

1 Pairwise quasi-orthogonal copies of $\mathcal{B}(H)$ in $\mathcal{B}(H) \otimes \mathcal{B}(H)$

Let \mathcal{H} and \mathcal{K} be two finite dimensional Hilbert spaces, with dimensions n and m , respectively. Fix a unitary $U \in \mathcal{B}(\mathcal{H})$ with the property that $U^m = \mathbf{1}$. Moreover, fix a base $\mathcal{E} = (e_1, \dots, e_m)$ in \mathcal{K} and consider the linear operators $E_{k,l}^{\mathcal{E}}$ ($k, l \in \{1, \dots, m\}$) defined by the formula

$$E_{k,l}^{\mathcal{E}} e_r = \delta_{l,r} e_k. \quad (1)$$

These operators form a complete set of so-called *matrix units*. We shall now use them to define a new set of matrix units. In fact, considering the elements of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ given by the formula

$$\tilde{E}_{k,l}^{U,\mathcal{E}} := U^{k-l} \otimes E_{k,l}^{\mathcal{E}} \quad (2)$$

(where $k, l \in \{1, \dots, m\}$), one has the relations

$$\tilde{E}_{k,l}^{U,\mathcal{E}} \tilde{E}_{r,s}^{U,\mathcal{E}} = \delta_{l,r} \tilde{E}_{k,s}^{U,\mathcal{E}}, \quad (\tilde{E}_{k,l}^{U,\mathcal{E}})^* = \tilde{E}_{l,k}^{U,\mathcal{E}}, \quad \sum_{k=1}^m \tilde{E}_{k,k}^{U,\mathcal{E}} = \mathbf{1}. \quad (3)$$

Thus the subspace spanned by these m^2 operators is a \mathcal{C}^* algebra which is in fact isomorphic to the algebra of m by m matrices; or, to put it in another way, this subalgebra of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ is of the form $W(\mathbf{1} \otimes \mathcal{B}(\mathcal{K}))W^*$, where $W \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ is a unitary operator.

In what follows, Hilbert spaces are always considered to be complex, and given a base $\mathcal{E} = \{e_1, \dots, e_m\}$ we shall use the symbol $\mathcal{A}^{\mathcal{E}}$ to denote the maximal abelian subalgebra isomorphic to \mathbb{C}^n , which is given by the linear operators that are diagonal in the base \mathcal{E} . In this algebra we may fix a unitary operator $Z \in \mathcal{A}^{\mathcal{E}}$ such that $Z^m = \mathbf{1}$ and its powers form an orthonormal base of $\mathcal{A}^{\mathcal{E}}$. Indeed, the linear operator given by the formula $Ze_k := e^{i\frac{2\pi}{m}k} e_k$ satisfies

- $Z \in \mathcal{A}^{\mathcal{E}}, \quad ZZ^* = \mathbf{1} = Z^m,$
- $\langle Z^k, Z^l \rangle = \tau(Z^{-k}Z^l) = \sum_{k=1}^m e^{i\frac{2\pi}{m}(l-k)} = \delta_{k,l}$ for all $k, l \in \{1, \dots, m\}$.

So let now $\mathcal{K} = \mathcal{H}$ and hence $m = n$, and consider a base \mathcal{E} of \mathcal{H} . Then, we can fix a unitary operator Z associated to the base \mathcal{E} in the above explained way, and considering the operators $\tilde{E}_{k,l}^{Z,\mathcal{E}} = Z^{k-l} \otimes E_{k,l}^{\mathcal{E}}$ we can define

$$\mathcal{M}^{\mathcal{E}} := \text{Span}\{\tilde{E}_{k,l}^{Z,\mathcal{E}} \mid k, l = 1, \dots, m\}, \quad (4)$$

which — as it was discussed — is a sub \mathcal{C}^* -algebra of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ which is isomorphic to $\mathcal{B}(H)$. Actually, we can state the followings.

