Sample exercises from the 1st part of the course

1. Let $U := \{x \in \mathbb{C}^3 \,|\, x_1 + ix_2 = 0\}$ and $W := \mathrm{Span}\{w\} \subset \mathbb{C}^3$ where $w_1 = w_2 = w_3 = 1$. Verify that U and W are complementary and calculate the matrix (in the std. basis) of the projection onto U along W.

2. Let P and Q be two projections of the vector space V. Prove that $P + Q$ is again a projection iff $\mathrm{Im}(P) \subset \mathrm{Ker}(Q)$ and $\mathrm{Ker}(P) \supset \mathrm{Im}(Q)$.

3. Let $x_1, x_2, \ldots, x_{n+1} \in V$ be $n+1$ vectors in the scalar product space V and assume that $\|x_k\| = 1$ and $|\langle x_k, x_j\rangle| < 1/n$ for all $k, j = 1, \ldots, n+1$. Show that in this case $x_1, \ldots, x_{n+1}$ must be linearly independent.

4. Let V be a finite dimensional vector space, and suppose that both $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ is a scalar product on V. Prove that there exists a bases $\mathcal{B} = (b_1, \ldots, b_n)$ such that the vectors of $\mathcal{B}$ are pairwise orthogonal with respect to both scalar products.

5. Let $A \in \mathcal{B}(\mathcal{H})$ be such $A^2 = 0$ and $AA^* + A^*A = \mathbb{1}$. Show that $\mathrm{Sp}(A + A^*) = \{1, -1\}$ and that the dimensions of the two eigenspaces of $A + A^*$ are equal.

6. Show that for an operator U on a finite dimensional Hilbert space $\mathcal{H}$ the following three properties are equivalent:

- U is invertable and $\|U\| = \|U^{-1}\| = 1$
- $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$
- U is a unitary operator.

7. Let A, B be two self-adjoint operators. Show that $\mathrm{Tr}((AB)^2) \in \mathbb{R}$ and $\mathrm{Tr}(A^2B^2) \in \mathbb{R}^+ \cup \{0\}$ and that

$$\mathrm{Tr}((AB)^2) \leq \mathrm{Tr}(A^2B^2)$$

with equality holding if and only if A and B commute.

8. Let A, B be two positive operators. Prove that $\mathrm{Ker}(A + B) = \mathrm{Ker}(A) \cap \mathrm{Ker}(B)$.

9. Show, by example, that if A, B are positive operators such that $A \geq B$, then it does *not* follow that $A^2 \geq B^2$. Decide if on the other hand the (weaker) inequality $\mathrm{Tr}(A^2) \geq \mathrm{Tr}(B^2)$ follows.