Approximation of a Class of Operators

V.E.S. Szabó

Department of Analysis, Budapest University of Technology and Economics, Hungary

July 8, 2008

MATEMATIKA INTÉZET
BUDAPESTI MŰSZAKI ÉS GAZDASÁGTUDOMÁNYI EGYETEM

Outline

Outline

System of Differential Equations.

Solve It.

$$\begin{aligned}\dot{x}_1(t) &= -te^{-t^2}x_1(t) + \frac{\sin(t)}{2t}x_2(t),\\ \dot{x}_2(t) &= \frac{\sin(t)}{2t}x_1(t) + \left(-te^{-t^2} + \frac{\sin(t)}{200t}\right)x_2(t); \quad t \in (0, \infty).\end{aligned}$$

Outline

Longstanding Approaches.

Symbolical Approximation.

Liouville-Green (WKBJ).

Numerical Approximation.

Runge-Kutta, e.t.c.

Longstanding Approaches.

Symbolical Approximation.

Liouville-Green (WKBJ).

Numerical Approximation.

Runge-Kutta, e.t.c.

Outline

Reformulating of the Problem.

Matrix notation.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),$$

where

- $\dot{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,
- $\mathbf{A}(t) = \begin{bmatrix} -te^{-t^2} & \frac{\sin(t)}{2t} \\ \frac{\sin(t)}{2t} & -te^{-t^2} + \frac{\sin(t)}{200t} \end{bmatrix}$.

Reformulating of the Problem.

Matrix notation.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t),$$

where

- $\dot{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$
- $\mathbf{A}(t) = \begin{bmatrix} -te^{-t^2} & \frac{\sin(t)}{2t} \\ \frac{\sin(t)}{2t} & -te^{-t^2} + \frac{\sin(t)}{200t} \end{bmatrix}.$

Outline

Redefinition.

Associated Matrix.

$$\mathbf{B}(t) := \begin{bmatrix} \dfrac{-e^{-t^2}}{2} & \dfrac{1}{2}\int_0^t \dfrac{\sin(u)}{u}\,du \\ \int_0^t \dfrac{\sin(u)}{2u}\,du & \dfrac{-e^{-t^2}}{2} + \dfrac{1}{200}\int_0^t \dfrac{\sin(u)}{u}\,du \end{bmatrix}.$$

Rewriting the Equation.

Since $\mathbf{A}(t) = \dot{\mathbf{B}}(t)$ we have

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{B}}(t)\mathbf{x}(t).$$

Redefinition.

Associated Matrix.

$$\mathbf{B}(t) := \begin{bmatrix} \frac{-e^{-t^2}}{2} & \frac{1}{2}\int_0^t \frac{\sin(u)}{u}du \\ \int_0^t \frac{\sin(u)}{2u}du & \frac{-e^{-t^2}}{2} + \frac{1}{200}\int_0^t \frac{\sin(u)}{u}du \end{bmatrix}.$$

Rewriting the Equation.

Since $\mathbf{A}(t) \equiv \dot{\mathbf{B}}(t)$ we have

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{B}}(t)\mathbf{x}(t).$$

Exact solution.

Since $\left[\mathbf{B}, \dot{\mathbf{B}}\right] \equiv \mathbf{0}$ we have

$$\mathbf{x}(t) \equiv e^{\mathbf{B}(t)}\mathbf{c}.$$

Spectrum estimation.

Since $\mathbf{B}^T \equiv \mathbf{B}$ the spectrum is real and for the spectral radius

$$r(\mathbf{B}) = \|\mathbf{B}\| \leq \|\mathbf{B}\|_F \leq 1.$$

Exact solution.

Since $\left[\mathbf{B}, \dot{\mathbf{B}}\right] \equiv \mathbf{0}$ we have

$$\mathbf{x}(t) \equiv e^{\mathbf{B}(t)}\mathbf{c}.$$

Spectrum estimation.

Since $\mathbf{B}^T \equiv \mathbf{B}$ the spectrum is real and for the spectral radius

$$r(\mathbf{B}) = \|\mathbf{B}\| \leq \|\mathbf{B}\|_F \leq 1.$$

Spectral decomposition.

$$e^{\mathbf{B}(t)} \equiv \int_{-1}^{1} e^{\lambda}\, dE(\lambda, t).$$

Approximation.

Let p be a polynomial of degree at most n such that

$$\max_{-1\le\lambda\le 1} \left| e^{\lambda} - p(\lambda) \right| \le c\,\omega\left(\exp, \frac{1}{n}\right).$$

Then

$$\left\| \mathbf{x}(t) - p(\mathbf{B}(t))\mathbf{c} \right\| = \left\| e^{\mathbf{B}(t)}\mathbf{c} - p(\mathbf{B}(t))\mathbf{c} \right\| \le \|\mathbf{c}\|\, c\,\omega\left(\exp, \frac{1}{n}\right).$$

Spectral decomposition.

$$e^{\mathbf{B}(t)} \equiv \int_{-1}^{1} e^{\lambda} dE(\lambda, t).$$

Approximation.

Let p be a polynomial of degree at most n such that

$$\max_{-1 \leq \lambda \leq 1} |e^{\lambda} - p(\lambda)| \leq c\omega \left(\exp, \frac{1}{n}\right).$$

Then

$$\|\mathbf{x}(t) - p(\mathbf{B}(t))\mathbf{c}\| \equiv \left\| e^{\mathbf{B}(t)}\mathbf{c} - p(\mathbf{B}(t))\mathbf{c} \right\| \leq \|\mathbf{c}\| \, c\omega \left(\exp, \frac{1}{n}\right).$$

Summary

- To find approximate solution of a class of linear o.d.e. systems we introduced a new method.

- Outlook
  - We need to extend the method for a wider class of linear o.d.e. systems.
  - We have to investigate the approximation of the exponential function.

Happy Birthday to Jóska!

Thanks for your attention!