Proposition 1.1. *Let E_1, \dots, E_N be a set of pairwise “mutually unbiased” bases in \mathcal{H} (i.e. such that $\mathcal{A}_{\mathcal{E}_k}$ ($k = 1, \dots, N$) are pairwise quasi-orthogonal). Then for all $\alpha, \beta \in \{1, \dots, N\}$*

- \mathcal{M}^{E_α} is quasi-orthogonal to $\mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$,
- $\mathcal{M}^{E_\alpha} \cap \mathbf{1} \otimes \mathcal{B}(\mathcal{H}) = \mathbf{1} \otimes \mathcal{A}^{\mathcal{E}_\alpha}$,
- \mathcal{M}^{E_α} and \mathcal{M}^{E_β} are quasi-orthogonal whenever $\alpha \neq \beta$.

Proof. We shall denote by Z_α the unitary associated (in the previously discussed way) to the base \mathcal{E}_α . Let $A \in \mathcal{B}(\mathcal{H})$ and $k, l \in \{1, \dots, n\}$. Then, using that the trace of the tensor product of operators is the product of the traces, and treating separately the cases $k = l$ and $k \neq l$, one can easily check that

$$\tau(AZ_\alpha^{k-l} \otimes E_{k,l}^{\mathcal{E}_\alpha}) = \tau(AZ_\alpha^{k-l}) \tau(E_{k,l}^{\mathcal{E}_\alpha}) = \tau(A \otimes \mathbf{1}) \tau(Z_\alpha^{k-l} \otimes E_{k,l}^{\mathcal{E}_\alpha}), \quad (5)$$

since for $k \neq l$ the trace of the matrix-unit $E_{k,l}^{\mathcal{E}_\alpha}$ is zero. Thus we obtain that

$$\tau((A \otimes \mathbf{1})(Z_\alpha^{k-l} \otimes E_{k,l}^{\mathcal{E}_\alpha})) = \tau(A \otimes \mathbf{1}) \tau(Z_\alpha^{k-l} \otimes E_{k,l}^{\mathcal{E}_\alpha}) \quad (6)$$

which implies that $\mathcal{M}^{\mathcal{E}_\alpha}$ is quasi-orthogonal to $\mathcal{B}(H) \otimes \mathbf{1}$. As for the second claim, consider an arbitrary element of $C \in \mathcal{M}^{\mathcal{E}_\alpha}$; that is, a linear combination of the form $C = \sum_{k,l} c_{k,l} Z_\alpha^{k-l} \otimes E_{k,l}^{\mathcal{E}_\alpha}$. We want to figure out the condition on the coefficients that will make C be in $\mathbf{1} \otimes \mathcal{B}(\mathcal{H})$, too. Rearranging the sum we have that

$$C = \sum_k \left(Z^k \otimes \left(\sum_l c_{l,k+l} E_{l,k+l}^{\mathcal{E}_\alpha} \right) \right) \quad (7)$$

where in the summation k, l still goes from 1 to n , but the $k + l$ appearing in the indices is meant modulo n . (Recall that $Z^n = \mathbf{1}$.) Then, as $\mathbf{1}$ is not included (infact orthogonal) to the subspace spanned by the powers Z^k for $k = 1, \dots, n - 1$, it follows that $C \in \mathbf{1} \otimes \mathcal{B}(\mathcal{H})$ implies that $\sum_l c_{l,k+l} E_{l,k+l}^{\mathcal{E}_\alpha}$ must be zero for all $k = 1, \dots, n - 1$. However, the matrix units are linearly independent, so this in turn implies that $c_{k,l} = 0$ whenever $k \neq l$. This shows that $\mathcal{M}^{E_\alpha} \cap \mathbf{1} \otimes \mathcal{B}(\mathcal{H}) \subset \mathbf{1} \otimes \mathcal{A}^{\mathcal{E}_\alpha}$. The inclusion in the other direction is trivial, since $\mathcal{A}^{\mathcal{E}_\alpha}$ is the linear span of the set of operators $\{E_{k,k}^{\mathcal{E}_\alpha} \mid k = 1, \dots, n\}$.

