Quasi-orthogonal decomposition of matrix algebras

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The concept of unbiased vectors and bases

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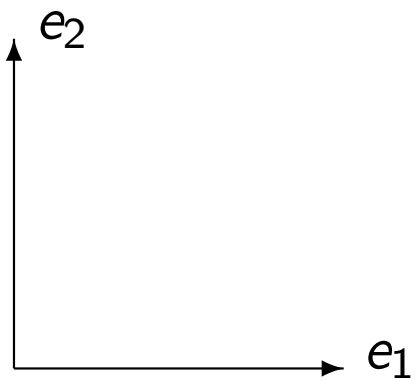

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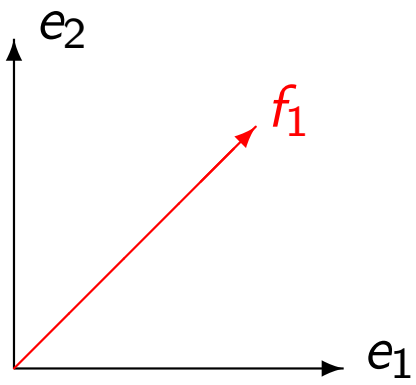

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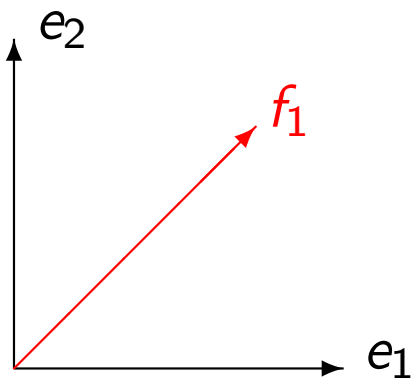

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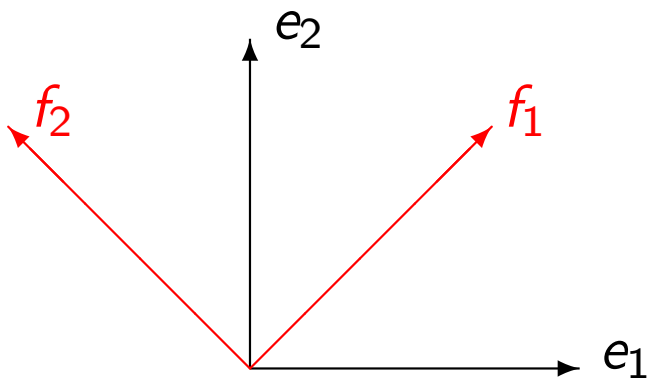

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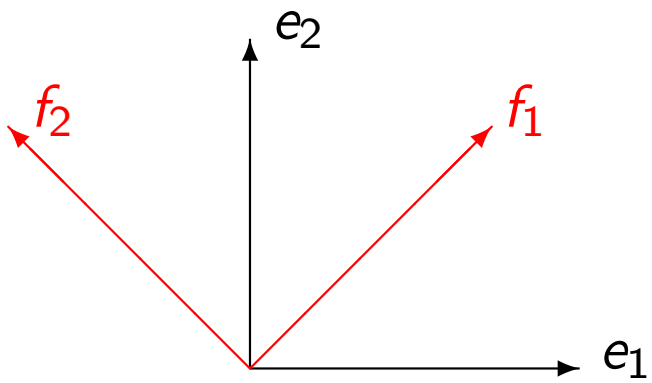

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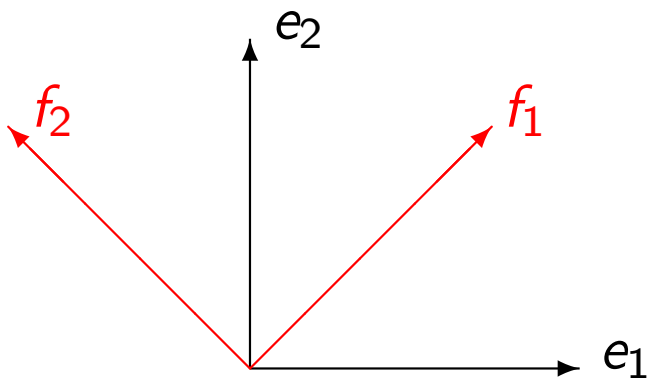

