

## List of material for the MAP final

Note that there are some “over-lappings” between the items; they were necessary so that the number of topics would be 8 and each topics would be approximately of the same size.

- (1) The general framework of Quantum Physics (or: how to model a system in Quantum Physics).
- (2) Quantum Physical description “in practice”: modelling a system of spin  $1/2$  particles. Physical facts (the Stern-Gerlach experiment), postulates and how they are turned into mathematical requirements. Possibilities regarding the number of solutions.
- (3) System of spin  $1/2$  particles: the mathematical requirements and their solutions (given in terms of Pauli matrices) in case of 2 particles. The difference between the 2 solutions, the existence of a probability law in which the “total spin is zero”.
- (4) The EPR experiment with 2 spin  $1/2$  particles: the “total spin is zero” probability law is unique and is infact a (pure) state; probabilities and correlations in this state and the arising “paradox”.
- (5) Bipartite systems in general; the partial state, product-states and entanglement; state-changes as positive maps and the “no-clone” theorem.
- (6) Symmetry transformations and Wigner’s theorem; rotations on spin  $1/2$  particles. Dense coding with 2 spin  $1/2$  particles.
- (7) Quantum cryptography and dense coding with 2 spin  $1/2$  particles.
- (8) Quantum gates and quantum computers; Grover’s algorithm.

# Detailed list of material

## 1st topic

§1. Fix a Hilbert space  $\mathcal{H}$ ; elements of the ortho-lattice  $\mathcal{P}(\mathcal{H})$  will stand for the possible events that can occur in our system. By Gleason's theorem, if  $\dim(\mathcal{H}) > 2$  (which we shall always assume from now on), then probability laws on  $\mathcal{P}(\mathcal{H})$  are in one-to-one connection with density operators on  $\mathcal{H}$  via the formula

$$p(Q) = \text{Tr}(\rho Q) \quad (Q \in \mathcal{P}(\mathcal{H}))$$

and pure probability laws are in one-to-one connection with density operators of the form  $|\Psi\rangle\langle\Psi|$  where  $\Psi \in \mathcal{H}$  is a vector of unit length. Note that in this case

$$p(Q) = \text{Tr}(|\Psi\rangle\langle\Psi|Q) = \langle\Psi, Q\Psi\rangle$$

and that  $|\Psi\rangle\langle\Psi| = |\Psi'\rangle\langle\Psi'|$  if and only if  $\Psi'$  is a (unit-) multiple of  $\Psi$ ; in other words, if and only if they only differ in their "phases".

§2. For a (real-, discrete-valued) physical quantity  $A$ , one can consider "A =  $\lambda$ ", that is, the event that the value of  $A$  is  $\lambda$ . In the model, for each such event we must choose an ortho-projection (that will stand for this event). Introducing  $A$  into our model means fixing each such projection.

The choice of these projections must satisfy certain rules. This is because events like "A =  $\lambda$ " and "A =  $\mu$ " where  $\lambda \neq \mu$  should be exclusive, moreover, as  $A$  must take *some* value,  $\vee_{\lambda}$  "A =  $\lambda$ " must be the total event  $\mathbb{1}$ . So denoting the projection associated to "A =  $\lambda$ " by  $P_{\text{"A = } \lambda}$ , we have the following natural requirements:

$$P_{\text{"A = } \lambda} P_{\text{"A = } \mu} = \delta_{\lambda, \mu} P_{\text{"A = } \mu}, \quad \sum_{\lambda} P_{\text{"A = } \lambda} = \mathbb{1}.$$

These requirements allow us to encode all of these projections into a single self-adjoint operator. By the above conditions, the sum defining  $\hat{A}$

$$\hat{A} = \sum_{\lambda} \lambda P_{\text{"A = } \lambda}$$

is actually also its spectral decomposition.

Instead of dealing with a certain labeled collection of projections, it is much more convenient to introduce  $A$  into the model by giving the single

self-adjoint operator  $\hat{A}$ . (Infact, in a physicist undergraduate course the usual treatment is to ignore the role of projections as events, and to take this fact — that physical quantities are described by self-adjoint operators — as a definition.) Note that by its definition,  $\text{Sp}(\hat{A})$  is the set of all values of  $A$  (that can possibly occure). If needed, the projection  $P_{\text{“}A=\lambda\text{”}}$  can be recovered by spectral decomposition of  $\hat{A}$ .

