A NOTE ON MUTUALLY ORTHOGONAL LATIN SQUARES

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SUMMARY. It is proved that the existence of a set of $(n-3)$ Mutually Orthogonal Latin Squares (MOLS) of order n implies the existence of a complete set of $(n-1)$ such squares and hence the existence of a finite projective plane $PG(2, n)$.

1. Introduction

A Latin Square of order n, is an $n \times n$ matrix whose entries are from a set of n distinct symbols such that each symbol occurs exactly once in each row and column. Two Latin Squares $A = (a_{ij})$, $B = (b_{ij})$ of order n are mutually orthogonal if the n^2 ordered pairs (a_{ij}, b_{ij}) are all distinct. A set $A_1, A_2, \ldots, A_m$ of Latin Squares of order n, is called orthogonal if A_i and A_j are orthogonal for all $i \neq j$. It is easy to show that $m \leqslant n-1$. An orthogonal set is said to be complete provided if $m = n-1$. Bose (1938) and Levi (1942) independently showed the equivalence of complete sets of such squares of order n to the finite projective planes $PG(2, n)$.

A Balanced Incomplete Block design (BIBD) is an arrangement of v elements into b sets of k $(<v)$ distinct elements such that every pair of difierent elements occurs in exactly λ sets. Then it is easy to show that every element occurs in exactly r of these sets. The numbers v, b, r, k, λ are called the parameters of the design. It is known that the existence of a complete set of squares of order n is equivalent to the existence of a BIBD with parameters

$$v = n^2, \quad b = n^2+n, \; r = n+1, k = n, \quad \lambda = 1. \qquad \ldots \quad (1.1)$$

Bose and Nair (1939) generalised the notion of a BIBD to a Partially Balanced Incomplete Block design (PBIBD). These new designs with 2-associate classes have been classified by Bose and Shimamoto (1952) according to their association schemes. Let $v = n^2$ treatments be exhibited in a square array of order n. If a set of $i-2$ MOLS of order n exists, then the association scheme of a PBIBD for these n^2 treatments is said to be of type L_i, if any two treatments are 2-associates, if and only if they occur in the same row, or column or correspond to the same symbol of any one of the Latin Squares. Then from Bose and Shimamoto (1952) we have

$$n_1 = (n-1)(n-i+1), \quad n_2 = i(n-1) \qquad \ldots \quad (1.2)$$

$$p^1{}_{11} = (n-i)^2+i-2 \; ; \; p^2{}_{11} = (n-i)(n-i+1)$$

2. Main result

Theorem : *Any set of* $(n-3)$ *MOLS of order* n *can be uniquely extended to a complete set of* $n-1$ *such squares, if* $n \neq 4$.

Proof: Suppose $(n-3)$ MOLS of order n exist. Then we can get a PBIB with association scheme L_{n-1} by forming blocks of size n, corresponding to the rows, columns or the same symbols of each of the $(n-3)$ Latin Squares. The parameters of this design are given by

$$v = n^2, \quad b = n^2-n, \quad r = n-1, \quad k = n, \quad \lambda_1 = 0, \quad \lambda_2 = 1, \quad \ldots \quad (2.1)$$
$$n_1 = 2(n-1), \quad n_2 = (n-1)^2; \; p^1_{11} = n-2, \; p^2_{11} = 2.$$

From Shrikhande (1959), it follows that the association scheme of the above design is of type L_2, i.e. the n^2 treatments can be uniquely arranged into a square array of order n, such that any two treatments in the same row or same column are 1-associates, otherwise they are 2-associates Forming $2n$ blocks corresponding to the rows and columns of this array we obviously get a BIBD with parameters

$$v = n^2, \quad b = n^2+n, \; r = n+1, \; k = n, \quad \lambda = 1$$

which in turn implies the existence of a complete set of squares of order n. It is obvious that if we apply the same correspondence which was used to obtain the blocks of (2.1), then the added n blocks corresponding to rows (columns) give a Latin Square, such that these Latin Squares are mutually orthogonal and also orthogonal to the previous set of $(n-3)$ such squares.

Using Bruck and Ryser's Theorem (1949), we have.

Corollary 1 : If $n \equiv 1$ or 2 (mod 4) and the square-free part of n contains a prime $4t+3$, then there do not exist $(n-3)$ MOLS of order n.

Similarly in the notation of Silverman (1960), we have the following refinement of his Theorem 4.8:

Corollary 2 : If $n \equiv 1$ or 2 (mod 4) and the square free part of n contains a prime $4t+3$, then $S^k(n)$ is not r-orthogonal for $k > n+r-3$

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