $(\cdot,\cdot)$-orthogonal of an ideal is still an ideal. It is not too hard to see, that having Wightmann-unitarity, the same applies to the $\langle\cdot,\cdot\rangle$-orthogonal of an ideal. (One may show the invariance of the orthogonal for Fourier modes of quasi-primary elements, and hence the invariance for all Fourier modes of all derivatives of quasi primary elements. But this is enough: by the mentioned fact about the representation type of the Virasoro algebra, every element is a linear combination of derivatives of quasi-primary elements.) Hence by the nondegeneracy of the scalar product V is simple. In turn, by the mentioned proposition we have that $(\cdot,\cdot)$ is nondegenerate, too. This implies that Θ is injective and hence, by dimensional reasons, that it is bijective (recall that V_n is finite dimensional for each integer n and that $\Theta V_n \subset V_n$).

Let now $a \in V$ be a quasi primary element of conformal weight d. Then $\overline{a}(z)$ is local so $\overline{a}(z) = Y(\overline{a}, z)$ where $\overline{a} \equiv \overline{a}(0)$ and for any $b, c \in V$ and $n \in \mathbb{Z}$ we have

$$\langle a_n \Theta b, c\rangle = \langle \Theta b, \overline{a}_{-n} c\rangle = (b, \overline{a}_{-n} c) = (-1)^d (\overline{a}_n b, c) = (-1)^d \langle \Theta \overline{a}_n b, c\rangle, \quad (21)$$

showing that $a_n \Theta = (-1)^d \Theta \overline{a}_n$. By the uniqueness of the vacuum, as $(\Omega, \Omega) = \langle \Omega, \Omega\rangle = 1$, we have that $\Theta\Omega = \Omega$. Therefore, using what we have already obtained, $a = a_{-d}\Theta\Omega = (-1)^d \Theta \overline{a}_{-d}\Omega = (-1)^d \Theta \overline{a}$ and so

$$\Theta^{-1} a = (-1)^d \overline{a} \quad (22)$$

and

$$\Theta^{-1} a_n \Theta = (-1)^d \overline{a}_n = (\Theta^{-1} a)_n. \quad (23)$$

This, together with the fact that Θ commutes with L_{-1} and that every element is a direct sum of derivatives of quasi-primary elements, shows that Θ preserves the n-products and so the (n)-products, too. □

1.4 A criteria for unitarity

Often one considers scalar products with which certain quasi-primary fields become hermitian. For example, for the Virasoro VOA with central charge $c \in \mathbb{R}^+$, one considers the normalized scalar product with which the condition $L_n^+ = L_{-n}$ (i.e. that $T(z)$ is hermitian) is satisfied. For the $U(1)$-current VOA, one considers the normalized scalar product that makes the current hermitian. (Recall that the fact whether a certain field is hermitian or not, depends on the choice of scalar product.) Are these scalar products indeed "invariant"? Certainly, if a quasi-primary field is hermitian, then its adjoint is still local (as it coincides with itself). However, the Wightmann-definition of unitary structure involved *all* quasi-primary fields. Nevertheless, we shall now show that in fact it is enough to consider a generating set. (Here and after, a set of fields is generating in V, if V is the smallest sub VA containing all of them; i.e. we consider "weak generation".)

Proposition 1.11. *Let V be a VOA with stress-energy $T(z)$ and suppose $\langle\cdot,\cdot\rangle$ is a normalized scalar product on V with which $T(z)$ is hermitian. Then the following three affirmations are equivalent.*

(i) There is a generating set of quasi-primary fields that are hermitian.

(ii) Every quasi-primary field is the sum of a hermitian and anti-hermitian ($\equiv$ i-times a hermitian) one.

(iii) The pair $(V, \langle\cdot,\cdot\rangle)$ is a unitary VOA.

Proof. $(iii) \Leftrightarrow (ii)$ is more or less trivial. Indeed, assuming unitarity, for any quasi-primary element $a \in V$ of conformal weight d, with setting

$$a^s(z) \equiv \frac{1}{2}(Y(a,z) + \overline{a}(z)) \text{ and } a^{as}(z) \equiv \frac{-i}{2}(Y(a,z) - \overline{a}(z)) \tag{24}$$

the decomposition $Y(a,z) = a^s(z) + ia^{as}(z)$ is exactly the one required by (ii). Vice versa, it is clear that assuming (ii) implies that the adjoint field of every quasi-primary field is local.