§3. Considering how physical quantites are described by self-adjoint operators and taking account of the meaning of *expected value* and some other notions such as *statistical sum*, *function of a physical quantity* and *simultaneous measurability*, one can arrive to the conclusion that

- in case the probability law is given by the density operator  $\rho$ , the expected value of  $A$  is  $\text{Tr}(\rho\hat{A})$ ,
- in particular, in case the system is in the state given by the unit vector  $\Psi$ , then the expected value of  $A$  is  $\langle\Psi, \hat{A}\Psi\rangle$
- $\widehat{f(A)} = f(\hat{A})$
- $\widehat{A_1 + A_2} = \hat{A}_1 + \hat{A}_2$  where on the left-hand side the sum is defined in the statistical sense,
- if  $A_1$  and  $A_2$  can be simultaneously measured without disturbing each other, then  $\hat{A}_1$  and  $\hat{A}_2$  must commute, and in this case  $\widehat{f(A_1, A_2)} = f(\hat{A}_1, \hat{A}_2)$ .

## 2nd topic

§1. By the Stern-Gerlach experiment, among others, the following important coclusions can be drawn.

- The electron’s spin (“intrinsic angular momentum”), when measured in a given direction, is always found to be either  $+\frac{1}{2}\hbar$  or  $-\frac{1}{2}\hbar$ .
- If in a given direction we measure the electron’s spin and obtain the value  $s$ , then, upon repeating the measurement — if the electron between the two measurements was free of external actions and disturbances — we obtain the same value  $s$ .

- Once the value of the electron’s spin in a certain direction is determined, the electron’s state (regarding its spin) cannot be further specified.

The first listed item needs no further comment, but there are certain things to remark about the second and third items. The fact that upon repeating measurement we re-find the previously obtained value does not seem too surprising. This is just what we would expect by our classical experience: if nothing acted on the electron, why would its spin’s value change? However, by the Stern-Gerlach experiment we also learn that some uncertainty is always present: if we know the value of the spin in a certain direction with certainty, then in an orthogonal direction its spin is just as likely to be found plus as minus; so no prediction can be given. Because of this nontrivial involvement of probabilities, one might think that if we repeat an experiment, in general we do not get the same value again. The point of the second item is that this is not so.

Finally, a remark about the third item. Suppose we take a beam of electrons and we perform a spin-measurement in direction  $\vec{v}$ . In general, the beam will split into two: electrons with spin “up” and with spin “down” (with respect to direction  $\vec{v}$ ) will go on different paths. After performing a further measurement of spin, this time in direction  $\vec{w}$  where  $\vec{w} \neq \pm\vec{v}$ , we will end up with four beams. One might think that electrons whose spin in direction  $\vec{w}$  is “up” and whose spin in direction  $\vec{v}$  was previously “up” would behave in a different way than those whose spin in direction  $\vec{w}$  is “up” but whose spin in direction  $\vec{v}$  was previously “down”. However, this is not so: if we were to perform further experiments, we would find no difference between the mentioned two beams. So after the electron’s spin is determined in the direction  $\vec{w}$ , the electron forgets about its “previous life”. Its state — regarding its spin — cannot be further specified; this is the conclusion drawn in the third item.

§2. We want to set up postulates regarding the spin-content of a system of  $n$  spin 1/2 particles. We want to ignore other parts of reality — so for example kinematics will be left out from our model. (This simplification makes things much more cleaner; moreover, it also makes possible to avoid the use of infinite dimensional Hilbert spaces.) Thus the basic quantities that we want to describe are the spin-values  $S_{k,\vec{v}}$ , where  $k = 1, 2, \dots, n$  and  $\vec{v}$  is a direction (that is, a unit vector) in our 3-dimensional Euclidean space. We postulate that

