Quantum Information Theory and pairwise quasi-orthogonal subalgebras

Mihály Weiner

Alfréd Rényi Institute of Mathematics, Budapest
Hungarian Academy of Sciences

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The concept of unbiased vectors and bases

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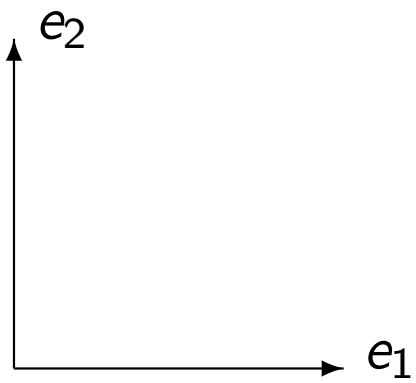

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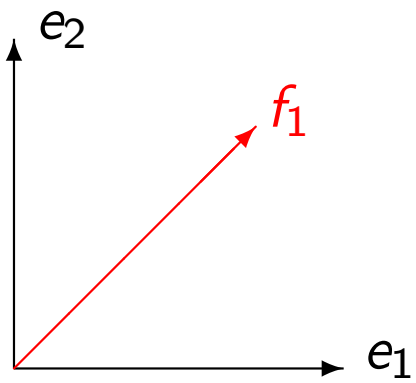

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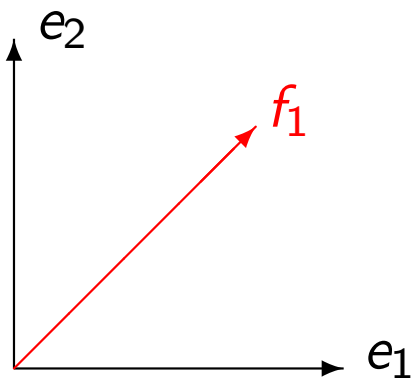

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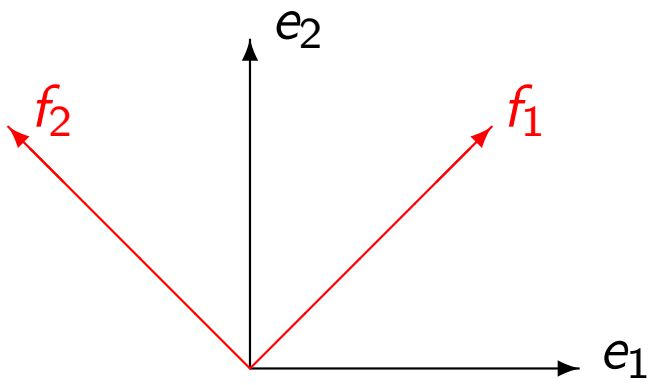

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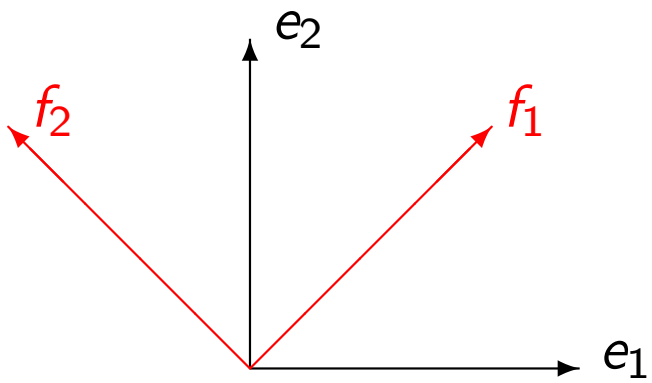

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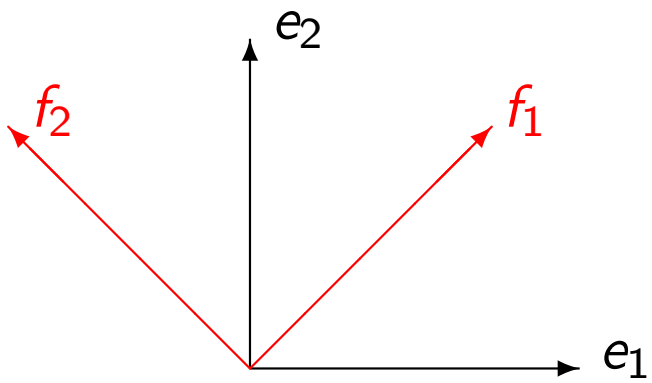

