

System of spin 1/2 particles

§1. Among others, the following important conclusions can be drawn from the results of the Stern-Gerlach experiment.

- The electron's spin (“intrinsic angular momentum”), when measured in a given direction, is always found to be either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$.
- If in a given direction we measure the electron's spin and obtain the value s , then, upon repeating the measurement — if the electron between the two measurements was free of external actions and disturbances — we obtain the same value s .
- Once the value of the electron's spin in a certain direction is determined, the electron's state (regarding its spin) cannot be further specified.

The first listed item needs no further comment, but there are certain things to remark about the second and third items. The fact that upon repeating measurement we find again the previously obtained value does not seem too surprising. This is just what we would expect by our classical experience: if nothing acted on the electron, why would its spin's value change? However, by the Stern-Gerlach experiment we also learn that some uncertainty is always present: if we know the value of the spin in a certain direction with certainty, then in an orthogonal direction its spin is just as likely to be found plus as minus; so no prediction can be given. Because of this nontrivial involvement of probabilities, one might think that if we repeat an experiment, in general we do not get the same value again. The point of the second item is that this is not so.

Finally, a remark about the third item. Suppose we take a beam of electrons and we perform a spin-measurement in direction \vec{v} . In general, the beam will split into two: electrons with spin “up” and with spin “down” (with respect to direction \vec{v}) will be deflected in different directions. After performing a further measurement of spin, this time in direction \vec{w} where $\vec{w} \neq \pm\vec{v}$, we will end up with four beams. One might think that electrons whose spin in direction \vec{w} is “up” and whose spin in direction \vec{v} was previously “up” would behave in a different way than those whose spin in direction \vec{w} is “up” but whose spin in direction \vec{v} was previously “down”. However, this is not so: if we were to perform further experiments, we would find no difference between the mentioned two beams. So after the electron's spin is determined

in the direction \vec{w} , the electron forgets about its “previous life”. Its state — regarding its spin — cannot be further specified; this is the conclusion drawn in the third item.

§2. We want to set up postulates regarding the spin-content of a system of n spin $1/2$ particles. We want to ignore other parts of reality — so for example kinematics will be left out from our model. (This simplification makes things much more cleaner; moreover, it also makes possible to avoid the use of infinite dimensional Hilbert spaces.) Thus the basic quantities that we want to describe are the spin-values $S_{k,\vec{v}}$, where $k = 1, 2, \dots, n$ and \vec{v} is a direction (that is, a unit vector) in our 3-dimensional Euclidean space. We postulate that

- (1) the possible values of $S_{k,\vec{v}}$ are $\pm\frac{1}{2}\hbar$,
- (2) if $k \neq l$, then $S_{k,\vec{v}}$ and $S_{l,\vec{w}}$ can be simultaneously measured without disturbing each other,
- (3) if $\vec{v} = \alpha\vec{u} + \beta\vec{w}$ for some $\alpha, \beta \in \mathbb{R}$ and $\vec{v}, \vec{u}, \vec{w}$ unit-vectors then $S_{k,\vec{v}} = \alpha S_{k,\vec{u}} + \beta S_{k,\vec{w}}$ in the statistical sense,
- (4) for each sequence of directions $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ there exists a unique probability law such that the probability of the event “ $S_{k,\vec{v}_k} = +\frac{1}{2}\hbar$ ” is exactly 1 for every $k = 1, 2, \dots, n$.

The first postulate is based on the results of the Stern-Gerlach experiment. The second is common sense: if we have two particles, one here the other on the other side of earth, then measurements on one of the particles should not affect measurements on the other one. (Actually, more than just common sense, if we except the theory of relativity this is must be so by the *causal* structure of spacetime.) The third postulate needs a little explanation, though it is a very natural one. In classical physics, the angular momentum \vec{S} of a spinning body is a *vectorial quantity*; it has a length and a direction (the axis of rotation). When we measure the angular momentum along direction \vec{v} (where \vec{v} is not necessarily the direction of the axis of rotation), what we get is

$$S_{\vec{v}} = \vec{v} \cdot \vec{S} = \cos(\theta) \|\vec{S}\|,$$

where θ is the angle between \vec{v} and the axis of rotation. In simple words, what we get is the component of \vec{S} that falls in direction \vec{v} . Thus if $\vec{v} = \alpha\vec{u} + \beta\vec{w}$

then

$$S_{\vec{v}} = \vec{v} \cdot \vec{S} = (\alpha\vec{u} + \beta\vec{w}) \cdot \vec{S} = \alpha\vec{u} \cdot \vec{S} + \beta\vec{w} \cdot \vec{S} = \alpha S_{\vec{u}} + \beta S_{\vec{w}}.$$

In case of a single electron's spin, in general we cannot establish such equality; simply because it might not be possible to measure together the spin values in all three directions in question. However, by putting many elementary particles together, we can form a macroscopic body. Thus our classical experience suggests that the above relation should be satisfied at least in the *statistical sense*.