- (1) the possible values of  $S_{k,\vec{v}}$  are  $\pm\frac{1}{2}\hbar$ ,
- (2) if  $k \neq l$ , then  $S_{k,\vec{v}}$  and  $S_{l,\vec{w}}$  can be simultaneously measured without disturbing each other,
- (3) if  $\vec{v} = \alpha\vec{u} + \beta\vec{w}$  for some  $\alpha, \beta \in \mathbb{R}$  and  $\vec{v}, \vec{u}, \vec{w}$  unit-vectors then  $S_{k,\vec{v}} = \alpha S_{k,\vec{u}} + \beta S_{k,\vec{w}}$  in the statistical sense,
- (4) for each sequence of directions  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  there exists a unique state such that the probability of the event " $S_{k,\vec{v}_k} = +\frac{1}{2}\hbar$ " is exactly 1 for every  $k = 1, 2, \dots, n$ .

The first postulate is based on the results of the Stern-Gerlach experiment. The second is common sense: if we have two particles, one here the other on the other side of earth, then measurements on one of the particles should not affect measurements on the other one. (Actually, more than just common sense, if we except the theory of relativity this is must be so by the *causal* structure of spacetime.) The third postulate needs a little explanation, though it is a very natural one. In classical physics, the angular momentum  $\vec{N}$  of a spinning body is a *vectorial quantity*; it has a length and a direction (the axis of rotation). When we measure the angular momentum along direction  $\vec{v}$  (where  $\vec{v}$  is not necessarily the direction of the axis of rotation), what we get is

$$N_{\vec{v}} = \vec{v} \cdot \vec{N} = \cos(\theta) \|\vec{N}\|,$$

where  $\theta$  is the angle between  $\vec{v}$  and the axis of rotation. In simple words, what we get is the component of  $\vec{N}$  that falls in direction  $\vec{v}$ . Thus if  $\vec{v} = \alpha\vec{u} + \beta\vec{w}$  then

$$N_{\vec{v}} = \vec{v} \cdot \vec{N} = (\alpha\vec{u} + \beta\vec{w}) \cdot \vec{N} = \alpha\vec{u} \cdot \vec{N} + \beta\vec{w} \cdot \vec{N} = \alpha N_{\vec{u}} + \beta N_{\vec{w}}.$$

In case of the electron's spin, in general we cannot establish such equality; simply because it might not be possible to measure together the spin values in all three directions in question. However, by putting many elementary particles together, we can form a macroscopic body. Thus our classical experience suggests that the above relation should be satisfied at least in the *statistical sense*.

Finally, the fourth postulate is essentially a rephrasing of some of the conclusions drawn from the Stern-Gerlach experiment. First, such a state — in which all those probabilities are 1 — exists. Indeed, all we have to

do is to measure the spins in the given directions, and then “turn” (with an appropriate magnetic field) the particles found with spins “down”. Then upon measuring the spins again in the given directions, we will get only “up”-s. Second, such a state is unique because once we determine the spin values in some given directions, no further specification can be made.

§3. Instead of the quantities  $S_{k,\vec{v}}$ , it is more convenient to work with the quantities  $\sigma_{k,\vec{v}} := \frac{2}{\hbar} S_{k,\vec{v}}$ . Considering the general framework of Quantum Physics and the postulates made, to model this system, we need to fix a Hilbert space  $\mathcal{H}$  and operators  $\hat{\sigma}_{k,\vec{v}}$  acting on  $\mathcal{H}$  (where  $k = 1, 2, \dots, n$  and  $\vec{v}$  runs over all unit vectors in our 3-dimensional Euclidean space) such that

- (0)  $\hat{\sigma}_{k,\vec{v}}^* = \hat{\sigma}_{k,\vec{v}}$ ,
- (1)  $\hat{\sigma}_{k,\vec{v}}^2 = \mathbb{1}$ ,
- (2) if  $k \neq l$  then  $\hat{\sigma}_{k,\vec{v}}\hat{\sigma}_{l,\vec{w}} = \hat{\sigma}_{l,\vec{w}}\hat{\sigma}_{k,\vec{v}}$ ,
- (3) if  $\vec{v} = \alpha\vec{u} + \beta\vec{w}$  for some  $\alpha, \beta \in \mathbb{R}$  and  $\vec{v}, \vec{u}, \vec{w}$  unit vectors then  $\hat{\sigma}_{k,\vec{v}} = \alpha\hat{\sigma}_{k,\vec{u}} + \beta\hat{\sigma}_{k,\vec{w}}$ ,
- (4) for every  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  sequence of unit vectors there exists a state  $p$  on  $\mathcal{P}(\mathcal{H})$  such that  $p(\frac{1}{2}(\mathbb{1} + \hat{\sigma}_{k,\vec{v}_k})) = 1$  for every  $k = 1, 2, \dots, n$ .