Finally, we shall justify the last claim by showing that for $\alpha \neq \beta$ and k, l, r, s , arbitrary we have

$$\tau\left((Z_\alpha^{k-l} \otimes E_{k,l}^{\mathcal{E}_\alpha})(Z_\beta^{r-s} \otimes E_{r,s}^{\mathcal{E}_\beta})\right) = \tau(Z_\alpha^{k-l} \otimes E_{k,l}^{\mathcal{E}_\alpha})\tau(Z_\beta^{r-s} \otimes E_{r,s}^{\mathcal{E}_\beta}) \quad (8)$$

This can be done by using the conditions of the proposition (and some well-known identities, e.g. that the trace of the tensor product of operators is the product of the traces) and treating the following two cases separately: 1. either $k \neq l$ or $r \neq s$, 2. both $k = l$ and $r = s$. Indeed, in the first case Z_α^{k-l} (and its adjoint, too) is orthogonal to Z_β^{r-s} , and hence both sides of the previous equation turn out to be zero. In the second case, $Z_\alpha^{k-l} = Z_\beta^{r-s} = \mathbf{1}$. Moreover, $E_{k,l}^{\mathcal{E}_\alpha} = E_{k,k}^{\mathcal{E}_\alpha} \in \mathcal{A}^{\mathcal{E}_\alpha}$ and $E_{r,s}^{\mathcal{E}_\beta} = E_{r,r}^{\mathcal{E}_\beta} \in \mathcal{A}^{\mathcal{E}_\beta}$; thus the two matrix units in question “live” in quasi-orthogonal subalgebras. Putting these facts together it is easy to check the equality in the second case, too. \square

Corollary 1.2. *If E_1, \dots, E_N are pairwise mutually unbiased bases in \mathcal{H} , then the following collection of subalgebras: $\mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$, $\mathcal{M}^{\mathcal{E}_\alpha}$ ($\alpha = 1, \dots, N$) are pairwise quasi-orthogonal. Hence if there are N mutually unbiased bases in \mathcal{H} , then there are at least $N + 1$ pairwise quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$.*

We shall end this section with some comments. In all comments n stands for the dimension of the Hilbert space \mathcal{H} .

The subspace of $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ orthogonal to $\mathbf{1}$ is $n^4 - 1$ dimensional, whereas the subspace of $\mathcal{B}(\mathcal{H})$ orthogonal to $\mathbf{1}$ is $n^2 - 1$ dimensional, thus by dimensional reasons, $\frac{n^4-1}{n^2-1} = n^2 + 1$ is an upper bound on the number of quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$.

There are at most $n + 1$ mutually unbiased bases in \mathcal{H} ; thus with the above construction at most we can present $n + 2$ pairwise mutually quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$. So there is a huge gap between the number achieved by our construction, and the upper bound given by the simple dimensional argument.

Infact, in the next section we shall give a much better construction — but only for the case when n is a prime number (though probably with some work that argument could be generalized for any prime-power.) What makes the construction presented in this section interesting is that its starting point is a collection of mutually unbiased bases, and thus it connects two problems. Note that it is exactly in dimensions which are not prime-powers, where we do not know the maximal number of mutually unbiased bases.

2 Pairwise quasi-orthogonal copies of $\mathcal{B}(H)$ in $\mathcal{B}(H) \otimes \mathcal{B}(H) \otimes \dots \otimes \mathcal{B}(\mathcal{H})$

Let \mathcal{H} be a finite dimensional (complex) Hilbert space, and consider the k -fold tensor product

$$\otimes_k \mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \otimes \dots \otimes \mathcal{B}(\mathcal{H}). \quad (9)$$

The traceless parts of pairwise quasi-orthogonal subalgebras must be linearly independent (as they are in particular pairwise orthogonal with respect to the Hilbert-Schmidt inner product.) Thus by simple dimensional arguments, the cardinality of any set of pairwise quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ in the algebra $\otimes_k \mathcal{B}(\mathcal{H})$, must be smaller or equal than

$$\frac{n^{2k} - 1}{n^2 - 1} = n^{2k-2} + n^{2k-4} + \dots + 1 \quad (10)$$

where n stands for the dimension of the Hilbert space \mathcal{H} . However, this upper bound is not necessarily optimal: for $n = 2, k = 2$ the dimensional upper bound is 5, but — as was already mentioned — the maximum number is only 4. So let us introduce the “deficit” number by the formula

$$\text{“deficit”} := \text{“dimensional upper bound”} - \text{“actual maximum”}. \quad (11)$$

We shall denote the deficit number by $D(k, n)$, as it may depend on the positive integers k and n . Let us summarize, what we know about it.