$(i) \Leftrightarrow (iii)$: evidently, (ii) is stronger than (i), so by what we have already obtained, all we have to show is $(i) \Leftrightarrow (ii)$. So suppose $\{a^\alpha\}$ is a generating set of quasi-primary, hermitian elements and let $c \in V$ be an arbitrary quasi-primary element of conformal weight d_c. We have to prove that $\overline{c}(z)$ is a local field; i.e. that for any $b \in V$ there exists an $N \in \mathbb{N}$ such that $(z-w)^N[\overline{c}(z), Y(b,w)] = 0$. By Dong's lemma (see e.g.[**?**]) it is enough to show this for all b from a generating set; say from $\{a^\alpha\}$. So let a be an arbitrary element in this set, and to simplify notations, set $a(z) \equiv Y(a,z)$

and $c(z) \equiv Y(c,z)$. As $\overline{a}(z)$ is local, in particular it is local to $c(z)$. But taking adjoints it is easy to see that

$$\overline{a}(z) \text{ is local to } c(z) \quad \Leftrightarrow \quad a(z) \text{ is local to } \overline{c}(z). \tag{25}$$

Hence $\overline{c}(z)$ is local to any of the fields associated to the elements of $\{a^\alpha\}$. $\square$

1.5 Unitary automorphisms and (essential) uniqueness of unitary structure

Let V be a VOA with stress-energy $T(z)$. A **VOA automorphism** of V is an automorphism of V that preserves $T(z)$. Without causing too much confusion, we shall denote the group of such automorphisms by $\mathrm{Aut}(V)$. (Purely VA automorphisms, that do not necessarily preserve $T(z)$, will be never considered further in this work and we always think of V with a *fixed* stress-energy. Recall however, that by Corollary 1.6, if the stress-energy is such that it allows the existence of a unitary structure, then $\mathrm{Aut}(V)$ is precisely the set of grade-preserving VA automorphisms of V.)

Let us assume now that $\langle\cdot,\cdot\rangle$ is normalized invariant scalar product on V; i.e. that $(V,\langle\cdot,\cdot\rangle)$ is a unitary VOA. Then we may consider the subgroup of $\mathrm{Aut}(V)$ that preserves the scalar product, in other words the set of elements $\alpha \in \mathrm{Aut}(V)$ for which $\langle\alpha\cdot,\alpha\cdot\rangle = \langle\cdot,\cdot\rangle$. We shall call it the subgroup of **unitary VOA automorphisms**, and we shall denote it by $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$. To put it in another way, $\alpha \in \mathrm{Aut}(V)$ belongs to $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$, if and only if α is a unitary operator with respect to $\langle\cdot,\cdot\rangle$.

By Corollary 1.6, elements of $\mathrm{Aut}(V)$ automatically preserve the normalized invariant bilinear form on V. Thus $\alpha \in \mathrm{Aut}(V)$ belongs to $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$, if and only if α commutes with the PCT operator Θ associated to $\langle\cdot,\cdot\rangle$.

In general $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$ is properly contained in $\mathrm{Aut}(V)$. If $\alpha \in \mathrm{Aut}(V)$ but $\alpha \notin \mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$, then $\langle\alpha(\cdot),\alpha(\cdot)\rangle$ is a normalized invariant scalar product on V different from $\langle\cdot,\cdot\rangle$. We shall now see that every normalized invariant scalar product on V arises in this way.

Proposition 1.12. *Let $(V,\langle\cdot,\cdot\rangle)$ be a unitary VOA with CPT operator Θ and let $\{\cdot,\cdot\}$ be another normalized invariant scalar product on V with corresponding CPT operator $\tilde{\Theta}$. Then there exists a $\beta \in \mathrm{Aut}(V)$ such that:*

- *(i) β is strictly positive with respect to $\langle\cdot,\cdot\rangle$; that is, $\langle a,\beta(b)\rangle = \langle\beta(a),b\rangle$ and $\langle a,\beta(a)\rangle > 0$ for all nonzero $a,b \in V$,*
- *(ii) $\tilde{\Theta} = \beta \circ \Theta \circ \beta^{-1}$.*

This β is unique, and with it we have that $\{\cdot,\cdot\} = \langle\beta^{-1}(\cdot),\beta^{-1}(\cdot)\rangle$.