§4.; It is now a mathematical problem to decide, whether such a system of operators exist or not and, in particular, how many solutions there are (if any). However, in reality we are not interested by the mere number of solutions. Two different solutions may be equivalent from the physical point of view. Without giving a rigorous definition, let us discuss now the concept of *physical equivalence*.

We want to find all system of operators satisfying the listed properties. Actually, to specify a solution it is enough to fix the operators  $\hat{\sigma}_{k,\vec{v}_j}$  where  $k = 1, 2, \dots, n$  and  $j = x, y, z$  and  $\vec{v}_x, \vec{v}_y, \vec{v}_z$  is a (previously chosen) ONB in our Euclidean space. Indeed, by property (3),

$$\hat{\sigma}_{k,\vec{v}} = a\hat{\sigma}_{k,\vec{v}_x} + b\hat{\sigma}_{k,\vec{v}_y} + c\hat{\sigma}_{k,\vec{v}_z}$$

where  $(a, b, c)$  are the coefficients of  $\vec{v}$  in our ONB; that is,  $\vec{v} = a\vec{v}_x + b\vec{v}_y + c\vec{v}_z$ . Second, we choose an ONB in  $\mathcal{H}$  (which we can indeed do, since — as we shall later see — it will turn out to be finite dimensional). Using our ONB we

may represent the  $3n$  operators  $\hat{\sigma}_{k,\vec{v}_j}$  (where  $n = 1, 2, \dots, n$  and  $j = x, y, z$ ) by  $3n$  matrices. To put it in another way, to give an actual solution we need to give  $3n$  matrices (satisfying certain properties).

Of course, one can choose different ONBs both in our Euclidean space and in  $\mathcal{H}$ . So a different list of  $3n$  matrices do not necessarily correspond to a different physical model; it may be that the difference is simply due to a different choice of coordinates (more precisely: a different choice of ONBs).

Accordingly, if we have two solutions, such that by a change of bases (either in our Euclidean space or in  $\mathcal{H}$ , or possibly both) the first one can be transformed into the second one, then we shall say that they are equivalent.

So what are the possible scenarios? In general, 3 things can happen.

- (1) There are no solutions.
- (2) There is a unique solution.
- (3) There are more, inequivalent solutions.

In the first case, we must conclude that the discussed framework of Quantum Physics is inadequate for describing the system in question; we should either modify our framework or our set of postulates. The second is the best possible case from a physicist point of view. In this case the (unique) model can be used to give predictions about reality. In the third case, we cannot give precise predictions, since each solution gives a (possibly) different one. We need to select the “good” model by confronting the different predictions against experimental data. To put it another way: we need further experimental data to formulate further postulates so that at the end there would be only a single solution. From the physicist point of view the best is when we use as little experimental input for setting up our model as possible. Ultimately, the aim to *use* our model to give as many predictions as possible; not the other way around (using experimental data to set up our model).

In our concrete case, when  $n = 2$  (that is, when we deal with a 2 particle system), we find two physically different solutions. We shall now describe an important difference between the two.

We shall say that a probability law is of “zero total spin”, if whenever we measure the two particles’ spin along the *same* directions, with probability 1 (that is: always) we get opposite values. That is,  $p$  is of “zero total spin” if and only if

$$p(\sigma_{1,\vec{v}}\sigma_{2,\vec{v}} = -1) = 1$$

for all  $\vec{v}$  unit vectors.

It turns out that in one of the solutions there is such a probability law, while in the other solution there is not.

In reality one can have a zero-spin particle decaying into two spin 1/2 particles. In this case — if the total angular momentum is to be conserved — the emerging particles' total spin (in the described sense) should be zero. Infact, experimental data tells us that this is indeed so. Thus postulating the existence of such a state reduces the number of the solutions to 1.