Apart from $D(1, n)$ (which is trivially zero), the only known value — as was mentioned — is $D(2, 2)$, which is 1. In [?] it is shown, that $D(k, n) \leq 1$ for all positive integers k . However, looking at the argument given in the mentioned article, one can see that what they show is slightly stronger. The algebra obtained by the $k + 1$ -th tensor product, as a vector space, can be considered to be the sum of the following two orthogonal subspaces:

$$\otimes_{k+1} \mathcal{B}(\mathcal{H}) = \left(\otimes_k \mathcal{B}(\mathcal{H}) \otimes \mathbf{1} \right) \oplus \left(\otimes_k \mathcal{B}(\mathcal{H}) \otimes \mathbf{1} \right)^\perp \quad (12)$$

Trying to relate the $k + 1$ -fold tensor product to the k -fold tensor product, at the above decomposition we may think of the first subspace as the “old

part”, and of the second one as the “new part”. In case of $n = 2$, what is actually shown is that the for $k \geq 2$

$$\text{“new part”} + \mathbb{C}\mathbf{1} = \text{sum of quasi-orthogonal copies of } \mathcal{B}(\mathcal{H}), \quad (13)$$

i.e. the new part can be divided into quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ with no deficit, and hence that $D(k+1, 2) \leq D(k, 2)$. Here we shall generalize this argument and show that it holds for all $n = p$ prime numbers. Using this, together with an explicit construction regarding the case of the 2-fold tensor product, we will then conclude that $D(k, p) \leq 1$ for all $k \leq 1$.

So let us assume that $\dim(\mathcal{H}) = p$, and fix a base (e_1, \dots, e_p) in \mathcal{H} . Consider the linear operators Z, X given by the formulas

$$Ze_j = e^{i\frac{2\pi}{p}j}e_j, \quad Xe_j = e_{j+1} \quad (14)$$

where of course the $j+1$ in the index is meant modulo p . Both of these operators are unitary. Moreover, they satisfy the relations

$$Z^p = X^p = \mathbf{1}, \quad XZ = e^{i\frac{2\pi}{p}}ZX, \quad (15)$$

and the operators $\{X^\alpha Z^\beta \mid (\alpha, \beta) \in \mathbb{F}_p^2\}$ form an orthonormed base in $\mathcal{B}(\mathcal{H})$. Note that as $Z^p = X^p = \mathbf{1}$, instead of thinking of the exponents as integers, we have chosen to think of them as elements of the finite field \mathbb{F}_p .

In order to move on to the case of the k -fold tensor product, we shall consider the unitary operators

$$W_{(\alpha, \beta)} = X^{\alpha_1} Z^{\beta_1} \otimes X^{\alpha_2} Z^{\beta_2} \otimes \dots \otimes X^{\alpha_k} Z^{\beta_k}, \quad (16)$$

where $(\alpha, \beta) \in \mathbb{F}_p^{2k}$. Again, it is easy to see, that $\{W_{(\alpha, \beta)} \mid (\alpha, \beta) \in \mathbb{F}_p^{2k}\}$ is an orthonormed base in $\otimes_k \mathcal{B}(\mathcal{H})$. To understand the multiplicative structure, we have to work out the product of two such unitary operators. By straightforward calculation, we obtain the following “Weyl-like” relation.

Lemma 2.1. *With the introduced notations,*

$$W_{(\alpha, \beta)} W_{(\eta, \zeta)} = e^{-i\frac{2\pi}{p} \sum_{j=1}^k \beta_j \eta_j} W_{(\alpha, \beta) + (\eta, \zeta)} = e^{i\frac{2\pi}{p} B((\alpha, \beta), (\eta, \zeta))} W_{(\eta, \zeta)} W_{(\alpha, \beta)}$$

where $B : \mathbb{F}_p^{2k} \times \mathbb{F}_p^{2k} \rightarrow \mathbb{F}_p$ is the bilinear form given by the formula

$$B((\alpha, \beta), (\eta, \zeta)) = \sum_{j=1}^k \det \begin{pmatrix} \alpha_j & \beta_j \\ \eta_j & \zeta_j \end{pmatrix}.$$

Let us see now how we can find quasi-orthogonal subalgebras using the “ W ” operators.