Proof. Let $\alpha \equiv \Theta\tilde{\Theta}$; then clearly $\alpha \in \mathrm{Aut}(V)$. We shall now show that α is a (strictly) positive hermitian operator with respect to $\langle\cdot,\cdot\rangle$. Indeed, with denoting the normalized invariant bilinear form by $(\cdot,\cdot)$, and using all previously listed properties of PCT operators as well as invariant bilinear forms (regarding anti-automorphisms), we have that

$$\langle a, \alpha(b)\rangle = (\Theta a, \Theta\tilde{\Theta}b) = \overline{(a, \tilde{\Theta}b)} = (\tilde{\Theta}a, b) = \{a, b\} \tag{26}$$

and that

$$\langle \alpha(a), b\rangle = \langle \Theta\tilde{\Theta}a, b\rangle = (\tilde{\Theta}a, b) = \{a, b\} \tag{27}$$

for all $a, b \in V$. Comparing the two equations we see that α is hermitian. Moreover, with letting $b = a$, we get that $\langle a, \alpha(a)\rangle = \{a, a\} \geq 0$ and equality holds if and only if $a = 0$.

We know that α preserves the finite dimensional spaces V_n ($n \in \mathbb{N}$). So by what was said, on each of them, the restriction of α is diagonalizable with positive eigenvalues. Hence we can take the square root of α and define $\beta \equiv \alpha^{-1/2}$. This β clearly satisfies requirement (i), but at this point we have not even shown that it is an automorphism. Nevertheless, by its construction, it surely preserves the grading (as α does so).

If $a, b \in V$ are eigenvectors of α with eigenvalues λ_a and λ_b respectively, then $\beta(a) = \sqrt{\lambda_a}a$, $\beta(b) = \sqrt{\lambda_b}b$ and for every $n \in \mathbb{Z}$ we have

$$\alpha(a_{(n)}b) = \alpha(a)_{(n)}\alpha(b) = \lambda_a\lambda_b\, a_{(n)}b. \tag{28}$$

Hence

$$\beta(a_{(n)}b) = \sqrt{\lambda_a\lambda_b}\, a_{(n)}b = \beta(a)_{(n)}\beta(b). \tag{29}$$

Thus by linearity it follows that $\beta \in \mathrm{Aut}(V)$ (as it is diagonalizable). Then using that β is hermitian, we have that for all $a, b \in V$,

$$\begin{aligned}(\beta\circ\Theta\circ\beta^{-1}(a), b) &= (\Theta\circ\beta^{-1}(a), \beta^{-1}(b)) = \langle\beta^{-1}(a), \beta^{-1}(b)\rangle = \langle\beta^{-2}(a), b\rangle \\ &= \langle\alpha(a), b\rangle = \langle\Theta\tilde{\Theta}a, b\rangle = (\tilde{\Theta}a, b)\end{aligned} \tag{30}$$

and thus (ii) follows by the nondegeneracy of $(\cdot,\cdot)$. Moreover, we have

$$\begin{aligned}\langle\beta^{-1}(a), \beta^{-1}(b)\rangle &= (\Theta\circ\beta^{-1}(a), \beta^{-1}b) = (\beta\circ\Theta\circ\beta^{-1}(a), b) \\ &= (\tilde{\Theta}(a), b) = \{a, b\}.\end{aligned} \tag{31}$$

Finally, for the uniqueness, assume that $\tilde{\beta} \in \mathrm{Aut}(V)$ satisfies (i, ii). Then for all $a, b \in V$,

$$\langle\Theta\tilde{\Theta}a, b\rangle = \{a, b\} = \langle\tilde{\beta}^{-2}(a), b\rangle \tag{32}$$

and hence $\tilde{\beta}^{-2} = \Theta\tilde{\Theta}$ and thus $\tilde{\beta} = \beta$ by the uniqueness of the (positive) square-root. □

As a consequence of the above proposition, up isomorphisms, at most there exists one unitary structure on a VOA. From the remark made before the proposition, we know that this structure is really unique (i.e. not only up unitary isomorphisms) iff every automorphism of V is unitary. Using [**?**, Remark 4.9c] one can easily give examples of non-unitary automorphism. However there are VOAs for which there exists a unique (normalized) invariant scalar product and we will give a characterization of this class using a natural the topological properties of $\mathrm{Aut}(V)$.