### 3rd topic

§1. As was discussed in the previous topic, we first choose an ONB  $\vec{v}_x, \vec{v}_y, \vec{v}_z$  in our Euclidean space and consider the generic operator  $\hat{\sigma}_{k,\vec{v}}$  as a linear combination of the operators  $\hat{\sigma}_{k,\vec{v}_j}$  where  $j = x, y, z$ . Then property (3) is automatically satisfied. On the other hand, after some work we find that property (0),(1) and (2) are satisfied if and only if

- (0)  $\hat{\sigma}_{k,\vec{v}_j}^* = \hat{\sigma}_{k,\vec{v}_j}$ ,
- (1)  $\hat{\sigma}_{k,\vec{v}_j}^2 = \mathbb{1}$ ,
- (2) if  $k \neq l$  then  $\hat{\sigma}_{k,\vec{v}_i} \hat{\sigma}_{l,\vec{v}_j} = \hat{\sigma}_{l,\vec{v}_j} \hat{\sigma}_{k,\vec{v}_i}$ ,
- (3) if  $i \neq j$  then  $\hat{\sigma}_{k,\vec{v}_i} \hat{\sigma}_{k,\vec{v}_j} = -\hat{\sigma}_{k,\vec{v}_j} \hat{\sigma}_{k,\vec{v}_i}$ .

Moreover, in this case these properties imply that the operators  $\{\hat{\sigma}_{k,\vec{v}_j}\}$  are linearly independent. Thus we should be able to give at least  $3n$  linearly independent operators on  $\mathcal{H}$ , so if  $n > 1$  (which we shall always assume from now on), then  $\dim(\mathcal{H}) > 2$  since  $\dim(\mathcal{B}(\mathcal{H})) = \dim(\mathcal{H})^2$ . In particular, we can apply Gleason's theorem and thus we can convert property (4) into the following one:

- (4) for every  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$  sequence of unit vectors there exists (an up-to-phase) unique vector  $\Psi \in \mathcal{H}$ ,  $\|\Psi\|^2 = 1$  such that  $\hat{\sigma}_{k,\vec{v}_k} \Psi = \Psi$  for all  $k = 1, 2, \dots, n$ .

In particular, up-to-phase, there exists a unique unit vector in  $\mathcal{H}$  that we shall denote by  $\Psi_{\uparrow\uparrow..\uparrow}$ , such that

$$\hat{\sigma}_{k,\vec{v}_z} \Psi_{\uparrow\uparrow..\uparrow} = \Psi_{\uparrow\uparrow..\uparrow}.$$

Let us now consider the two-particle case (that is, when  $n = 2$ ). Then, assuming that our operators satisfy the newly derived properties, we find that the 4 vectors

$$\Psi_{\uparrow\uparrow}, \Psi_{\downarrow\uparrow} := \hat{\sigma}_{1,\vec{v}_x} \Psi_{\uparrow\uparrow}, \Psi_{\uparrow\downarrow} := \hat{\sigma}_{2,\vec{v}_x} \Psi_{\uparrow\uparrow}, \Psi_{\downarrow\downarrow} := \hat{\sigma}_{2,\vec{v}_x} \Psi_{\downarrow\uparrow}$$

must form an ONB in  $\mathcal{H}$ . Using this ONB it turns out that the matrices of the operators  $\hat{\sigma}_{1,\vec{v}_x}, \hat{\sigma}_{1,\vec{v}_y}$  and  $\hat{\sigma}_{1,\vec{v}_z}$  must be

$$\sigma_x \otimes \mathbb{1}, \pm\sigma_z \otimes \mathbb{1}, \sigma_z \otimes \mathbb{1},$$

respectively, where  $\sigma_x, \sigma_y, \sigma_z$  are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