Lemma 2.2. *If $V \subset \mathbb{F}_p^{2k}$ is an \mathbb{F}_p -linear subspace, then*

$$\mathcal{A}(V) := \text{Span}\{W_{(\alpha,\beta)} \mid (\alpha,\beta) \in V\}$$

is a \mathbb{C}^ subalgebra of $\otimes_k \mathcal{B}(\mathcal{H})$ of dimension $p^{\dim(V)}$. Moreover,*

- (i) $\mathcal{A}(V)$ is Abelian if and only if $B|_{V \times V} = 0$,
- (ii) $\mathcal{A}(V_1)$ is quasi-orthogonal to $\mathcal{A}(V_2)$ if and only if $V_1 \cap V_2 = \{0\}$,
- (iii) $\mathcal{A}(V) \simeq \mathbb{C}^{p^r}$ if and only if $\dim(V) = r$ and $B|_{V \times V} = 0$.
- (iv) $\mathcal{A}(V) \simeq \mathcal{B}(\mathcal{H})$ if and only if $\dim(V) = 2$ and $B|_{V \times V} \neq 0$.

Proof. By the product formula given in Lemma 2.1 we have that $W_{(\alpha,\beta)}W_{(\eta,\zeta)}$ is proportional to $W_{(\alpha,\beta)+(\eta,\zeta)}$. Since the “ W ” operators are unitary and $W_{(0,0)} = \mathbf{1}$, it also follows that $W_{(\alpha,\beta)}^*$ is proportional to $W_{-(\alpha,\beta)}$. For an \mathbb{F}_p -linear subspace $V \subset \mathbb{F}_p^{2k}$ the above facts show that $\mathcal{A}(V)$ is a \mathbb{C}^* subalgebra. Moreover, as the “ W ” operators are linearly independent (infact they are pairwise orthogonal), the dimension of $\mathcal{A}(V)$ is just simply the cardinality of V , which is $p^{\dim(V)}$.

Claim (i) and the “only if” part of claim (iv) is again just a trivial consequence of our product formula, whereas claim (ii) follows from the fact that the “ W ” operators are pairwise orthogonal. Claim (iii) follows from the so-far obtained properties since the only abelian \mathbb{C}^* algebra of dimension p^r is (isomorphic to) \mathbb{C}^{p^r} .

There remains to prove the “if” part of claim (iv). So let $V \subset \mathbb{F}_p^{2k}$ be a 2-dimensional \mathbb{F}_p -linear subspace, and suppose that $B|_{V \times V} \neq 0$. Then there exists a pair of elements $x, y \in V$ such that $B(x, y) \neq 0$; infact we may assume that $B(x, y) = 1$ (if it was originally not true, we may redefine x as $x := x_{\text{old}}/B(x_{\text{old}}, y)$ to obtain such a pair). By the antisymmetry of B , x and y must be linearly independent, and hence they must form a base in V . Moreover, using our product formula we have that $W_x W_y = e^{i\frac{2\pi}{p}} W_y W_x$. Hence by what was said it is easy to see, that the map

$$X \mapsto W_x, \quad Z \mapsto W_z \tag{17}$$

extends to a \mathbb{C}^* isomorphism between $\mathcal{B}(\mathcal{H})$ and $\mathcal{A}(V)$. \square

The above lemma enables us to apply arguments coming from finite geometry. As is well known, in a $2k$ -dimensional vector space, up to isomorphism, there is a unique non-degenerate antisymmetric bilinear form. (Note that B is non-degenerate). The *symplectic polar space* is the $2k - 1$ -dimensional finite geometry, whose points, lines, planes, ect. are the 1-dimensional, 2-dimensional, 3-dimensional ect. subspaces of our vectorspace on which B vanishes (with the incidence-structure given by the inclusion).

