A dichotomy in the maximal number of mutually unbiases basis

Mihály Weiner

Abstract

A collection of pairwise mutually unbiased bases in d dimensions can consists of at most $d+1$ bases. Such maximal collections are known to exists in $\mathbb{C}^d$ when d is a power of a prime. However, in general little is known about the maximal number $N(d)$ of bases that a collection of pairwise mutually unbiased bases in $\mathbb{C}^d$ can have.

In this work it is proved that given a collection of d pairwise mutually unbiased bases in $\mathbb{C}^d$, one can always find and add one more basis in such a way that the collection will still consists of pairwise mutually unbiased bases. Hence there is a dichotomy: *either* $N(d) = d+1$ (so that there exists a so-called "complete set" of pairwise mutually unbiased bases), *or* $N(d) \leq d-1$. In other words, $N(d) \neq d$.

1 Preliminaries

Throughout this work, the expression "a subalgebra of $M_d(\mathbb{C})$" will always stand for a $*$-subalgebra containing $\mathbb{1} \in M_d(\mathbb{C})$. Up to unitary equivalence, such a subalgebra $\mathcal{A} \subset M_d(\mathbb{C})$, is always of the form

$$\mathcal{A} = \oplus_j \left(M_{a_j}(\mathbb{C}) \otimes \mathbb{1}_{b_j} \right) \tag{1}$$

where $\sum_j a_j^2 = \dim(\mathcal{A})$ and $\sum_j a_j b_j = \dim(\mathbb{C}^d) = d$. The commutant of $\mathcal{A}$ —that is, the set of matrices that commute with each element of $\mathcal{A}$ — is again a subalgebra. If $\mathcal{A}$ is of the above form, then

$$\mathcal{A}' = \oplus_j \left(\mathbb{1}_{a_j} \otimes M_{b_j}(\mathbb{C}) \right). \tag{2}$$

$\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$, that is, the center of $\mathcal{A}$, is the set of elements of the form

$$\oplus_j z_j \left(\mathbb{1}_{a_j} \otimes \mathbb{1}_{b_j} \right), \tag{3}$$

where $z_j \in \mathbb{C}$. The above element is a projection if and only if every z_j is either 1 or 0. Hence the dimensions of the images of the minimal projections of $\mathcal{Z}(\mathcal{A})$ are exactly the values $a_j b_j$.

In case b_j/a_j is independent of j, we shall say that $\mathcal{A}$ is **homogeneously ballanced**. Of course $\mathcal{A}$ is homogeneously ballanced if and only if so is $\mathcal{A}'$, too. Note that if $\mathcal{A}$ is homogeneously ballanced, $b_j/a_j = \lambda$ for every j, then from one hand

$$d = \sum_j a_j b_j = \lambda \sum_j a_j^2 = \lambda \mathrm{dim}(\mathcal{A}), \tag{4}$$

but on the other hand,

$$d = \sum_j a_j b_j = \frac{1}{\lambda} \sum_j b_j^2 = \frac{1}{\lambda} \mathrm{dim}(\mathcal{A}'), \tag{5}$$

and hence $\mathrm{dim}(\mathcal{A})\mathrm{dim}(\mathcal{A}') = d^2$. Note also, that $\mathcal{A}$ is abelian, that is, $\mathcal{A} = \mathcal{Z}(\mathcal{A})$, if and only if $a_j = 1$ for every j. So an abelian subalgebra is homogeneously ballanced if and only if the dimensions of the images of its minimal projections are all equal.

There is a natural scalar product on $M_d(\mathbb{C})$; the so-called *Hilbert-Schmidt* scalar product, defined by the formula

$$\langle A, B \rangle = \mathrm{Tr}(A^* B) \qquad (A, B \in M_d(\mathbb{C})). \tag{6}$$

In this sense, if $\mathcal{A} \subset M_d(\mathbb{C})$ is a linear subspace, one can consider the orthogonal projection $E_{\mathcal{A}}$ onto $\mathcal{A}$. In case $\mathcal{A}$ is a subalgebra, $E_{\mathcal{A}}$ is nothing else than the *trace-preserving conditional expectation* onto $\mathcal{A}$.