At this point, we shall use our freedom of choosing our ONB in our Euclidean space so that the matrix of  $\hat{\sigma}_{1,\vec{v}_y}$  is  $+\sigma_z \otimes \mathbb{1}$ . (By changing our ONB from  $\vec{v}_x, \vec{v}_y, \vec{v}_z$  to  $\vec{v}_x, -\vec{v}_y, \vec{v}_z$ , the matrix of  $\hat{\sigma}_{1,\vec{v}_y}$  is multiplied by  $-1$ . Note that in contrast, exchanging  $\vec{v}_x$  to  $-\vec{v}_x$  would not cause a change in the matrix of  $\hat{\sigma}_{1,\vec{v}_x}$ . This is due to the fact that  $\hat{\sigma}_{1,\vec{v}_x}$  was actually used at fixing our ONB in  $\mathcal{H}$ , whereas  $\hat{\sigma}_{1,\vec{v}_y}$  was not.)

It then turns out that the matrices of  $\hat{\sigma}_{2,\vec{v}_x}, \hat{\sigma}_{2,\vec{v}_y}$  and  $\hat{\sigma}_{2,\vec{v}_z}$  must be

$$\mathbb{1} \otimes \sigma_x, \pm\mathbb{1} \otimes \sigma_y, \mathbb{1} \otimes \sigma_z,$$

respectively. By direct check, regardless whether we choose the  $+$  or  $-$  sign for the matrix of  $\hat{\sigma}_{2,\vec{v}_y}$ , we get a solution.

§2. In the previous topic we defined the concept of “zero total spin”. By using Gleason’s theorem and making some easy transformations, we obtain that a “zero total spin” probability law exists if and only if there is a density operator  $\rho \geq 0, \text{Tr}(\rho) = 1$  such that

$$\text{Tr}(\rho \hat{\sigma}_{1,\vec{v}} \hat{\sigma}_{2,\vec{v}}) = -1$$

or equivalently, such that

$$\text{Tr}(\rho(\mathbb{1} + \hat{\sigma}_{1,\vec{v}} \hat{\sigma}_{2,\vec{v}})) = 0$$

for all  $\vec{v}$  unit vectors. However,  $\hat{\sigma}_{1,\vec{v}}$  and  $\hat{\sigma}_{2,\vec{v}}$  commute so their product is self-adjoint. Moreover, we find that the square of their product is  $\mathbb{1}$ . Considering spectrums, it is then easy to see that

$$\mathbb{1} + \hat{\sigma}_{1,\vec{v}}\hat{\sigma}_{2,\vec{v}} \geq 0$$

so since  $\rho$  is also positive, by the previous “trace = 0” equality

$$\rho(\mathbb{1} + \hat{\sigma}_{1,\vec{v}}\hat{\sigma}_{2,\vec{v}}) = 0$$

or equivalently, that

$$\text{Ker}(\rho) \supset \text{Im}(\mathbb{1} + \hat{\sigma}_{1,\vec{v}}\hat{\sigma}_{2,\vec{v}}).$$

However, the images of the matrices

$$\mathbb{1} + (\sigma_x \otimes \sigma_x), \mathbb{1} - (\sigma_y \otimes \sigma_y), \mathbb{1} + (\sigma_z \otimes \sigma_z)$$

actually span the full space  $\mathbb{C}^4$ . Thus  $\text{Ker}(\rho)$  should be the full space, or equivalently,  $\rho$  should be zero — which is impossible, since its trace should be 1. On the other hand, the images of the matrices

$$\mathbb{1} + (\sigma_x \otimes \sigma_x), \mathbb{1} + (\sigma_y \otimes \sigma_y), \mathbb{1} + (\sigma_z \otimes \sigma_z)$$

span only a 3-dimensional subspace; its orthogonal is 1-dimensional and is spanned by (the coordinate vector of) the unit vector

$$\Psi = \frac{1}{\sqrt{2}}(\Psi_{\uparrow\downarrow} - \Psi_{\downarrow\uparrow}).$$

Thus the image of  $\rho$  must be the 1-dimensional subspace  $\mathbb{C}\Psi$ . This shows that even in case of using the solution “with the plus sign”, there can be at most 1 such  $\rho$ . Indeed, there is only a single density operator whose image is exactly  $\mathbb{C}\Psi$  — namely,  $|\Psi\rangle\langle\Psi|$ ; the ortho-projection onto  $\mathbb{C}\Psi$ . On the other hand, by straightforward substitution this density operator, for the solution “with the plus sign”, indeed satisfies the required condition.