A well-studied question regarding polar spaces is the existence of *spreads*. A μ -spread is a partitioning of the polar space into disjoint μ -dimensional hyperplanes. By the above lemma, each spread gives us a way to “decompose” $\otimes_k \mathcal{B}(\mathcal{H})$ into quasi-orthogonal abelian subalgebras; i.e. to give a set of quasi-orthogonal abelian subalgebras whose span is the full algebra.

In the $2k - 1$ -dimensional polar space a hyperplane can be at most $k - 1$ -dimensional. By the theorem of [1], the $2k - 1$ dimensional symplectic polar space always admits a $k - 1$ -spread. This shows that $\otimes_k \mathcal{B}(\mathcal{H})$ can always be decomposed into quasi-orthogonal maximal abelian subalgebras; i.e. that above \mathbb{C} , in $n = p^k$ dimension, there indeed exist $n + 1$ *mutually unbiased bases*. This was of course well-known [2, 3, 4]. [5] EZ MOST EGY UJ KONSTRUKCIO [6] However, there are several other type of spreads that are known to exist. What we shall actually use here is that in a symplectic polar space there always exists a 1-spread, see [7, 8]. This means, that in we can always find a collection of pairwise independent (i.e. any pair of them intersects in the zero, only) 2-dimensional subspaces such that on each of them B vanishes, and there union is the full space.

Theorem 2.3. *Let $\dim(H) = p$ be a prime, and $k \geq 2$. Then the “new part” of $\otimes_{k+1} \mathcal{B}(\mathcal{H})$ can be divided into quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ without deficit, i.e. $(\otimes_k \mathcal{B}(\mathcal{H}) \otimes \mathbf{1})^\perp \oplus \mathbf{1}$ is the span of quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$. Thus the deficit cannot grow: $D(k + 1, p) \leq D(k, p)$.*

Proof. Since the dimension of $(\otimes_k \mathcal{B}(\mathcal{H}) \otimes \mathbf{1})^\perp$ is $p^{2(k+1)} - p^{2k}$ whereas the dimension of the traceless part of $\mathcal{B}(\mathcal{H})$ is $p^2 - 1$, the statement can be proved by exhibiting

$$\frac{p^{2(k+1)} - p^{2k}}{p^2 - 1} = p^{2k} \tag{18}$$

pairwise quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ such that each of them is quasi-orthogonal to $\otimes_k \mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$. For $p = 2$ this has been done in [9], so we may assume that $p \geq 3$. Then the square function $\mathbb{F}_p \rightarrow \mathbb{F}_p, s \mapsto s^2$ is not a

bijection, since $1^2 = (-1)^2 = 1$. Hence \mathbb{F}_p has at least one element, say $t \in \mathbb{F}_p$, which is not a square. Note that the fact that t is not a square implies that for two elements $\alpha, \beta \in \mathbb{F}_p$, the equality $\alpha^2 - t\beta^2 = 0$ holds if and only if $\alpha = \beta = 0$.

As was explained, we know that there exist collection of 2-dimensional \mathbb{F}_p -linear subspaces $V_1, V_2, \dots, V_r \subset \mathbb{F}_p^{2k}$ where $r = \frac{p^{2k}-1}{p^2-1}$, such that their union is the full space \mathbb{F}_p^{2k} , $B|_{V_j \otimes V_j} = 0$ and $V_j \cap V_l = \{0\}$ for all $j \neq l$. Let us fix a base (x_j, y_j) in each subspace V_j ($j = 1, \dots, r$), and let $\tilde{V} := \{0\} \times \mathbb{F}_p^2 \subset \mathbb{F}_p^{2(k+1)}$ and for each $j = 1, \dots, r$ and $(\alpha, \beta) \in \mathbb{F}_p^2$, $(\alpha, \beta) \neq 0$, let

$$\tilde{V}_{j,(\alpha,\beta)} := \text{Span} \left\{ \tilde{x}_{j,(\alpha,\beta)} := \begin{pmatrix} x_j \\ (\alpha, \beta) \end{pmatrix}, \tilde{y}_{j,(\alpha,\beta)} := \begin{pmatrix} y_j \\ (t\beta, \alpha) \end{pmatrix} \right\} \subset \mathbb{F}_p^{2(k+1)}. \quad (19)$$