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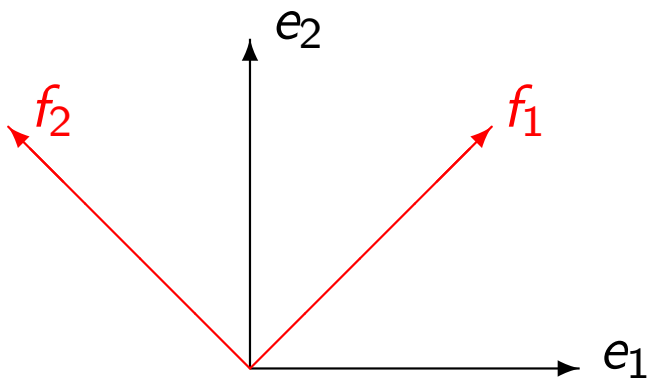

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Note: (e_1, e_2) is also unbiased for (f_1, f_2). Unbiasedness is automatically **mutual**.

Definition

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Two bases $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ in $\mathcal{H}$ satisfying

$$|\langle e_j, f_k \rangle| = |\langle e_{j'}, f_{k'} \rangle|$$

for all $j, k, j', k' \in \{1, \ldots n\}$ are called **mutually unbiased**.

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Remark

If $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are mutually unbiased, then infact $|\langle e_j, f_k \rangle| = \frac{1}{\sqrt{n}}$ for all $j, k \in \{1, \ldots n\}$.

Problem

In an n-dimensional (complex) space, at most how many ONB can be given, such that any two of them is mutually unbiased?

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- $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is such an algebra if and only if it is a maximal abelian C^*-subalgebra of $\mathcal{B}(\mathcal{H})$
- Two maximal abelian C^*-subalgebras $\mathcal{A}, \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ correspond to mutually unbiased bases if and only if

$$\tau(AB) = \tau(A)\tau(B)$$

for all $A \in \mathcal{A}, B \in \mathcal{B}$ ("**statistical independence**")

$\mathcal{A}$ and $\mathcal{B}$ are statistically independent

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- Note that $\{\mathbb{1}\}^\perp =$ traceless operators, so: $\mathcal{A}$ and $\mathcal{B}$ are quasi-orthogonal $\Leftrightarrow$ their traceless parts are orthogonal

Mutually unbiased bases and quasi-orthogonality

MUB $\Leftrightarrow$ quasi-orthogonal maximal abelian C^*-subalgebras

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Complete collection of MUB: the corresponding subalgebras linearly span $\mathcal{B}(\mathcal{H}) \Leftrightarrow$ their number is $n + 1$.

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- numerical calculations done with computers seem to indicate that $N(6) = 3$

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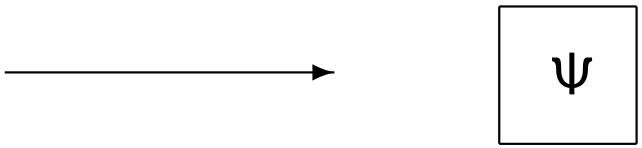

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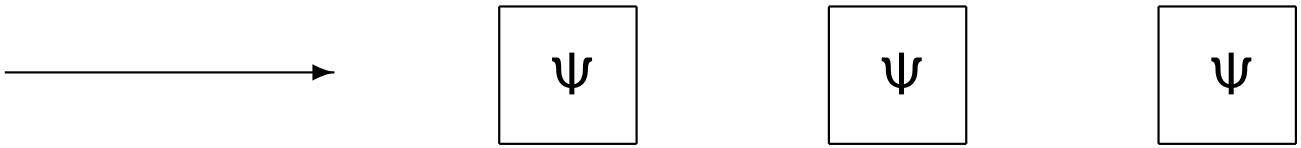

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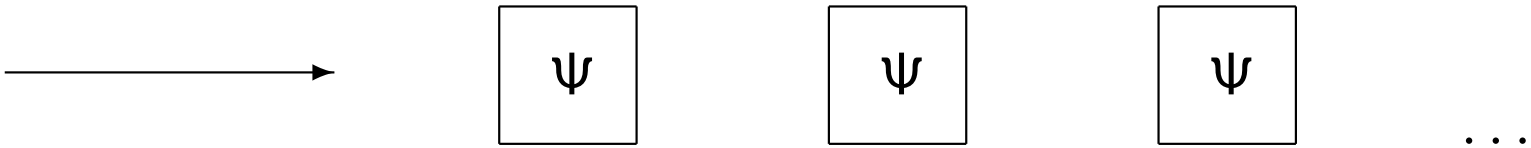