## 4th topic

## 5th topic

§1. We want to treat the two systems  $I$  and  $II$  together as a single (composite) system  $I + II$ . In particular, we need to think of an event regarding any

of these systems as infact an event regarding the composite system. Thus, from the mathematical point of view, we will have two sub ortho-lattices  $\mathcal{L}_1, \mathcal{L}_2$  of a (bigger) ortho-lattice  $\mathcal{L}$  (associated to  $I + II$ ).

Any event of the composite system should be possible to describe in terms of events regarding  $I$  and events regarding  $II$  (simply because  $I + II$  has no “further parts”: it is really composed of  $I$  and  $II$ , only). Thus, by a repeated use of the operations  $\vee, \wedge$  and  $\neg$ , starting from the elements of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  we should be able to get any element of  $\mathcal{L}$ .

One also expects a certain kind of independence of the two parts. Consider a certain ensemble of the composite system  $I + II$  prepared in such a way that (together) they follow the statistics given by probability law  $p_1$ . Consider also another, independent ensemble, which is governed by probability law  $p_2$ . Of course, the pairing between  $I$  and  $II$  is not something physical; it is just that we treat certain pairs as a single system. But we are also allowed to pair (in our head) the given copies of  $I$  and  $II$  in a different way and treat the thus obtained pairs as examples of the composite system  $I + II$ . In particular, we shall consider copies of the composite system  $I + II$  where  $I$  is from the ensemble (of composite systems) governed by  $p_1$  and  $II$  is from the ensemble governed by  $p_2$ . In this new ensemble, the two parts should be independent; if  $x \in \mathcal{L}_1$  and  $y \in \mathcal{L}_2$  then the probability of  $x \wedge y$  should be  $p_1(x)p_2(y)$ .

Apart from these two properties, one may also expects that an event regarding  $I$  can be simultaneously checked with an event regarding  $II$ . Of course, in the classical case this is a tautology, since any two events can be simultaneously checked. However, in the quantum case this gives a nontrivial condition.

So we shall say that  $\mathcal{L}$  is the composite of its two sub ortho-lattices  $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{L}$ , iff

- (1)  $\mathcal{L}_1$  together with  $\mathcal{L}_2$  generate  $\mathcal{L}$ ,
- (2) for every pair of probability laws  $p_1, p_2$  on  $\mathcal{L}$  there exists a *product probability law*  $p$ ; that is, a probability law  $p$  satisfying  $p(x \wedge y) = p_1(x)p_2(y)$  for all  $x \in \mathcal{L}_1, y \in \mathcal{L}_2$ ,
- (3)  $x$  and  $y$  can be simultaneously checked whenever  $x \in \mathcal{L}_1$  and  $y \in \mathcal{L}_2$ .

§2. *Example: the classical case.* Suppose we cast two dice. The ortho-lattice of events is the power set of  $H \times H$  where  $H = \{1, 2, 3, 4, 5, 6\}$ . That is, each

event will be thought as a subset of  $H \times H$ . For example, the event “the sum of the two outcomes is greater than 9” is represented by the subset

$$\{(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}.$$

An event regarding the first die only, is of the form  $X \times H$  where  $X \subset H$ . For example, the event that “the outcome on the first die is exactly 4” is represented by the subset

$$\{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\} = \{4\} \times \{1, 2, 3, 4, 5, 6\}.$$

So the first die corresponds to the sub ortho-lattice  $\mathcal{P}(H) \times H \subset \mathcal{P}(H \times H)$  whereas the second one to  $H \times \mathcal{P}(H) \subset \mathcal{P}(H \times H)$  (where “ $\mathcal{P}$ ” stands for “the power set of”). It is fairly easy to check that properties (1), (2) and (3) are satisfied and so that  $\mathcal{P}(H \times H)$  can be regarded to be the composite of  $\mathcal{P}(H) \times H$  and  $H \times \mathcal{P}(H)$  in the precise sense which we have just defined. Note that for a single die,  $H$  is the total event  $\mathbb{1}$ , so we could actually write that the two sub ortho-lattices in question are:  $\mathcal{P}(H) \times \mathbb{1}$  and  $\mathbb{1} \times \mathcal{P}(H)$ , which would bring the notation even closer to the one used in Quantum Physics.

