Linear algebra exercises

These exercises should not all be considered as homeworks. (We will discuss in class which one you can handle in.) Rather, they are for you to see wether you need a further brush-up in linear algebra or not. Feel free to ask me questions about the problems and have a good fun with them!

Problem 2.1. *Consider the linear subspaces* $U, W \subset \mathbb{R}^4$ *defined as* $U := \mathrm{Span}\{x, y\}$ *and* $W := \mathrm{Span}\{z, v\}$*, where*

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix},\ y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ z = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 1 \end{pmatrix},\ v = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$$

Give a basis in $U \cap W$ *and calculate the dimension of* $U + W$*.*

Problem 2.2. *Consider the subspaces* $U, W \subset \mathbb{R}^4$ *defined as*

$$\begin{aligned} U : &= \{x \in \mathbb{R}^4 \,|\, x_1 + x_4 = 3x_2 - 2x_3 = 0\} \\ W : &= \{x \in \mathbb{R}^4 \,|\, x_1 + x_2 + x_3 + 6x_4 = 3x_3 + x_4 = 0\}. \end{aligned}$$

Give a basis in $U + W$ *and calculate the dimension of* $U \cap W$*.*

Problem 2.3. *Let* V *be a 10-dimensional vectorspace and suppose that* U_1, U_2 *and* U_3 *are all 8-dimensional subspaces of* V*. How much can be the dimension of* $U_1 \cap U_2 \cap U_3$*? Give an example for all possible values (explaining also why the dimension cannot take other values).*

Problem 2.4. *Consider the vector space* V_3 *of polynomials of order* ≤ 2*. Verify that the polynomials* p_1, p_2, p_3 *given by the formulas*

$$p_1(x) = x^2,\ p_2(x) = (x + 1)^2,\ p_3(x) = x$$

form a basis in V_3*, and that the derivation is a linear map from* V_3 *to* V_3*. Write down the matrix of the derivation in the given basis.*

Problem 2.5. *Let* $U := \{x \in \mathbb{C}^3 \,|\, x_1 + ix_2 = 0\}$ *and* $W := \mathrm{Span}\{w\} \subset \mathbb{C}^3$ *where* $w_1 = w_2 = w_3 = 1$*. Verify that* U *and* W *are complementary and calculate the matrix (in the std. basis) of the projection onto* U *along* W*.*

Problem 2.6. *Let* P *and* Q *be two projections of the vector space* V*. Prove that* $P + Q$ *is again a projection iff* $\mathrm{Im}(P) \subset \mathrm{Ker}(Q)$ *and* $\mathrm{Ker}(P) \supset \mathrm{Im}(Q)$*.*

Problem 2.7. *Let P be the projection onto U along W. Show that the projection onto W along U is precisely $\mathbb{1} - P$.*

Problem 2.8. *Let $A, B : V \to W$ be two linear maps, and suppose that* $\dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(B)) = \dim(\mathrm{Im}(A + B)) = 1$. *Prove that either* $\mathrm{Im}(A) = \mathrm{Im}(B)$ *or* $\mathrm{Ker}(A) = \mathrm{Ker}(B)$.

Problem 2.9. *Suppose the linear operator A is such that $A^n = 0$. Show that $\mathbb{1} - A$ is invertable and express its inverse as a polynomial of A.*

Problem 2.10. *Alice was given a linear map $A : V \to W$. She fixed a bases in V and another one in W, and then began to work out with her choices the matrix of A. The same linear map was given to Bob, too. He also fixed a bases in V and another one in W, and he also began to work out (using his choices) the matrix of A. After some time they arrived to the following partial results:*

$$\text{Alice} : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & ? \end{pmatrix}, \quad \text{Bob} : \begin{pmatrix} 1 & 1 & 1 \\ 4 & ? & ? \end{pmatrix}.$$

- *Suppose the missing element in Alice's matrix is* 5*. Then, by the information so far given, is it possible to determine the mixing elements in Bob's matrix? And if the missing element in Alice's matrix is* 6*?*
- *Can you give a coordinate triple in such a way, that — regardless of the missing elements in the given matrices — it surely represents different vectors of V for Alice and Bob? If yes, do so, if no, explain why not.*

Problem 2.11. *Calculate* $\det(AB + A)$ *where*

$$A = \begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & 3 & 5 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and B is the transpose of A.

Problem 2.12. *Is there an $n \times n$ real matrix whose square is $-\mathbb{1}$? More precisely, for what values of n there is, and for what values n there is not?*

Problem 2.13. *Suppose the sum of each row in a square-matrix is* 1*. Does this reveal something about its eigenvalues and eigenvectors? Find all eigenvalues and eigenspaces of the matrix*

$$\begin{pmatrix} 0 & 2-i & -1+i \\ 2+i & 3 & -4-i \\ -1-i & -4+i & 6 \end{pmatrix}.$$

Problem 2.14. *Suppose A, B are two commuting operators. Prove that each eigenspace of A is an invariant space for B.*

Problem 2.15. *Let V be a finite dimensional vectorspace, and $A : V \to V$ a linear operator. Prove that* $\mathrm{Ker}(A)$ *and* $\mathrm{Im}(A)$ *are complementary if and only if* $\mathrm{Ker}(A) = \mathrm{Ker}(A^2)$ *which is further if and only if* $\mathrm{Im}(A) = \mathrm{Im}(A^2)$*. Conclude that if in particular V is a vectorspace over $\mathbb{C}$, then A is diagonalizable if and only if* $\mathrm{Ker}(A - \lambda \mathbb{1}) = \mathrm{Ker}((A - \lambda \mathbb{1})^2)$ *for every $\lambda \in \mathbb{C}$.*

Problem 2.16. *Fill in the third column of the matrix*

$$\begin{pmatrix} i & 2 & ? \\ 1 & i & ? \\ 0 & 0 & ? \end{pmatrix}$$

in such a way that it will **not** *be diagonalizable.*

Problem 2.17. *Let A be a linear map such that $A^2 = A^3$. Show that A is diagonalizable if and only if A is a projection. Give an example for a linear map A that satisfies $A^2 = A^3$ but is not a projection.*