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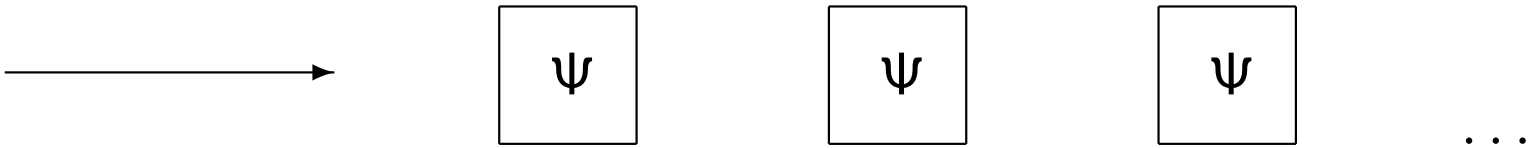

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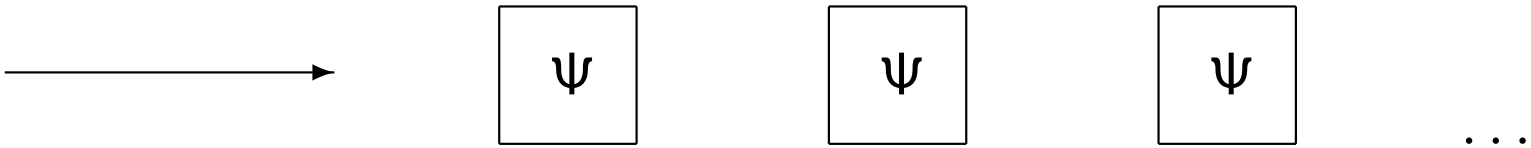

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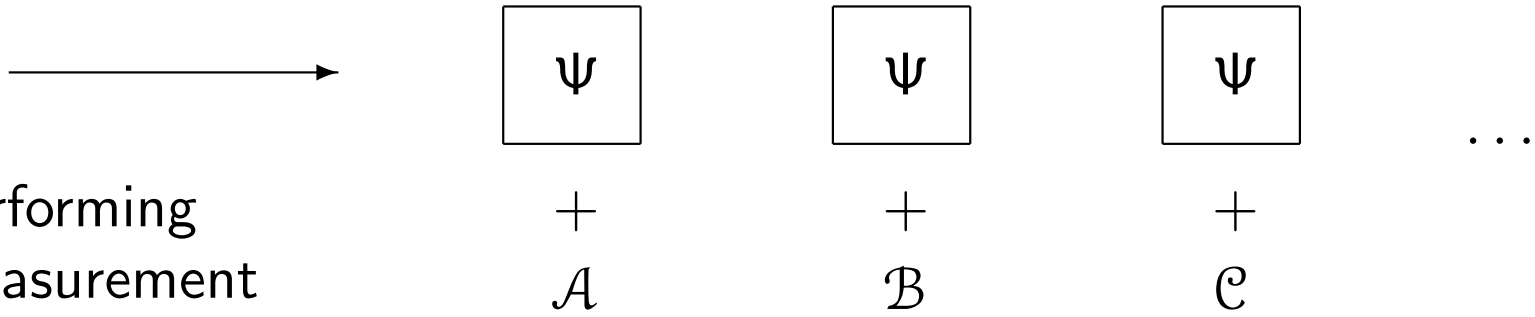

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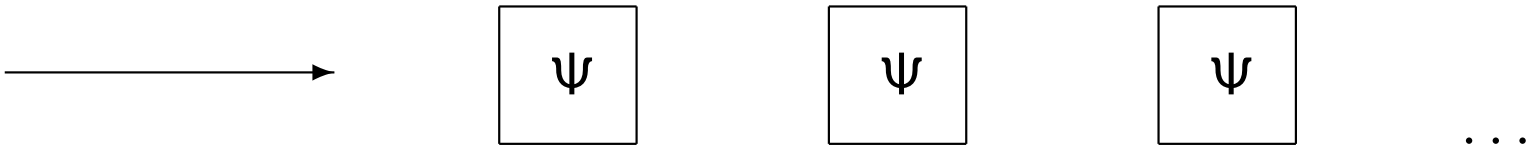