Of course $\dim(\tilde{V}_0) = 2$ and $B|_{\tilde{V}_0 \times \tilde{V}_0} \neq 0$. However, the same is true for $V_{j,(\alpha,\beta)}$. Indeed, as $B|_{V_j \times V_j} = 0$, we have that $B(x_j, y_j) = 0$ and consequently that $B(\tilde{x}_{j,(\alpha,\beta)}, \tilde{y}_{j,(\alpha,\beta)}) = B((\alpha, \beta), (t\beta, \alpha)) = \alpha^2 - t\beta^2 \neq 0$. Thus by Lemma 2.2, the subalgebras $\mathcal{A}(\tilde{V}_0), \mathcal{A}(\tilde{V}_{j,(\alpha,\beta)})$ where $(\alpha, \beta) \in \mathbb{F}_p^2 \setminus \{0\}, j \in \{1, \dots, r\}$, form a collection of $1 + \frac{p^{2k}-1}{p^2-1}(p^2-1) = p^{2k}$ copies of $\mathcal{B}(\mathcal{H})$. We shall justify our statement by proving that they are pairwise quasi-orthogonal (which by the mentioned lemma is equivalent to the fact that for all $(j, (\alpha, \beta)) \neq (j', (\alpha', \beta'))$, the intersection $\tilde{V}_0 \cap \tilde{V}_{j,(\alpha,\beta)} = \tilde{V}_{j,(\alpha,\beta)} \cap \tilde{V}_{j',(\alpha',\beta')} = \{0\}$), and that they are all quasi-orthogonal to $\otimes_k \mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$ (which by the mentioned lemma is equivalent to the fact that any of these subspaces intersect $\mathbb{F}^{2k} \times \{0\}$ in the point 0, only).

Of course the fact that $\tilde{V}_0 \cap \mathbb{F}^{2k} \times \{0\} = \{0\}$ is trivial. So consider a vector $v \in \tilde{V}_{j,(\alpha,\beta)}$. We know that it must have the form $v = n\tilde{x}_{j,(\alpha,\beta)} + m\tilde{y}_{j,(\alpha,\beta)}$. If $v \in \tilde{V}_0$, then $nx_j + my_j = 0$ and hence $n = m = 0$ by the linear independence of x_j and y_j . If $v \in \mathbb{F}^{2k} \times \{0\}$, then $n(\alpha, \beta) + m(t\beta, \alpha) = 0$, or equivalently:

$$\begin{pmatrix} \alpha & \beta \\ t\beta & \alpha \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = 0. \quad (20)$$

As the determinant of our matrix equals to $\alpha^2 - t\beta^2 \neq 0$, it follows that $n = m = 0$. Thus $\tilde{V}_{j,(\alpha,\beta)} \cup \tilde{V}_0 = \{0\}$. Finally, suppose that $v \in \tilde{V}_{j',(\alpha',\beta')}$. Then there exists some $n', m' \in \mathbb{F}_p$ such that $v = n'\tilde{x}_{j',(\alpha',\beta')} + m'\tilde{y}_{j',(\alpha',\beta')}$. Then $n'x'_{j'} + m'y'_{j'} = nx_j + my_j$, so if $v \neq 0$ then $j' = j$, $n' = n$ and $m' = m$ since $V_j \cap V_{j'} = \{0\}$ whenever $j \neq j'$, and the vectors x_j, y_j form a base in

V_j . In turn, it then follows that $n(\alpha, \beta) + m(t\beta, \alpha) = n(\alpha', \beta') + m(t\beta', \alpha')$, or equivalently, that

$$\begin{pmatrix} (\alpha - \alpha') & (\beta - \beta') \\ t(\beta - \beta') & (\alpha - \alpha') \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} = 0. \quad (21)$$