Finally, the fourth postulate is essentially a rephrasing of some of the conclusions drawn from the Stern-Gerlach experiment. First, such a state — in which all those probabilities are 1 — exists. Indeed, all we have to do is to measure the spins in the given directions, and then “turn” (with an appropriate magnetic field) the particles found with spins “down”. Then upon measuring the spins again in the given directions, we will get only “up”-s. Second, such a state is unique because once we determine the spin values in some given directions, no further specification¹ can be made.

§3. Instead of the quantities $S_{k,\vec{v}}$, it is more convenient to work with the quantities $\sigma_{k,\vec{v}} := \frac{2}{\hbar} S_{k,\vec{v}}$. Considering the general framework of quantum physics and the postulates made, to model this system, we need to fix a Hilbert space \mathcal{H} and operators $\hat{\sigma}_{k,\vec{v}}$ acting on \mathcal{H} (where $k = 1, 2, \dots, n$ and \vec{v} runs over all unit vectors in our 3-dimensional Euclidean space) such that

- (0) $\hat{\sigma}_{k,\vec{v}}^* = \hat{\sigma}_{k,\vec{v}}$,
- (1) $\hat{\sigma}_{k,\vec{v}}^2 = \mathbb{1}$,
- (2) if $k \neq l$ then $\hat{\sigma}_{k,\vec{v}}\hat{\sigma}_{l,\vec{w}} = \hat{\sigma}_{l,\vec{w}}\hat{\sigma}_{k,\vec{v}}$,
- (3) if $\vec{v} = \alpha\vec{u} + \beta\vec{w}$ for some $\alpha, \beta \in \mathbb{R}$ and $\vec{v}, \vec{u}, \vec{w}$ unit vectors then $\hat{\sigma}_{k,\vec{v}} = \alpha\hat{\sigma}_{k,\vec{u}} + \beta\hat{\sigma}_{k,\vec{w}}$,
- (4) for every $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ sequence of unit vectors there exists a probability law p on $\mathcal{P}(\mathcal{H})$ such that $p(\frac{1}{2}(\mathbb{1} + \hat{\sigma}_{k,\vec{v}_k})) = 1$ for every $k = 1, 2, \dots, n$.

¹Of course we could further specify the electron's position, momentum, etc. However, we do not want to consider kinematics: for us, two states with the same spin-content (but possibly different kinematics) are regarded as identical. So this last condition is in some sense also a *definition* of what we consider part of the system.

§4.; It is now a mathematical problem to decide, whether such a system of operators exist or not and, in particular, how many solutions there are (if any). However, in reality we are not interested by the mere number of solutions. Two different solutions may be equivalent from the physical point of view. Without giving a rigorous definition, let us discuss now the concept of *physical equivalence*.

We want to find all system of operators satisfying the listed properties. Actually, to specify a solution it is enough to fix the oerators $\hat{\sigma}_{k,\vec{v}_j}$ where $k = 1, 2, \dots, n$ and $j = x, y, z$ and $\vec{v}_x, \vec{v}_y, \vec{v}_z$ is a (previously chosen) ONB in our Euclidean space. Indeed, by property (3),

$$\hat{\sigma}_{k,\vec{v}} = a\hat{\sigma}_{k,\vec{v}_x} + b\hat{\sigma}_{k,\vec{v}_y} + c\hat{\sigma}_{k,\vec{v}_z}$$

where (a, b, c) are the coefficients of \vec{v} in our ONB; that is, $\vec{v} = a\vec{v}_x + b\vec{v}_y + c\vec{v}_z$. Second, we choose an ONB in \mathcal{H} (which we can indeed do, since — as we shall later see — it will turn out to be finite dimensional). Using our ONB we may represent the $3n$ operators $\hat{\sigma}_{k,\vec{v}_j}$ (where $n = 1, 2, \dots, n$ and $j = x, y, z$) by $3n$ matrices. To put it in another way, to give an actual solution we need to give $3n$ matrices (satisfying certain properties).