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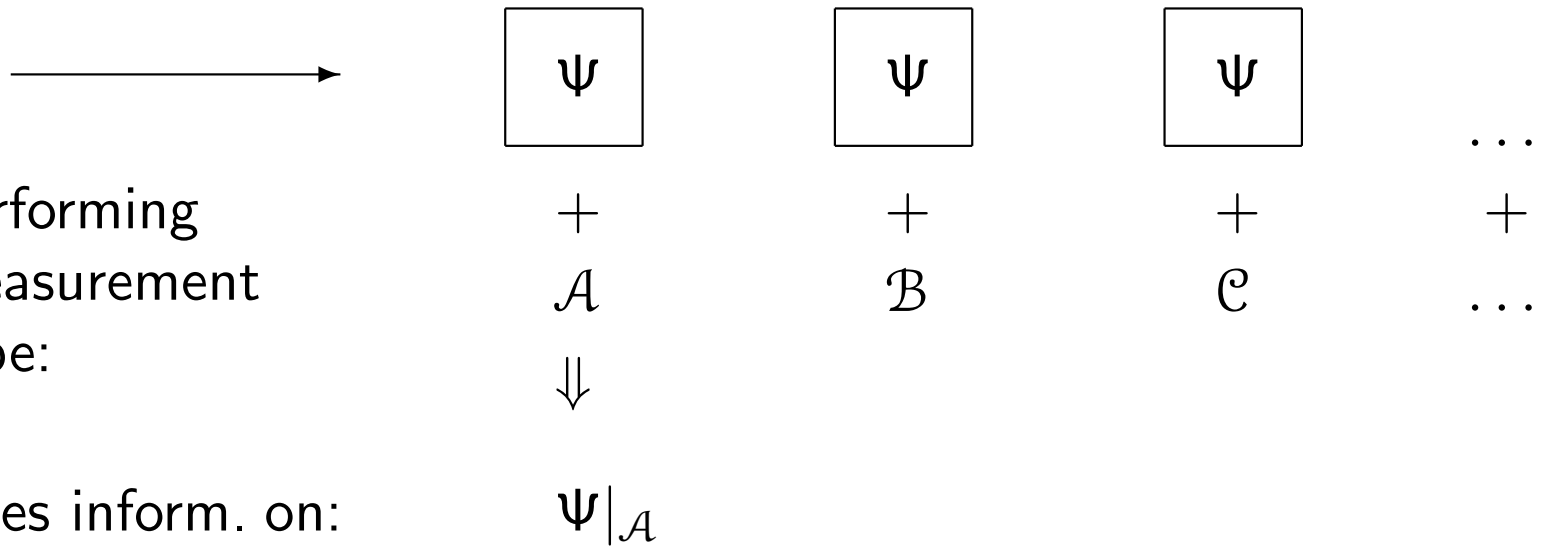

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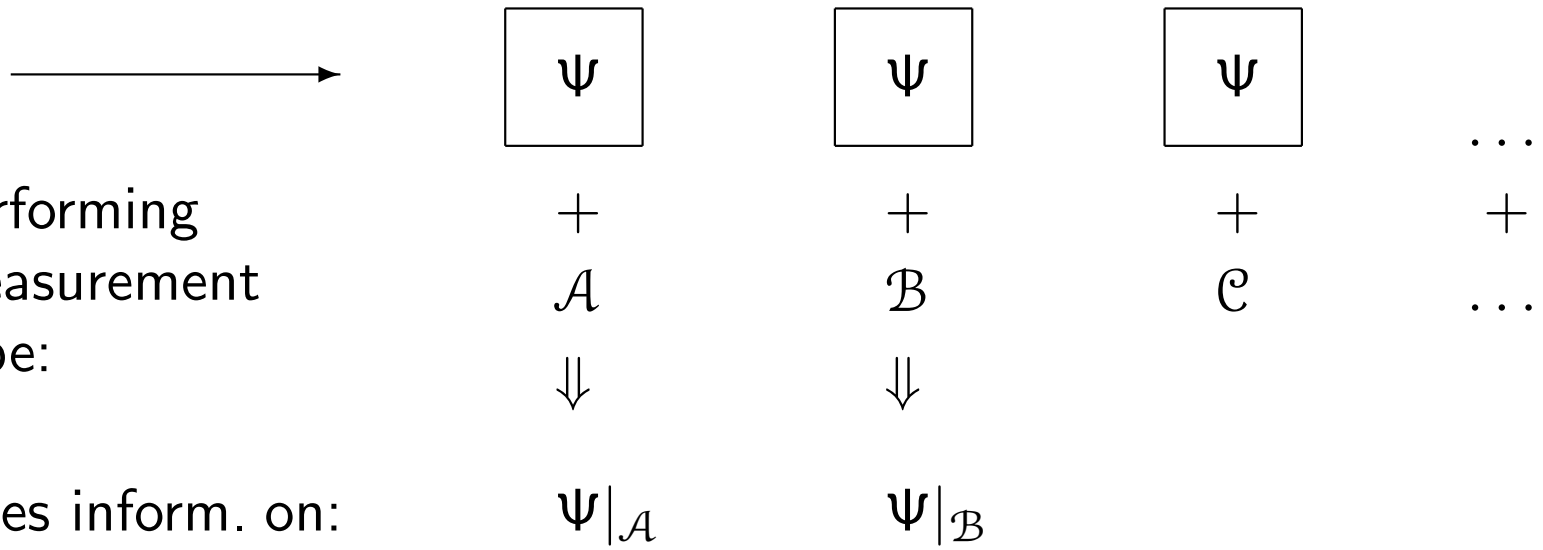

