Summary of the general framework of quantum physics, or: how to model a system in quantum physics

§1. Fix a Hilbert space $\mathcal{H}$; elements of the ortho-lattice $\mathcal{P}(\mathcal{H})$ will stand for the possible events that can occur in our system. By Gleason's theorem, if $\dim(\mathcal{H}) > 2$ (which we shall always assume from now on), then probability functions on $\mathcal{P}(\mathcal{H})$ are in one-to-one connection with density operators on $\mathcal{H}$ via the formula

$$p(Q) = Tr(\rho Q) \qquad (Q \in \mathcal{P}(\mathcal{H}))$$

and states of the system (i.e. pure probability functions) are in one-to-one connection with density operators of the form $|\Psi\rangle\langle\Psi|$ where $\Psi \in \mathcal{H}$ is a vector of unit length. Note that in this case

$$p(Q) = Tr(|\Psi\rangle\langle\Psi|Q) = \langle\Psi, Q\Psi\rangle$$

and that $|\Psi\rangle\langle\Psi| = |\Psi'\rangle\langle\Psi'|$ if and only if Ψ' is a (unit-) multiple of Ψ; in other words, if and only if they only differ in their "phases".

§2. For a (real-, discrete-valued) physical quantity A, one can consider "$A = \lambda$", that is, the event that the value of A is λ. In the model, for each such event we must choose an ortho-projection (that will stand for this event). Introducing A into our model means fixing each such ortho-projection.

The choice of these projections must satisfy certain rules. This is because upon measurement, a *single* value of A is obtained, and hence

(i) "$A = \lambda$" and "$A = \mu$" where $\lambda \neq \mu$ must be exclusive.

Moreover, as A must take *some* value,

(ii) $\vee_\lambda$"$A = \lambda$" must be the total event $\mathbb{1}$.

Let us denoting the projection associated to "$A = \lambda$" by $P_{``A=\lambda"}$. Then the natural requirements (i) and (ii) can be turned into the following algebraic requirements:

$$\text{(i) } \mathrm{P}_{``\mathrm{A}=\lambda"}\mathrm{P}_{``\mathrm{A}=\mu"} = \delta_{\lambda,\mu}\mathrm{P}_{``\mathrm{A}=\mu"}, \qquad \text{(ii) } \sum_\lambda \mathrm{P}_{``\mathrm{A}=\lambda"} = \mathbb{1}.$$

These requirements allow us to encode all of these projections into a single self-adjoint operator. Indeed, by the above conditions,

$$\hat{A} := \sum_\lambda \lambda P_{``A=\lambda"}$$

is a self-adjoint operator and the sum appearing in its definition is actually also its spectral decomposition.

Instead of dealing with a certain labeled collection of projections, it is much more convenient to introduce A into the model by giving the single self-adjoint operator $\hat{A}$. (Infact, in a physicist undergraduate course the usual treatment is to ignore the role of projections as events, and to take this fact — that physical quantities are described by self-adjoint operators — as a definition.) Note that by its definition, $\mathrm{Sp}(\hat{A})$ is the set of all values of A (that can possibly occure). If needed, the projection $P_{\text{“}A=\lambda\text{”}}$ can be recovered by spectral decomposition of $\hat{A}$.

§3. Considering how physicial quantites are described by self-adjoint operators and taking account of the meaning of *expected value* and some other notions such as *statistical sum, function of a physical quantity* and *simultaneous measurability*, one can arrive to the conclusion that

- in case the probability function is given by the density operator ρ, the expected value of A is $\mathrm{Tr}(\rho\hat{A})$,
- in particular, in case the system is in the state given by the unit vector Ψ, the expected value of A is $\langle\Psi, \hat{A}\Psi\rangle$
- $\widehat{f(A)} = f(\hat{A})$
- $\widehat{A_1 + A_2} = \hat{A}_1 + \hat{A}_2$ where on the left-hand side the sum is defined in the statistical sense,
- $\hat{A}_1 \geq \hat{A}_2$ iff $A_1 \geq A_2$ where the first inequality is meant in the operator inequality sense, whereas the second in the statistical sense,
- if A_1 and A_2 can be simultaneously measured without disturbing each other, then $\hat{A}_1$ and $\hat{A}_2$ must commute, and in this case the statistical sum $A_1 + A_2$ coincides with the "true" sum. Moreover, still in this case, one can also consider the product of these quantities for which it turns out that $\widehat{(A_1 A_2)} = \hat{A}_1\hat{A}_2$.