Of course, one can choose different ONBs both in our Euclidean space and in \mathcal{H} . So a different list of $3n$ matrices do not necessarily correspond to a different physical model; it may be that the difference is simply due to a different choice of coordinates (more precisely: a different choice of ONBs).

Accordingly, if we have two solutions, such that by a change of bases (either in our Euclidean space or in \mathcal{H} , or possibly both) the first one can be transformed into the second one, then we shall say that they are equivalent.

So what are the possible scenarios? In general, 3 things can happen.

- (1) There are no solutions.
- (2) There is a unique solution.
- (3) There are more, inequivalent solutions.

In the first case, we must conclude that the discussed framework of quantum physics is inadequate for describing the system in question; we should either modify our framework or our set of postulates. The second is the best possible case from a physicist point of view. In this case the (unique) model can be used to give predictions about reality. In the third case, we cannot give precise predictions, since each solution gives a (possibly) different one.

We need to select the “good” model by confronting the different predictions against experimental data. To put it another way: we need further experimental data to formulate further postulates so that at the end there would be only a single solution. From the physicist point of view the best is when we use as little experimental input for setting up our model as possible. Ultimately, the aim to *use* our model to give as many predictions as possible; not the other way around (using experimental data to set up our model).

In our concrete case, when $n = 2$ (that is, when we deal with a 2 particle system), we find two physically different solutions. We shall now describe these solutions and analyze the difference between them.

§5. As was explained, we first choose an ONB $\vec{v}_x, \vec{v}_y, \vec{v}_z$ in our Euclidean space and consider the generic operator $\hat{\sigma}_{k,\vec{v}}$ as a linear combination of the operators $\hat{\sigma}_{k,\vec{v}_j}$ where $j = x, y, z$. Then property (3) is automatically satisfied. On the other hand, after some work we find that property (0),(1) and (2) are satisfied if and only if

- (0) $\hat{\sigma}_{k,\vec{v}_j}^* = \hat{\sigma}_{k,\vec{v}_j}$,
- (1) $\hat{\sigma}_{k,\vec{v}_j}^2 = \mathbb{1}$,
- (2) if $k \neq l$ then $\hat{\sigma}_{k,\vec{v}_i} \hat{\sigma}_{l,\vec{v}_j} = \hat{\sigma}_{l,\vec{v}_j} \hat{\sigma}_{k,\vec{v}_i}$,
- (3) if $i \neq j$ then $\hat{\sigma}_{k,\vec{v}_i} \hat{\sigma}_{k,\vec{v}_j} = -\hat{\sigma}_{k,\vec{v}_j} \hat{\sigma}_{k,\vec{v}_i}$.

Moreover, in this case these properties imply that the operators $\{\hat{\sigma}_{k,\vec{v}_j}\}$ are linearly independent. Thus we should be able to give at least $3n$ linearly independent operators on \mathcal{H} , so if $n > 1$ (which we shall always assume from now on), then $\dim(\mathcal{H}) > 2$ since $\dim(\mathcal{B}(\mathcal{H})) = \dim(\mathcal{H})^2$. In particular, we can apply Gleason’s theorem and thus we can convert property (4) into the following one:

- (4) for every $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ sequence of unit vectors there exists (an up-to-phase) unique vector $\Psi \in \mathcal{H}$, $\|\Psi\|^2 = 1$ such that $\hat{\sigma}_{k,\vec{v}_k} \Psi = \Psi$ for all $k = 1, 2, \dots, n$.