Two subalgebras $\mathcal{A}, \mathcal{B} \subset M_d(\mathbb{C})$, as subspaces, cannot be orthogonal, since $\mathcal{A} \cap \mathcal{B} \neq \{0\}$ as $\mathbb{1} \in \mathcal{A} \cap \mathcal{B}$. At most, the subspaces $\mathcal{A} \cap \{\mathbb{1}\}^\perp$ and $\mathcal{B} \cap \{\mathbb{1}\}^\perp$ can be orthogonal, in which case we say that the subalgebras $\mathcal{A}$ and $\mathcal{B}$ are **quasi-orthogonal**. Note that $\mathcal{A}$ and $\mathcal{B}$ are quasi-orthogonal subalgebras of $M_d(\mathbb{C})$ if and only if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\tau(AB) = \tau(A)\tau(B), \tag{7}$$

where $\tau = \frac{1}{d}\mathrm{Tr}$ is the normalized trace. Note also that if $\mathcal{A}$and $\mathcal{B}$ are quasi-orthogonal, and $\tilde{A}$ and $\tilde{B}$ are subalgebras included in $\mathcal{A}$ and $\mathcal{B}$, respectively, then $\tilde{A}$ and $\tilde{B}$ are also quasi-orthogonal.

Let us see now how the problem of mutually unbiased bases can be rephrased in terms of subalgebras. Let $\mathcal{E} = (\mathbf{e}_1, \ldots, \mathbf{e}_d)$ be an orthonormed bases (ONB) in $\mathbb{C}^d$, and denote the ortho-projection onto the one-dimensional subspace $\mathbb{C}\mathbf{e}_j$ by $P_{\mathbf{e}_j}$ for each $j = 1, \ldots, d$. Then we may consider

$$\mathcal{A}_{\mathcal{E}} = \mathrm{Span}\{\, P_{\mathbf{e}_j} \,|j = 1, \ldots, d\}, \tag{8}$$

that is, the subspace of $M_d(\mathbb{C})$ spanned linearly by the ortho-projections $P_{\mathbf{e}_j}$ ($j = 1, \ldots, d$). It is a maximal abelian subalgebra (MASA), and actually, if $\mathcal{A} \subset M_d(\mathbb{C})$ is a MASA, then there exists an ONB $\mathcal{E}$ such that $\mathcal{A} = \mathcal{A}_{\mathcal{E}}$.

As is well-known, (but in any case it can be obtained by a straightforward calculation), two MASAs $\mathcal{A}_{\mathcal{E}}$ and $\mathcal{A}_{\mathcal{F}}$ are quasi-orthogonal if and only if $\mathcal{E}$ and $\mathcal{F}$ are mutually unbiased. So the problem of finding a certain number of pairwise mutually unbiased ONBs is equivalent to finding the same number of pairwise quasi-orthogonal MASAs.

The dimension of $\mathcal{A} \cap \{\mathbb{1}\}^{\perp}$ is $\dim(\mathcal{A}) - 1 = d - 1$ for a MASA $\mathcal{A}$, whereas the dimension of $M_d(\mathbb{C}) \cap \{\mathbb{1}\}^{\perp}$ is $d^2 - 1$. If $d > 1$, then at most

$$\frac{d^2 - 1}{d - 1} = d + 1 \tag{9}$$

pairwise orthogonal, $(d-1)$-dimensional subspaces can be fitted in a (d^2-1)-dimensional space. So a collection of pairwise quasi-orthogonal MASAs can have at most $d + 1$ elements; this is one of the ways one can obtain the well-known upper limit on the number of mutually unbaised bases.

The main result of this work is achieved by a — some sense tricky, but in another sense straightforward — application of a formula and an inequality derived in the previous paper [**?**] of the author. For self-containmennt, let us review the main points.

If we have a pair of quasi-orthogonal subalgebras $\mathcal{A}, \mathcal{B} \subset M_d(\mathbb{C})$, we may consider their commutant $\mathcal{A}'$ and $\mathcal{B}'$. In case $\mathcal{A}$ and $\mathcal{B}$ are MASAs, we have $\mathcal{A}' = \mathcal{A}$ and $\mathcal{B}' = \mathcal{B}$, so we do not seem to get anything new. However, if our motivation is operator algebras (and quantum information theory) in general, rather than just the (geometrical) problem of mutually unbiased bases, then it is a natural step to consider the commutant. So one may ask: if $\mathcal{A}$ and $\mathcal{B}$ are quasi-orthogonal, does the same hold for $\mathcal{A}'$ and $\mathcal{B}'$? The answer, in general, is no. Actually, it is possible to give a nice characterization of the case when the commutants are again quasi-orthogonal; see [**?**, Proposition ?.?]. However, at this point, what we want now is to measure "how much" the subalgebras $\mathcal{A}'$ and $\mathcal{B}'$ are *not* quasi-orthogonal. For this reason, for two subalgebras $\mathcal{A}$ and $\mathcal{B}$ consider the value $\mathrm{Tr}(E_{\mathcal{A}}E_{\mathcal{B}})$; it is a positive real number, infact

$$1 \leq \mathrm{Tr}(E_{\mathcal{A}}E_{\mathcal{B}}) \tag{10}$$

with equality holding if and only if $(\mathcal{A},\mathcal{B})$ is a quasi-orthogonal pair of subalgebras; see [**?**, Lemma ?.?]. Note that in the expression $\mathrm{Tr}(E_{\mathcal{A}}E_{\mathcal{B}})$ we need to take the trace of a linear map from $M_d(\mathbb{C})$ to $M_d(\mathbb{C})$; so although the same symbol "Tr" appears both here and in the definition of the Hilbert-Schmidt scalar product, they correspond to different objects.