Let $(V, \langle\cdot,\cdot\rangle$ be a unitary VOA. Then V is a normed space with the norm defined by the formula $\|a\| = \langle a, a\rangle^{1/2}$. However V cannot be a Banach space because, being union of finite dimensional subspace it is of first category. Using the norm on V we can topologize $\mathrm{End}(V)$ with the strong operator topology. With this topology $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$ becomes a topological group (this need not to be the case for $\mathrm{Aut}(V)$ since its elements could be unbounded).

Lemma 1.13. *Let $(V, \langle\cdot,\cdot\rangle)$ be a unitary VOA. Then $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$ is compact (and second countable) in the strong operator topology.*

Proof. Denote by $\mathcal{U}_H$ the group of unitary grading preserving, elements of $\mathrm{End}(V)$. The subspace V_n is finite dimensional for all $n \in \mathbb{Z}$, thus by standard arguments $\mathcal{U}_H$ is strong-operator compact. Hence as $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V) \subset \mathcal{U}_H$, we only need to show that $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$ is strongly closed. So let α_λ be a net in $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$ with strong limit $\alpha \in \mathcal{U}_H$. For all $k, n, m \in \mathbb{Z}$, the bilinear map

$$V_k \times V_m \ni (a, b) \mapsto a_{(n)}b \in V \tag{33}$$

is jointly continuous since V_k and V_m are finite dimensional. It follows that, for all homogeneous $a, b \in V$ we have

$$\alpha(a_{(n)}b) = \lim \alpha_\lambda(a_{(n)}b) = \lim_\lambda \alpha_\lambda(a)_{(n)}\alpha_\lambda(b) = \alpha(a)_{(n)}\alpha(b). \tag{34}$$

Hence, by linearity $\alpha \in \mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$. □

Theorem 1.14. *Let $(V, \langle\cdot,\cdot\rangle)$ be a unitary VOA and let Θ be the corresponding PCT operator. Then the following affirmations are equivalent.*

(i) *$\langle\cdot,\cdot\rangle$ is the unique (normalized) invariant scalar product on V.*

(ii) $\mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V) = \mathrm{Aut}(V)$.

(iii) *Every $\alpha \in \mathrm{Aut}(V)$ commutes with Θ.*

(iv) $\mathrm{Aut}(V)$ *is strong-operator compact.*

(v) $\mathrm{Aut}(V)|_{V_n}$ *has compact closure in* $\mathrm{GL}(V_n)$ *for all* $n \in \mathbb{N}$.

Proof. $(i) \Leftrightarrow (ii)$: it follows from Prop. 1.12 and the comments made before the cited proposition.

$(ii) \Leftrightarrow (iii)$: it is an easy consequence of the fact that elements of $\mathrm{Aut}(V)$ automatically preserve the normalized invariant bilinear form (Corollary 1.6).

$(iv) \Rightarrow (v)$ is trivial and $(ii) \Rightarrow (iv)$ follows from our previous lemma. So by what we have so far obtained, all we need is to prove implication $(v) \Rightarrow (i)$. So assume by contradiction that $\{\cdot,\cdot\}$ is a (normalized) invariant scalar product on V different from $\langle\cdot,\cdot\rangle$. By Proposition 1.12 there is an $\beta \in \mathrm{Aut}(V)$ such that it is diagonalizable with positive eigenvalues and $\{\cdot,\cdot\} = \langle\beta^{-1}(\cdot), \beta^{-1}(\cdot)\rangle$. As by assumption $\beta \neq \mathbf{1}$ and it preserves the grading, there must exist a homogenous eigenvector $v \in V_n$ for some $n \in \mathbb{N}$ such that the corresponding positive eigenvalue $\lambda \neq 0$. Thus either the sequence $\beta^k(v) = \lambda^k v$ $(k = 1, 2, ..)$ or the sequence $\beta^{-k}(v) = \lambda^{-k} v$ $(k = 1, 2, ..)$ must be unbounded in V_n; hence (v) cannot hold. □