§3. *Example: the quantum case.* We have seen that in case of a system of 2 spin 1/2 particles, regarding the spin-content only, the Hilbert space of the system is of the form  $\mathcal{H} \otimes \mathcal{H}$ ; spin quantities associated to the first particle are described by operators of the form  $A \otimes \mathbb{1}$ , whereas spin quantities associated to the second particle are described by operators of the form  $\mathbb{1} \otimes A$ . Actually, in this example it is easy to see, that in fact events regarding the spin content of the first particle are in one-to-one connection with projections of the form  $Q \otimes \mathbb{1}$ , whereas events regarding the spin content of the second particle are in one-to-one connection with projections of the form  $\mathbb{1} \otimes Q$ .

Without further explanation, in this course we shall simply except that this is the general situation. Namely, when modelling a system composed of two parts, in Quantum Physics we use a Hilbert space of the form

$$\mathcal{H}_1 \otimes \mathcal{H}_2$$

with the sub ortho-lattices

$$\mathcal{P}(\mathcal{H}_1) \otimes \mathbb{1} \subset \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) \quad \text{and} \quad \mathbb{1} \otimes \mathcal{P}(\mathcal{H}_2) \subset \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

corresponding to the first and second subsystems, respectively.

So how about the required properties? Property (3) is satisfied because operators of the form  $A \otimes \mathbb{1}$  commute with all operators of the form  $\mathbb{1} \otimes A$ . Strictly speaking, in case of infinite dimensions, property (1) is not true: starting with projections of the form  $Q \otimes \mathbb{1}$  and  $\mathbb{1} \otimes Q$ , one cannot obtain all projections by a *finite* use of the operators  $\vee, \wedge, \neg$ , so infact our definition should be slightly modified to allow limits, etc. but at this course we shall only work with finite-level quantum systems so we ignore such delicate details and stick with our original definition. If  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are finite dimensional, then property (1) can be checked (though it is not completely trivial); however we shall not discuss the proof of this.

It is easy to check that

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\mathrm{Tr}(B)$$

and that if  $\rho_1 \in \mathcal{B}(\mathcal{H}_1)$  and  $\rho_2 \in \mathcal{B}(\mathcal{H}_2)$  are density operators, then  $\rho_1 \otimes \rho_2$  is a density operator on  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Let  $p$  be the probability law given by  $\rho_1 \otimes \rho_2$  and suppose  $x \in \mathcal{P}(\mathcal{H}) \otimes \mathbb{1}$  and  $y \in \mathbb{1} \otimes \mathcal{P}(\mathcal{H})$ . Then  $x = Q_1 \otimes \mathbb{1}$  and  $y = \mathbb{1} \otimes Q_2$  for some  $Q_1 \in \mathcal{P}(\mathcal{H}_1)$  and  $Q_2 \in \mathcal{P}(\mathcal{H}_2)$  and

$$p(x) = \mathrm{Tr}((\rho_1 \otimes \rho_2)(Q_1 \otimes \mathbb{1})) = \mathrm{Tr}(\rho_1 Q_1 \otimes \rho_2) = \mathrm{Tr}(\rho_1 Q_1)\mathrm{Tr}(\rho_2) = \mathrm{Tr}(\rho_1 Q_1).$$

Similarly, one gets that  $p(y) = \mathrm{Tr}(\rho_2 Q_2)$ . Thus, since

$$x \wedge y = (Q_1 \otimes \mathbb{1})(\mathbb{1} \otimes Q_2) = Q_1 \otimes Q_2,$$

we have that

$$\begin{aligned} p(x \wedge y) &= \mathrm{Tr}((\rho_1 \otimes \rho_2)(Q_1 \otimes Q_2)) = \mathrm{Tr}(\rho_1 Q_1 \otimes \rho_2 Q_2) \\ &= \mathrm{Tr}(\rho_1 Q_1)\mathrm{Tr}(\rho_2 Q_2) = p(x)p(y), \end{aligned}$$

so  $p$  is a product probability law.