Hence, using again the determinant and the special property of t , we obtain that either $v = 0$, or $\alpha - \alpha' = \beta - \beta' = 0$, and so $\tilde{V}_{j,(\alpha,\beta)} \cap \tilde{V}_{j',(\alpha',\beta')} = \{0\}$ for all $(j, (\alpha, \beta)) \neq (j', (\alpha', \beta'))$. \square

Theorem 2.4. *Let $\dim(H) = p$ be a prime, and $k \geq 2$. Then the deficit number $D(k, p) \leq 1$; i.e. there exist a family of $\frac{p^{2k}-1}{p^2-1} - 1$ quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$ in $\otimes_k \mathcal{B}(\mathcal{H})$.*

Proof. Again, we may assume that $p \geq 3$ (since for $p = 2$ the result has been already obtained in [?]), and fix a $t \in \mathbb{F}_p$ non-square. By the previous theorem it is enough to show that $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ admits a family of $\frac{p^4-1}{p^2-1} - 1 = p^2$ quasi-orthogonal copies of $\mathcal{B}(\mathcal{H})$, and by Lemma 2.2 such a family surely exists if we can exhibit a collection of 2-dimensional \mathbb{F}_p -linear subspaces $V_1, \dots, V_{p^2} \subset \mathbb{F}_p^4$ such that $B|_{V_j \times V_{j'}} \neq 0$ and $V_j \cap V_{j'} = \{0\}$ for all $j \neq j'$. An example for such a family is:

$$V_{(\alpha,\beta)} := \text{Span} \left\{ x_{(\alpha,\beta)} := \begin{pmatrix} (1, \alpha) \\ (0, \beta) \end{pmatrix}, y_{(\alpha,\beta)} := \begin{pmatrix} (0, t\beta + 1) \\ (t, \alpha) \end{pmatrix} \right\} \subset \mathbb{F}_p^4, \quad (22)$$

where $(\alpha, \beta) \in \mathbb{F}_p^2$. Indeed, by direct calculation $B(x_{(\alpha,\beta)}, y_{(\alpha,\beta)}) = 1$. On the other hand, if $nx_{(\alpha,\beta)} + mx_{(\alpha,\beta)} = n'x_{(\alpha',\beta')} + m'y_{(\alpha',\beta')}$, then clearly it follows that $n = n'$ and $m = m'$. Moreover, resolving the linear equation results again the same expression as that of equation (21). Hence, by the special property of t we get that either $\alpha - \alpha' = \beta - \beta' = 0$, or $n = m = 0$. This shows that $V_{(\alpha,\beta)} \cap V_{(\alpha',\beta')} = \{0\}$ for all $(\alpha, \beta) \neq (\alpha', \beta')$. \square

As was mentioned, in case of $p = k = 2$ the deficit number is *exactly* 1 (i.e. one cannot give more than 4 quasi-orthogonal copies of $M_2(\mathbb{C})$ in $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, as was proved in [?]). This was achieved by showing that if $\mathcal{A}_1, \mathcal{A}_2 \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ are quasi-orthogonal copies of $M_2(\mathbb{C})$, then $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \mathbb{C}\mathbf{1}$. If the same was true for $k = 2$ in general, then by simple dimensional reasoning, it would follow that $D(2, p) = 1$ for all prime numbers p . However, it is easy to see, that this property about the commutants does not even hold for $p = 3$. Indeed, for $p = 3$ the only non-square is $t = 2$, and by

direct calculation the subspace $V_{(1,2)}$ appearing in the previous construction intersects both $\mathbb{F}_p^2 \times \{0\}$ and $\{0\} \times \mathbb{F}_p^2$ in the zero, only. Hence $\mathcal{A}(V_{(1,2)})$ is quasi-orthogonal to both $\mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$ and $\mathbf{1} \otimes \mathcal{B}(\mathcal{H}) = (\mathcal{B}(\mathcal{H}) \otimes \mathbf{1})'$; so in particular, $\mathcal{A}(V_{(1,2)}) \cap (\mathcal{B}(\mathcal{H}) \otimes \mathbf{1})' = \mathbb{C}\mathbf{1}$. So while it still seems likely that the deficit number is exactly 1, the proof of this fact must be totally different from the one given for $p = 2$.