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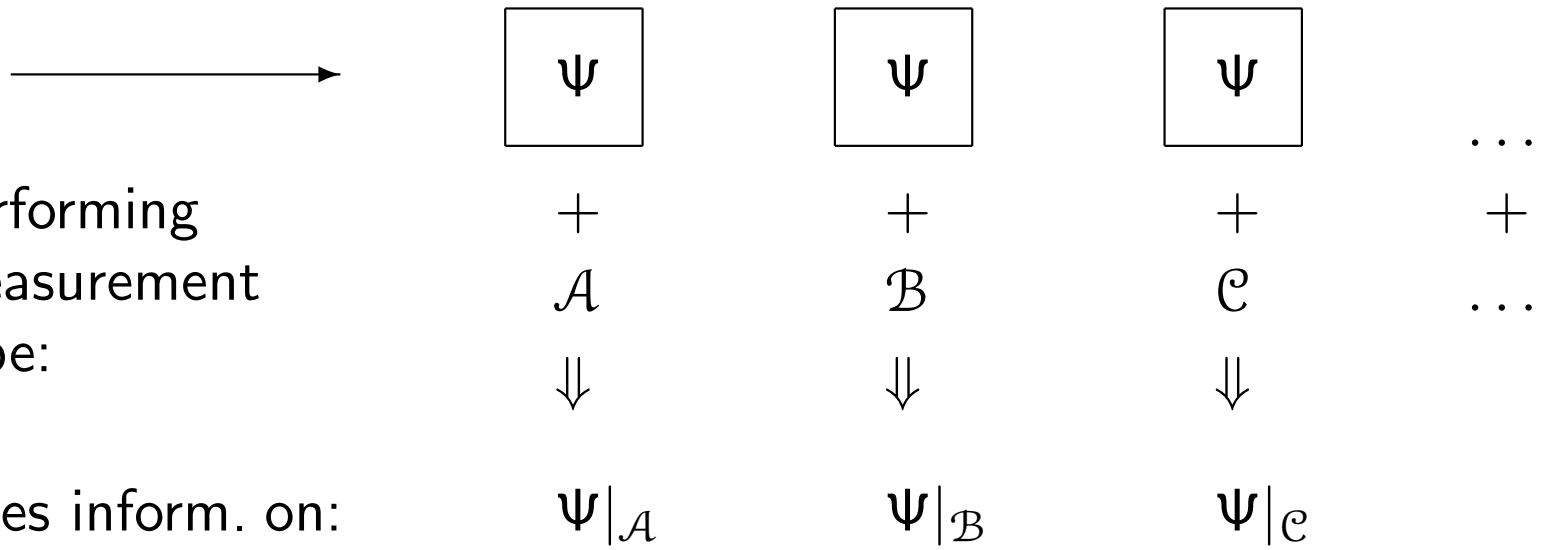

