Positive operators, density operators, and the lattice of projections

Throught this exercise sheet $\mathcal{H}$ stands for a finite dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the *-algebra of linear operators on $\mathcal{H}$, and $\mathcal{P}r(\mathcal{H})$ denotes the ortho-lattice formed by the orthogonal projections of $\mathcal{B}(\mathcal{H})$.

Problem 8.1. *Show that the set of orthogonal projections span $\mathcal{B}(\mathcal{H})$, and that as a consequence, if $A, B \in \mathcal{B}(\mathcal{H})$ are two operators such that $\mathrm{Tr}(AP) = \mathrm{Tr}(BP)$ for all orthogonal projections $P \in \mathcal{B}(\mathcal{H})$, then $A = B$. Conclude that if p is a probability law on $\mathcal{P}r(\mathcal{H})$, then there exists at most one density operator $D \in \mathcal{B}(\mathcal{H})$ such that $p(\cdot) = \mathrm{Tr}(D\cdot)$.*

Problem 8.2. *Let $P, Q \in \mathcal{P}r(\mathcal{H})$. Show that $P \wedge Q = \lim_n (PQ)^n$ where convergence is understood in the operator-norm sense. Would this theorem remain true in infinite dimensions? How should the notion of convergence be changed for the infinite dimensional case?*

Problem 8.3. *Show that the probability laws of the lattice $\mathcal{P}r(\mathcal{H})$ separate the elements of $\mathcal{P}r(\mathcal{H})$; that is, if $P \neq Q$ then there exists a probability law $p : \mathcal{P}r(\mathcal{H}) \to [0, 1]$ such that $p(P) \neq p(Q)$.*

Problem 8.4. *Suppose $e_1, e_2, e_3 \in \mathcal{H}$ is an ONB in $\mathcal{H}$ and P_1, P_2 and Q are the orthogonal projections onto $\mathbb{C}e_1, \mathbb{C}e_2$ and $\mathbb{C}((2+i)e_1 + 2e_2)$, respectivelly. Suppose $p : \mathcal{P}r(\mathcal{H}) \to [0, 1]$ is a probability law for which $p(P_1) = 1/2$ and $p(P_2) = 1/2$. Using Gleason's theorem, work out* all *possible values of $p(Q)$.*

Problem 8.5. *Let V be a finite dimensional vector space, and suppose that both $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ is a scalar product on V. Prove that there exists a bases $\mathcal{B} = (b_1, \ldots, b_n)$ such that the vectors of $\mathcal{B}$ are pairwise orthogonal with respect to both scalar products.*

Problem 8.6. *Show, by example, that if $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators such that $A \geq B$, then it does* not *follow that $A^2 \geq B^2$. Show also, that the implication becomes true if the extra condition $AB = BA$ is imposed.*

Problem 8.7. *Show, that if $A, B \in \mathcal{B}(\mathcal{H})$ are two positive operators for which $A^2 \geq B^2$, then $A \geq B$.*