1.6 Unitary subalgebras

A subalgebra of a Vertex Algebra V is a linear subspace $W \subset V$ such that it is closed under the (n)-products, it contains the vacuum and it is invariant under T. Evidently, W may be viewed as a VA on its own. Let now $(V, \langle\cdot,\cdot\rangle)$ be a unitary VOA, and suppose $W \subset V$ is a subalgebra. Recall that the scalar product permits us to consider the (formal) adjoint of a vertex operator. Obviously, if $a \in W$ is a homogenous element, then the product $a_n b$ remains in W for every $n \in \mathbb{Z}, b \in W$, but there is no guarantee that $a_n^+ b$ is in W, too. We shall say that a subalgebra $W \subset V$ is a **unitary subalgebra**, iff it is

(1) compatible with the grading: $W = \oplus_{n\in\mathbb{N}}(W \cap V_n)$,

(2) "adjoint-invariant": $a_n^+ b \in W$ for all $a, b \in W$ and $n \in \mathbb{Z}$.

Note that (1) is equivalent to saying that $H(W) \subset W$. To see the mathematical consequences of (2), consider that for e homogeneous element $a \in V_k$, we have that

$$\begin{aligned} \langle a_n^+ b, c\rangle &= \langle b, a_n c\rangle = (\Theta b, a_n c) = (-1)^k \sum_{j=0}^{\infty} \frac{1}{j!}((L_1^j a)_{-n}\Theta b, c) \\ &= (-1)^k \sum_{j=0}^{\infty} \frac{1}{j!}\langle \Theta(L_1^j a)_{-n}\Theta b, c\rangle, \end{aligned} \tag{35}$$

for every $b, c \in V$ and $n \in \mathbb{Z}$, where of course Θ, L_1 and $(\cdot,\cdot)$ stand for what they stood before. It follows that

$$a_n^+ = (-1)^k \sum_{j=0}^{\infty} \frac{1}{j!} \Theta(L_1^j a)_{-n} \Theta = (-1)^k \sum_{j=0}^{\infty} \frac{1}{j!} (L_1^j \Theta a)_{-n}, \tag{36}$$

and hence that if $a \in W_k \equiv W \cap V_n$, then $\sum_{j=0}^{\infty} \frac{1}{j!}(L_1^j \Theta a)_{-n}\Omega \in W$ for every $n \in \mathbb{Z}$ (since $\Omega \in W$). From here, by varying n, one can easily deduce that in fact $L_1^j \Theta a \in W$ for each $j \in \mathbb{N}$. This, in turn (by letting first $j = 0$ and then $j = 1$) implies that W is invariant under both Θ and L_1:

$$\Theta(W) \subset W \quad \text{and} \quad L_1(W) \subset W. \tag{37}$$

Lemma 1.15. *With everything as before, a subalgebra $W \subset V$ is unitary if and only if it is invariant under both Θ and L_1.*

Proof. We have just seen the "only if" part. As for the "if" part, assume that $\Theta(W) \subset W$ and $L_1(W) \subset W$. Then, as $L_{-1}(W) = T(W) \subset W$, we have that $H(W) \subset W$ since $H = L_0 = \frac{1}{2}[L_1, L_{-1}]$; thus W is compatible with the grading. Property (2), appearing in the definition of unitary subalgebras, is a simple consequence of formula (36). □

The above lemma provides, in particular, two simple ways of exhibiting a unitary subalgebra of a unitary VOA. First, for a subgroup $G \subset \mathrm{Aut}_{\langle\cdot,\cdot\rangle}(V)$, the *orbifold* subalgebra (i.e. the set of fixed elements of V under the action of elements of G) is clearly unitary. Second, if $\{a_\lambda\}$ is a collection of hermitian quasi-primary elements, then the subalgebra generated by them is again clearly unitary.

Suppose $W \subset V$ is a unitary subalgebra. Then W, viewed in itself, is an H-graded VA satisfying properties (1,2,3) and (4) of a VOA (see them again in section 1.2), which is already endowed with a normalized scalar product $\langle\cdot,\cdot\rangle_W$; namely the restriction of the scalar product on V. This leaves us with the natural question: can we choose a stress-energy tensor (by choosing a Virasoro element) in W, with which $(W, \langle\cdot,\cdot\rangle_W)$ becomes a unitary VOA? By Prop. 1.5, at most there can be one such Virasoro element.