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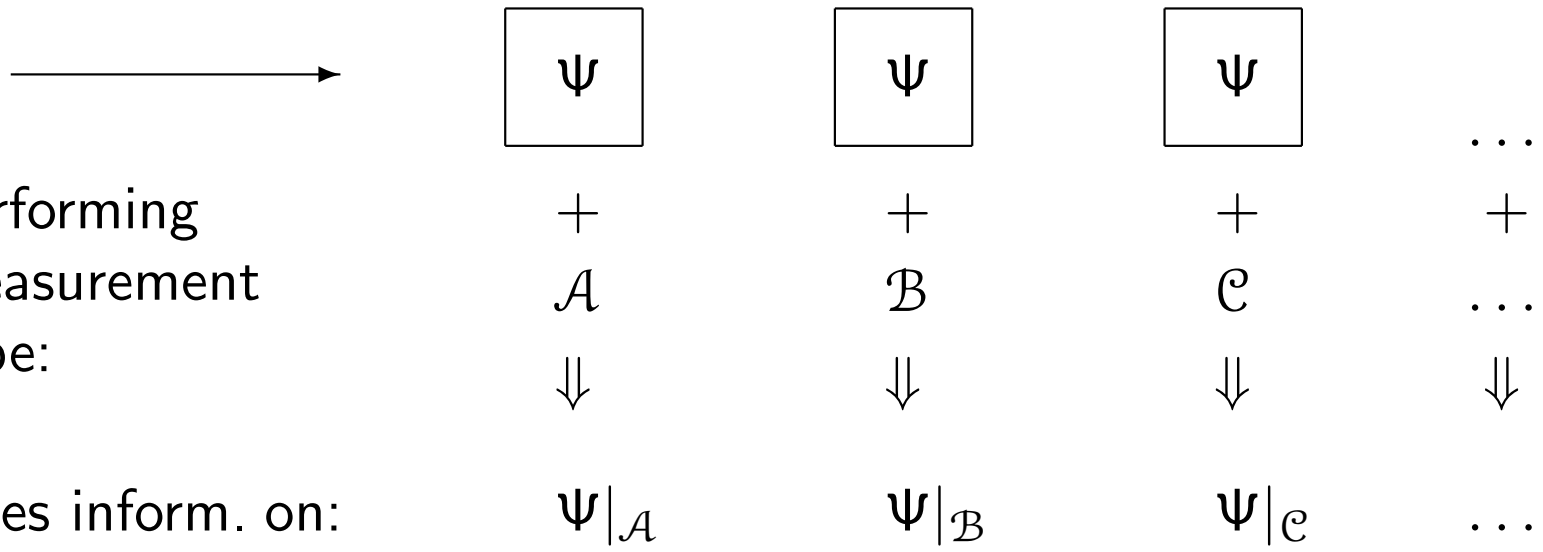

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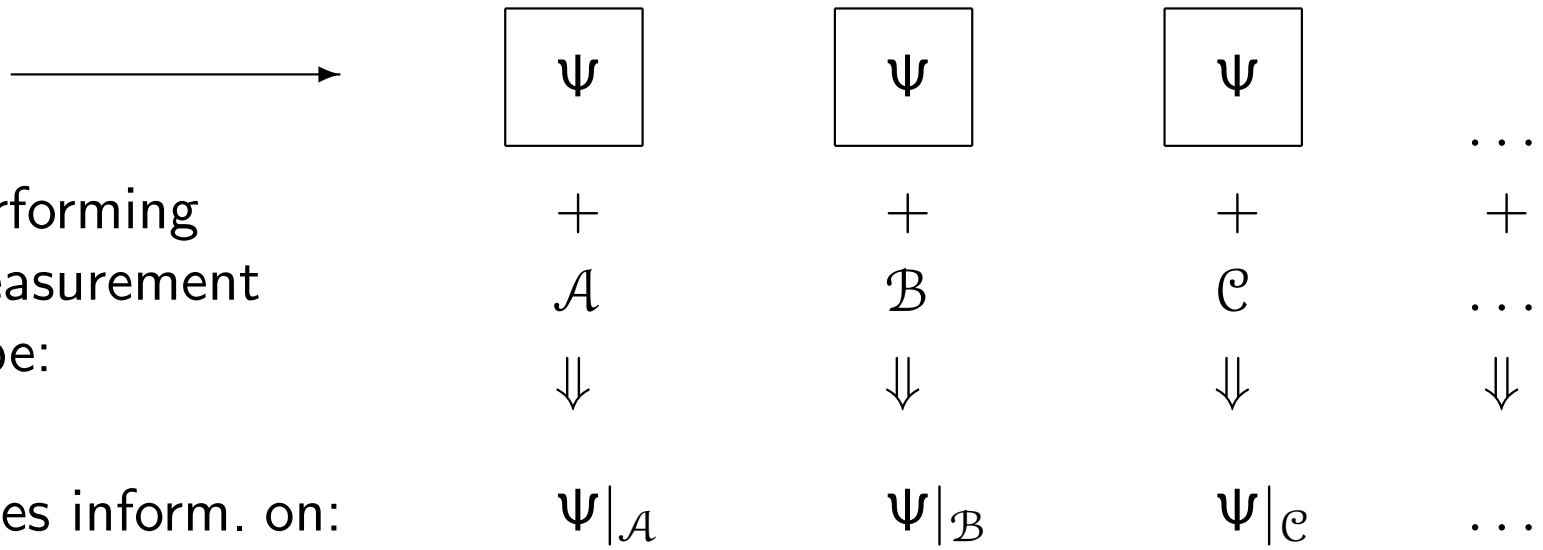

