Operators on finite dimensional Hilbert spaces: exercises with spectral decompositions

Throught this exercise sheet $\mathcal{H}$ stands for a finite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the *-algebra of linear operators on $\mathcal{H}$.

Problem 7.1. *Let $X, N \in \mathcal{B}(\mathcal{H})$ and suppose that N is normal. Show that X commutes with N if and only if it commutes with every single spectral projection of N.*

Problem 7.2. *Let $X, N \in \mathcal{B}(\mathcal{H})$ and suppose that N is normal. Using the result of the previous exercise, conclude that X commutes with N if and only if it commutes with N^*. Does this remain valid, if the assumption of the normality of N is dropped? If yes, prove it, if no, give a counter-example.*

Problem 7.3. *Let $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ be two commuting normal operators. Prove that N_1 and N_2 may be considered two be functions of a single normal operator; that is, there exists a normal operator $N \in \mathcal{B}(\mathcal{H})$ and two functions f_1 and f_2 such that $N_1 = f_1(N)$ and $N_2 = f_2(N)$.*

Problem 7.4. *Considering $\mathbb{C}^3$ with its standard scalar product, let the self-adjoint operator $M : \mathbb{C}^3 \to \mathbb{C}^3$ be given by its matrix in the standard bases of $\mathbb{C}^3$ as*

$$\begin{pmatrix} 1 & 0 & 4-3i \\ 0 & 2 & 0 \\ 4+3i & 0 & 1 \end{pmatrix}.$$

Is it possible to simply re-order the bases vectors in the standard bases so that the matrix of M becomes block-diagonal? Work out the spectral decomposition of M.

Problem 7.5. *Find the solution of the system of differential equations*

$$x'(t) = 2x(t) + y(t), \quad y'(t) = x(t) + 2y(t)$$

where $x, y : \mathbb{R} \to \mathbb{C}$.

Problem 7.6. *Consider $\mathbb{C}^n$ with its standard scalar product and suppose that $x \in \mathbb{C}^n$ is a vector of unit length. Show that the matrix*

$$\begin{pmatrix} \overline{x_1}x_1 & \overline{x_2}x_1 & \dots & \\ \overline{x_1}x_2 & \overline{x_2}x_2 & \dots & \\ \dots & \dots & \dots & \\ & & & \end{pmatrix}$$

is exactly the matrix of the orthogonal projection onto $\mathbb{C}x$.

Problem 7.7. *Consider* $\mathbb{C}^4$ *with its standard scalar product and suppose that* M *is a self-adjoint operator on* $\mathbb{C}^4$ *such that* $\mathrm{Sp}(M) = \{0, 1/2, 1, 2\}$ *and the vector*

$$\begin{pmatrix} 1+3i \\ 3-2i \\ 0 \\ 1+i \end{pmatrix}$$

is an eigenvector for M *with eigenvalue* 1. *Calculate, if it is possible by the given information, the matrix (in the standard bases of* $\mathbb{C}^4$*) of*

$$25\sin(\pi M) = 25\left(\pi M - \frac{(\pi M)^3}{3!} + \frac{(\pi M)^5}{5!}\dots\right).$$

Problem 7.8. *Let* $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ *be two normal operators. We say that* N_1 *and* N_2 *have equivalent spectral measures, if both their spectrum and the dimensions of the corresponding eigenspaces coincide. Prove that the following three conditions are equivalent:*

- $\mathrm{Tr}(N_1^k) = \mathrm{Tr}(N_2^k)$ *for all* $k \in \mathbb{N}$,
- *there exists a unitary* $U \in \mathcal{B}(\mathcal{H})$ *such that* $N_2 = UN_1U^*$,
- *the spectral measures of* N_1 *and* N_2 *are equivalent.*

Problem 7.9. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be such* $A^2 = 0$ *and* $AA^* + A^*A = \mathbf{1}$. *Show that* $\mathrm{Sp}(A + A^*) = \{1, -1\}$ *and that the dimensions of the two eigenspaces of* $A + A^*$ *are equal.*

Problem 7.10. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Prove that if* $A, B \geq 0$, *then* $\mathrm{Ker}(A + B) = \mathrm{Ker}(A) \cap \mathrm{Ker}(B)$.

Problem 7.11. *Let* $A \in \mathcal{B}(\mathcal{H})$. *Show that if* $A \geq 0$, *then* $\mathrm{Tr}(A) \geq 0$ *and equality holds if and only* $A = 0$.

Problem 7.12. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Show that if* $A, B \geq 0$ *then* $\mathrm{Tr}(AB)$ *is a real, nonnegative number.*

Problem 7.13. *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *two self-adjoint operators. Show that both* $\mathrm{Tr}(A^2B^2)$ *and* $\mathrm{Tr}((AB)^2)$ *are reals,*

$$\mathrm{Tr}(A^2B^2) \geq \mathrm{Tr}((AB)^2),$$

and equality holds if and only if $AB = BA$.