Physical quantities and self-adjoint operators

Throught this exercise sheet capital letters like A and B stand for (real-valued) physical quantities of an n-level quantum physical system, and capital letters with "hats" like $\hat{A}$ and $\hat{B}$ denote the corresponding self-adjoint operators acting on the $n < \infty$ dimensional Hilbert space $\mathcal{H}$ associated to the system.

Problem 6.1. *Suppose $C = A + B$ in the statistical sense. Show that $\min(A) + \min(B) \leq \min(C)$ and $\max(A) + \max(B) \geq \max(C)$ where $\min(X)$ and $\max(X)$ stand for the minimum and maximum values that X can possibly take.*

Problem 6.2. *Suppose $C = A + B$ in the statistical sense, and that in a certain (fixed) ensemble $A = a$ with probability 1. Show that $E(C^2) = a^2 + 2aE(B) + E(B^2)$ where "E" stands for the expected value in this ensemble. Decide if it also follows that $E(C^3) = a^3 + 3a^2E(B) + 3aE(B^2) + E(B^3)$.*

Problem 6.3. *By performing experiments on randomly chosen members of a large ensemble of identical quantum physical systems, it was concluded that in this ensemble the physical quantities A, B, C, D, E take values with corresponding probabilities as follows.*

- $A:\ -3\,(25\%),\ 3\,(50\%),\ 5\,(25\%)$;
- $B:\ -4\,(50\%),\ 4\,(50\%)$;
- $C:\ 2\,(100\%)$;
- $D:\ -1\,(25\%),\ 4\,(75\%)$;
- $E:\ -5(25\%),\ 1\,(25\%),\ 5\,(25\%),\ 7\,(25\%)$.

By the above data, can $A + B$ be equal to C in the statistical sense? And D or E? In each case either show that equality can hold by giving an example, or prove the contrary.

Problem 6.4. *Recall that the variance of a quantity X in a given state is*

$$\triangle^2(X) \equiv E((X - E(X))^2) = E(X^2) - E(X)^2$$

where E stands for the expected value in the state in question. Prove Heisenberg's uncertanity relation:

$$\triangle^2(A)\triangle^2(B) \geq |\mathrm{Tr}(D[\hat{A}, \hat{B}])|^2$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ and D is the density operator associated to the state in question.

Problem 6.5. *Consider the following properties:*

(i) $p(\text{"}A = a\text{"}) = 1 \implies E_p(C) = aE_p(B)$,

(ii) $p(\text{"}A = a\text{"}) = 1 \implies p(\text{"}C = c\text{"}) = p(\text{"}aB = c\text{"})$.

(Here p stands for a generic probability function, and E_p denotes the corresponding expected value.) If A and B can be measured together and $C = AB$ (in the usual sense), then both (i) and (ii) hold. So in case A and B do not necessarily have a simultaneous meaning, we could try to give some meaning to the equation $C = AB$ by requiring property (i) or (ii).

Discuss existence and uniqueness of a C satisfying property (k) for both k=i and k=ii in the concrete example in which $\hat{A}$ and $\hat{B}$ are the self-adjoint operators whose matrices wrt. a certain ONB $\mathcal{E}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

respectively.

Problem 6.6. *Concerning a certain large organic molecule, theory (regarding macroscopical averages) tells us that the quantities A, B and C should satisfy the equality $A = B^2 + C^2$ where the sum is meant in the statistical sense. A research team using random samples drawn from a specific ensemble (i.e. a quantity of this organic material at specified temperature and other conditions) concluded that in this particular ensemble A and B take values with corresponding probabilities as follows:*

- $A:\ 1\,(52\%),\ 4\,(24\%),\ 16\,(24\%)$;
- $B:\ -1\,(50\%),\ 3\,(50\%)$.

Because of its importance in biology, the findings were sent for peer-review to a journal of biology. After some time, the following answer arrived from the editor.

"Dear authors, I am sorry but the referee did not recommend publication of your work. According to the referee's report, your finding, if correct, would contradict to theory (and you do not even seem to realize this!) of which there are fairly strong evidences. Therefore the referee advises a careful check of your laboratory devices. Should you confirm your results, you may resend your work after a serious revision (you will have to clearly state that the findings contradict a so-far accepted theory)."

The team leader, considering that the actual journal is a journal of biology, concluded that the referee was probably not an expert of quantum physics, and hence sent one more letter.

"Dear editor, if our organic molecule was a classical system, our findings would indeed contradict the theory $A = B^2 + C^2$, if it was a classical system. However, this system, though large for a molecule, is still a microscopic one. Thus the noted (apparent) contradiction may well disappear when quantum physics is taken account."

In turn, the editor responded (after some delay due to summer vacation) by the following rather short letter.

"Dear authors, our referee, who is an expert of quantum physics states that your results contradict theory even if quantum physics is taken account. Therefore, I am sorry to say, but my decision has remained the same."

At this point the team has decided to check again their results. They have then quickly realized that because of a mistake in editing, the probabilities regarding the outcomes of A were wrongly given; correctly they are: $1\,(50\%),\ 4\,(5\%),\ 16\,(45\%)$. After correcting the mistake, the work was sent again to the journal, where this time it was excepted for publication. Explain the mathematics behind the story.