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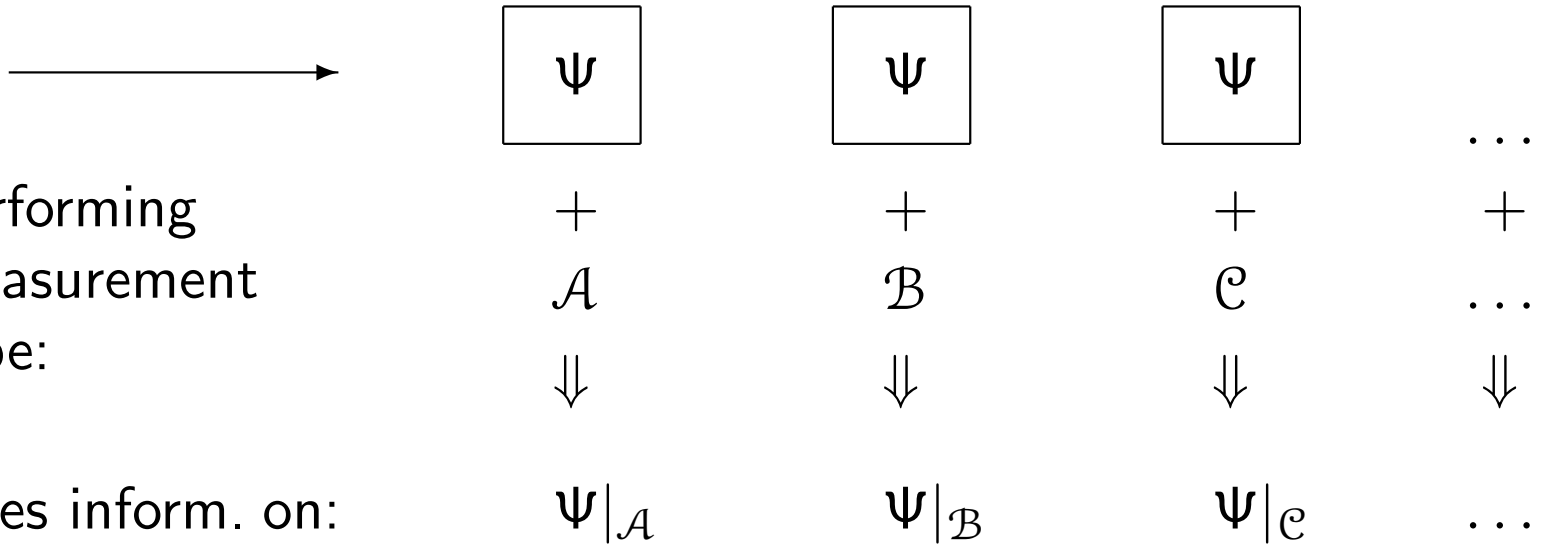

The best is to choose $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ so that they are statistically independent. Moreover, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ is a **complete set of measurables** $\Leftrightarrow$ they span $\mathcal{B}(\mathcal{H})$.

Historical remarks

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- a new indication: consider the problem of finding a complete set of quasi-orthogonal copies of $\mathbb{C}^n$ in $\mathbb{C}^{n^2}$

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Some even resolve some problems — but they are not the mainstream.