We shall use the value $\mathrm{Tr}(E_{\mathcal{A}}E_{\mathcal{B}})$ to measure "how far away" the subalgebras $\mathcal{A}$ and $\mathcal{B}$ are from quasi-orthogonality. By what was explained, the natural question is the following: what is the relation between $\mathrm{Tr}(E_{\mathcal{A}}E_{\mathcal{B}})$ and $\mathrm{Tr}(E_{\mathcal{A}'}E_{\mathcal{B}'})$? It turns out, that in general not much. However, if $\mathcal{A},\mathcal{B} \subset M_d(\mathbb{C})$ are homogeneously ballanced, then

$$\mathrm{Tr}(E_{\mathcal{A}'}E_{\mathcal{B}'}) = \frac{d^2}{\dim(E_{\mathcal{A}})\dim(E_{\mathcal{B}})}\mathrm{Tr}(E_{\mathcal{A}}E_{\mathcal{B}}), \tag{11}$$

see [**?**, Theorem ?.?].

At a first glance one would think that the cited formula is of no use when we deal with MASAs, only. Infact, if the author came to consider this formula exactly because he was *not* thinking about the problem of mutually unbiased bases, but the problem of finding quasi-orthogonal system of (non necessarily abelian) subalgebras. This topic was initiated by D. Petz; see for example [**?**, **?**].

So what is the use of the cited formula in the maximal abelian case? We may consider homogeneously ballanced subalgebra *included* in $\mathcal{A}$ and $\mathcal{B}$. They are of course still abelian, but they are not maximal abelian. In this way — as we shall see — one can obtain some nontrivial inequalities regarding a system of MASAs.

For an effective use of our formula we shall also need an inequality concerning ortho-projections. In the already cited paper of the author, it is shown, that as a consequence of the *Welch inequalities* [**?**], if $E_1,\ldots,E_n$ are ortho-projections in a D-dimensional Hilbert space, then

$$D\sum_{j,k}\mathrm{Tr}(E_jE_k) \geq \left(\sum_j \mathrm{Tr}(E_j)\right)^2, \tag{12}$$

see [**?**, Lemma ?.?]

2 The solution to the MUB 6 problem

Let $\mathcal{E} = (\mathbf{e}_1, \ldots, \mathbf{e}_6)$ be an ONB in $\mathbb{C}^6$, and denote the ortho-projection onto the one-dimensional subspace $\mathbb{C}\mathbf{e}_j$ by $P_{\mathbf{e}_j}$ for each $j = 1, \ldots, 6$. We have already introduced the MASA

$$\mathcal{A}_{\mathcal{E}} = \mathrm{Span}\{P_{\mathbf{e}_j} | j = 1, \ldots, 6\}. \tag{13}$$

We shall now introduce two subspaces of $\mathcal{A}_{\mathcal{E}}$. Let

$$\mathcal{B}_{\mathcal{E}} = \mathrm{Span}\{P_{\mathbf{e}_1} + P_{\mathbf{e}_3} + P_{\mathbf{e}_5},\, P_{\mathbf{e}_2} + P_{\mathbf{e}_4} + P_{\mathbf{e}_6}\,\} \tag{14}$$

and

$$\mathcal{C}_{\mathcal{E}} = \mathrm{Span}\{P_{\mathbf{e}_1} + P_{\mathbf{e}_2},\, P_{\mathbf{e}_3} + P_{\mathbf{e}_4},\, P_{\mathbf{e}_5} + P_{\mathbf{e}_6}\,\}. \tag{15}$$

Clearly, $\mathcal{B}_{\mathcal{E}}$ is a 2-dimensional, and $\mathcal{C}_{\mathcal{E}}$ is a 3-dimensional subspace of $\mathcal{A}_{\mathcal{E}}$.

Lemma 2.1. *$\mathcal{B}_{\mathcal{E}}$ and $\mathcal{C}_{\mathcal{E}}$ are quasi-orthogonal homogeneously ballanced subalgebras.*

Proof. Since $P_{\mathbf{e}_j} P_{\mathbf{e}_k} = \delta_{j,k} P_{\mathbf{e}_k} = \delta j, k P^*_{\mathbf{e}_k}$, it is evident that they are subalgebras. Quasi-orthogonality can be checked by verifying (**??**). Indeed, if $X \in \mathcal{A}_{\mathcal{B}}$ and $Y \in \mathcal{A}_{\mathcal{C}}$ are one of the □