In what follows, we shall positively answer this question by giving an explicit construction of the Virasoro element in question. In order to do so, let us first note that the orthogonal projection P_W onto W, is a well-defined element in $\mathrm{End}(V)$. (In general — as V is infinite dimensional — a subspace and its orthogonal in V may not be complementary.) This is a simple consequence of the fact that W is compatible with the grading, and that the subspaces V_n ($n \in \mathbb{N}$) are finite dimensional.

Lemma 1.16. *With everything as before, let $W \subset V$ be a unitary subalgebra. Then $P_W a_{(n)} = a_{(n)} P_W$ for every $a \in W$ and $n \in \mathbb{Z}$.*

Proof. By linearity and the fact that W is compatible with the grading, we may assume that a is homogeneous and so that $a_{(n)} = a_m$ for a certain $m \in \mathbb{Z}$. If $b \in V$; then $a_m P_W b \in W$ as $P_W b \in W$ and W is a subalgebra, and of course also $P_W a_m b \in W$. Hence they are equal if and only if their scalar products, with elements of W, coincide. So let $c \in W$, then $a_m^+ c \in W$ (since W is unitary), and

$$\begin{aligned} \langle c, a_m P_W b\rangle &= \langle P_W a_m^+ c, b\rangle = \langle a_m^+ c, b\rangle = \langle c, a_m b\rangle \\ &= \langle P_W c, a_m b\rangle = \langle c, P_W a_m b\rangle, \end{aligned} \tag{38}$$

which, by what was explained, concludes our proof. □

Proposition 1.17. *With everything as before, let $W \subset V$ be a unitary subalgebra. Then $\nu \in W$ is a Virasoro element, and $(W, \langle\cdot,\cdot\rangle_W)$ is a unitary VOA when W is considered with stress-energy $T_W(z) \equiv Y_W(P_W \nu, z)$.*

Proof. $T(W) \subset W$, $H(W) \subset W$ and moreover by Lemma 1.15, $L_1(W) \subset W$ and $\Theta(W) \subset W$. In addition, we have that $L_0^+ = L_0, L_{\pm1}^+ = L_{\mp1}$ and that Θ is anti-unitary. The listed facts imply that P_W commutes with the operators $T = L_{-1} \equiv \nu_{-1}$, $H = L_0 \equiv \nu_0$, $L_1 \equiv \nu_1$ and with Θ. It also follows that the restrictions of $(P_W\nu)_{-1}, (P_W\nu)_0$ and $(P_W\nu)_1$ onto W are exactly $T|_W$, $H|_W$ and $L_1|_W$, respectively.

This has two immediate consequences. First, by Prop. 1.1, it implies that $\nu_W \equiv P_W \nu$ is a Virasoro element. Second, as the definition of a hermitian quasi-primary elements may be rephrased with the help of H, L_1 and Θ only, we have that if $a \in V$ is quasi-primary and hermitian, then $P_W a$ is quasi-primary and hermitian in W, too. In particular, the stress-energy $T_W(z)$ is a hermitian field and moreover, by Prop. 1.11, we have that $(W, \langle\cdot,\cdot\rangle_W)$ is a unitary VOA. □

As ν is invariant under the action of elements of $\mathrm{Aut}(V)$, a unitary orbifold subalgebra is always a unitary VOA *with the same Virasoro element.* However, it is easy to find unitary subalgebras that do not contain ν. For example, let V and $\tilde{V}$ be two unitary VOAs with corresponding scalar products $\langle\cdot,\cdot\rangle$ and $\{\cdot,\cdot\}$ and Virasoro elements ν and $\tilde{\nu}$. Then $V \otimes \tilde{V}$ can be naturally endowed with the structure of a unitary VOA, where the Virasoro element fixing the stress-energy field is $\nu_\otimes \equiv \nu \otimes \tilde{\Omega} + \Omega \otimes \tilde{\nu}$. Clearly, the subalgebra $V \otimes \tilde{\Omega} \subset V \otimes \tilde{V}$ is a unitary subalgebra, which is isomorphic to V. However, $\nu_\otimes \notin V \otimes \tilde{\Omega}$. In stead, the unitary subalgebra $V \otimes \tilde{\Omega}$ is to be considered with its *own* Virasoro element, which is $\nu \otimes \tilde{\Omega}$.