Sample exercises from the 2nd part of the course

In the first 2 exercises capital letters like A and B stand for (real-valued) physical quantities of an *n*-level quantum physical system, and capital letters with "hats" like \hat{A} and \hat{B} denote the corresponding self-adjoint operators acting on the $n < \infty$ dimensional Hilbert space \mathcal{H} associated to the system.

1. Suppose C = A + B in the statistical sense, and that in a certain (fixed) ensemble A = a with probability 1. Show that $E(C^2) = a^2 + 2aE(B) + E(B^2)$ where "E" stands for the expected value in this ensemble. Decide if it also follows that $E(C^3) = a^3 + 3a^2E(B) + 3aE(B^2) + E(B^3)$.

- **2.** Consider the following properties:
- (i) $p(``A = a'') = 1 \implies E_p(C) = aE_p(B),$
- (ii) $p(``A = a'') = 1 \implies p(``C = c'') = p(``aB = c'').$

(Here p stands for a generic probability function, and E_p denotes the corresponding expected value.) If A and B can be measured together and C = AB (in the usual sense), then both (i) and (ii) hold. So in case A and B do not necessarily have a simultaneous meaning, we could try to give some meaning to the equation C = AB by requiring property (i) or (ii). Discuss existence and uniqueness of a C satisfying property (k) for both k=i and k=ii in the concrete example in which \hat{A} and \hat{B} are the self-adjoint matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

respectively.

3. A bipartite system described by the tensor-product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is in a pure state. Show that the partial states are either both pure or both not pure.

4. A system of 3 spin 1/2 particles is in the state given by the unit vector

$$\Psi = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle).$$

Calculate the partial states of the following couples: first two, last two, and first + last spins. Are all spins entangled with the other two?

5. A system of 3 spin 1/2 particles is in a pure state. Is it possible, that there is no entanglement between the first and the second spin, and neither between the first and third, but that there is entanglement between the first and the rest?

6. A system of 4 spin 1/2 particles is in the state given by the unit vector $|\uparrow\uparrow\uparrow\uparrow\rangle$. An invertable physical operation is performed on the first two spins, then another one on the couple composed of the second and third spins, and finally, one more on the couple composed of the third and fourth spins. Can

$$\Psi = \frac{1}{\sqrt{3}} (|\uparrow\uparrow\uparrow\uparrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$$

be the unit vector describing the final state? If yes, give an example for the operations resulting Ψ . If no, explain why not.