Non-classical measurements on a q-system

commutative C^*-subalgebra $\Leftrightarrow$ "classical" measurement
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$\Rightarrow$ study quasi-orthogonal copies of $M_2(\mathbb{C})$ in $\otimes_k M_2(\mathbb{C})$

The case of a 2 q-bits long register

$$\frac{\dim(M_2(\mathbb{C}) \otimes M_2(\mathbb{C})) - 1}{\dim(M_2(\mathbb{C})) - 1} = \frac{4^2 - 1}{2^2 - 1} = 5,$$

so by dimensional reasons, there could be quasi-orthogonal copies of $M_2(\mathbb{C})$ in $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \simeq M_4(\mathbb{C})$. Less then 5 copies do not form a complete set.

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Obstruction found: if $\mathcal{A}$ and $\mathcal{B}$ are quasi-orthogonal copies of $M_2(\mathbb{C})$ in $M_4(\mathbb{C})$, then

$$\mathcal{A}' \cap \mathcal{B} \neq \mathbb{C}\mathbb{1}$$

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- construction relying on (tensor products of) Pauli-matrices $\rightsquigarrow$ show that the "defect number" is always ≤ 1
- conjecture that it is actually 1
- but unless $k = 2$, they could not find an obstruction

Further questions

- q-computer with "n-level bits" sends an n-level bit, (or a "normal" q-computer sends a packet containing more than 1 bit)

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- q-computer with "n-level bits" sends an n-level bit, (or a "normal" q-computer sends a packet containing more than 1 bit) $\rightsquigarrow$ quasi-orthogonal copies of $M_n(\mathbb{C})$ in $\otimes_k M_n(\mathbb{C})$?
- relationship between MUB and the new questions?

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Proposition (First generalisation)

There exists a collection of $N(n) + 1$ quasi-orthogonal copies of $M_n(\mathbb{C})$ in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$

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Proposition (First generalisation)

There exists a collection of $N(n) + 1$ quasi-orthogonal copies of $M_n(\mathbb{C})$ in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$

Proposition (Second generalisation)

For $n = p$, there exists a collection of n^2 quasi-orthogonal copies of $M_n(\mathbb{C})$ in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$.

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Example is found for $\mathcal{A}, \mathcal{B} \subset M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$ two copies of $M_3(\mathbb{C})$ such that $\mathcal{A}, \mathcal{B}$ and $\mathcal{A}'$ are all quasi-orthogonal.

Constructions for $n =$ prime number

Let $e_1, \dots e_p$ an ONB, and define $X, Z \in \mathcal{B}(\mathcal{H})$ by

$$Xe_j = e_{j+1}, \quad Ze_j = e^{i\frac{2\pi}{p}j} e_j.$$

Then $XX^* = ZZ^* = X^p = Z^p = \mathbb{1}$, $ZX = e^{i\frac{2\pi}{p}} XZ$ and the unitary operators $X^\alpha Z^\beta$ where $(\alpha, \beta) \in \mathbb{F}_p^2$, form an ONB in $M_p(\mathbb{C})$.

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Generalisation: the unitary operators

$$W_{(\alpha,\beta)} := X^{\alpha_1} Z^{\beta_1} \otimes X^{\alpha_2} Z^{\beta_2} \otimes \dots \otimes X^{\alpha_k} Z^{\beta_k}$$

where $(\alpha, \beta) \in \mathbb{F}_p^{2k}$, form an ONB in $\otimes_k M_p(\mathbb{C})$.

With $B((\alpha,\beta),(\eta,\zeta)) := \sum_{j=1}^{k} \det \begin{pmatrix} \alpha_j & \beta_j \\ \eta_j & \zeta_j \end{pmatrix}$ bilinear form,

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Lemma

If $V \subset \mathbb{F}_p^{2k}$ is an $\mathbb{F}_p$-linear subspace, then

$$\mathcal{A}(V) := \mathrm{Span}\{W_{(\alpha,\beta)} \,|\, (\alpha,\beta) \in V\}$$

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- $\mathcal{A}(V_1)$ is quasi-orthogonal to $\mathcal{A}(V_2)$ iff $V_1 \cap V_2 = \{0\}$,
- $\mathcal{A}(V) \simeq M_p(\mathbb{C})$ iff $\dim(V) = 2$ and $B|_{V\times V} \neq 0$

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So the problem is reduced to finite geometry. (This is how that "second generalisation" was also obtained.) Further results:

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Proposition

The "deficit number" of $M_3(\mathbb{C})$ in $\otimes_k M_3(\mathbb{C})$ is ≤ 1.