In particular, up-to-phase, there exists a unique unit vector in \mathcal{H} that we shall denote by $\Psi_{\uparrow\uparrow..\uparrow}$, such that

$$\hat{\sigma}_{k,\vec{v}_z} \Psi_{\uparrow\uparrow..\uparrow} = \Psi_{\uparrow\uparrow..\uparrow}.$$

Let us now consider the two-particle case (that is, when $n = 2$). Then, assuming that our operators satisfy the newly derived properties, we find that the 4 vectors

$$\Psi_{\uparrow\uparrow}, \Psi_{\downarrow\uparrow} := \hat{\sigma}_{1,\vec{v}_x} \Psi_{\uparrow\uparrow}, \Psi_{\uparrow\downarrow} := \hat{\sigma}_{2,\vec{v}_x} \Psi_{\uparrow\uparrow}, \Psi_{\downarrow\downarrow} := \hat{\sigma}_{2,\vec{v}_x} \Psi_{\downarrow\uparrow}$$

must form an ONB in \mathcal{H} . Using this ONB it turns out that the matrices of the operators $\hat{\sigma}_{1,\vec{v}_x}, \hat{\sigma}_{1,\vec{v}_y}$ and $\hat{\sigma}_{1,\vec{v}_z}$ must be

$$\sigma_x \otimes \mathbb{1}, \pm\sigma_y \otimes \mathbb{1}, \sigma_z \otimes \mathbb{1},$$

respectively, where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

At this point, we shall use our freedom of choosing our ONB in our Euclidean space so that the matrix of $\hat{\sigma}_{1,\vec{v}_y}$ is $+\sigma_y \otimes \mathbb{1}$. (By changing our ONB from $\vec{v}_x, \vec{v}_y, \vec{v}_z$ to $\vec{v}_x, -\vec{v}_y, \vec{v}_z$, the matrix of $\hat{\sigma}_{1,\vec{v}_y}$ is multiplied by -1 . Note that in contrast, exchanging \vec{v}_x to $-\vec{v}_x$ would not cause a change in the matrix of $\hat{\sigma}_{1,\vec{v}_x}$. This is due to the fact that $\hat{\sigma}_{1,\vec{v}_x}$ was actually used at fixing our ONB in \mathcal{H} , whereas $\hat{\sigma}_{1,\vec{v}_y}$ was not.)

It then turns out that the matrices of $\hat{\sigma}_{2,\vec{v}_x}, \hat{\sigma}_{2,\vec{v}_y}$ and $\hat{\sigma}_{2,\vec{v}_z}$ must be

$$\mathbb{1} \otimes \sigma_x, \pm\mathbb{1} \otimes \sigma_y, \mathbb{1} \otimes \sigma_z,$$

respectively. By direct check, regardless whether we choose the $+$ or $-$ sign for the matrix of $\hat{\sigma}_{2,\vec{v}_y}$, we get a solution.

§6. We shall say that a probability law is of “zero total spin”, if whenever we measure the two particles’ spin along the *same* directions, with probability 1 (that is: always) we get opposite values. That is, p is of “zero total spin” if and only if

$$p(\sigma_{1,\vec{v}}\sigma_{2,\vec{v}} = -1) = 1$$

for all \vec{v} unit vectors. By using Gleason’s theorem and making some easy transformations, we obtain that a “zero total spin” probability law exists if and only if there is a density operator $\rho \geq 0, \text{Tr}(\rho) = 1$ such that

$$\text{Tr}(\rho\hat{\sigma}_{1,\vec{v}}\hat{\sigma}_{2,\vec{v}}) = -1 \quad (*)$$

for all \vec{v} unit vectors. So if ρ is to describe such a probability law for the model with the “-” sign, then in particular, with denoting the matrix of ρ in our chosen ONB with ρ , again, we should have

$$\text{Tr}(\rho(\sigma_x \otimes \sigma_x)) = j\text{Tr}(\rho(\sigma_y \otimes \sigma_y)) = \text{Tr}(\rho(\sigma_z \otimes \sigma_z)) = -1.$$

where $j = -1$. However, there is no density operator ρ satisfying the above equation with $j = -1$. On the other hand, there is a unique density operator satisfying the equation with $j = +1$: namely, $\rho = |\Psi\rangle\langle\Psi|$ where

$$\Psi = \frac{1}{\sqrt{2}} (\Psi_{\uparrow\downarrow} - \Psi_{\downarrow\uparrow}).$$

Direct check shows then that for the model with the “+” sign, this density operator satisfies (*) for all directions (not only for the 3 coordinate-axes). So as we have seen, in one of the models one there is such a probability law, while in the other there is not.

In reality one can have a zero-spin particle decaying into two spin 1/2 particles. In this case — if the total angular momentum is to be conserved — the emerging particles’ total spin (in the described sense) should be zero. Infact, experimental data tells us that this is indeed so. Thus postulating the existence of such a state selects a unique model.