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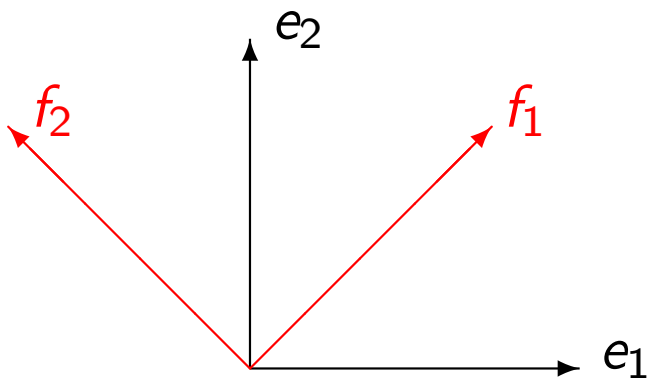

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Note: (e_1, e_2) is also unbiased for (f_1, f_2). Unbiasedness is automatically **mutual**.

Definition

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Two bases $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ in $\mathcal{H}$ satisfying

$$|\langle e_j, f_k \rangle| = |\langle e_{j'}, f_{k'} \rangle|$$

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Remark

If $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are mutually unbiased, then infact $|\langle e_j, f_k \rangle| = \frac{1}{\sqrt{n}}$ for all $j, k \in \{1, \ldots n\}$.

Problem

In an n-dimensional (complex) space, at most how many ONB can be given, such that any two of them is mutually unbiased?

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- Instead of a basis consider the algebra of operators spanned by the ortho-projections onto the basis vectors
- $\mathcal{A} \subset M_n(\mathbb{C})$ is such an algebra if and only if it is a maximal abelian *-subalgebra (MASA)
- Two maximal abelian *-subalgebras $\mathcal{A}, \mathcal{B} \subset M_n(\mathbb{C})$ correspond to mutually unbiased bases if and only if

$$\tau(AB) = \tau(A)\tau(B)$$

for all $A \in \mathcal{A}, B \in \mathcal{B}$ ("**statistical independence**")

Quasi-orthogonality

The formula

$$\langle A, B \rangle := \tau(A^* B)$$

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- Note that $\{\mathbb{1}\}^\perp =$ traceless operators, so: $\mathcal{A}$ and $\mathcal{B}$ are quasi-orthogonal $\Leftrightarrow$ their traceless parts are orthogonal

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Complete collection of MUB: the corresponding subalgebras linearly span $M_n(\mathbb{C}) \Leftrightarrow$ their number is $n + 1$.

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$N(n) :=$ maximum number of MUB in dimension n. Then $N(1) = 1$ and by what was said $N(n) \leq n + 1$. Moreover

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- i.e. already in 6 dimensions the question is unsolved: $N(6) =?$
- numerical calculations done with computers seem to indicate that $N(6) = 3$

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- better understanding of the role of symmetries

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- best measurements $\Leftrightarrow$ maximal abelian C^*-subalgebras

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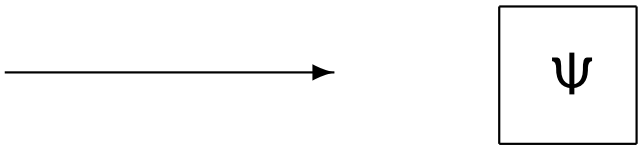

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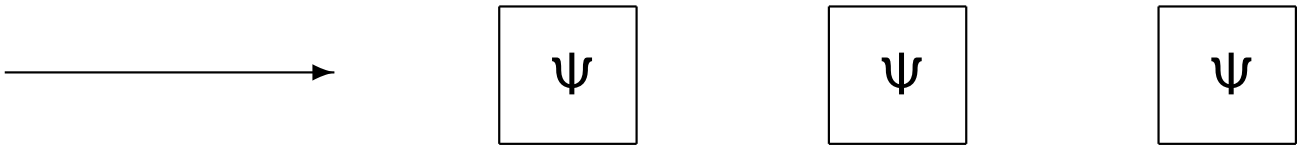

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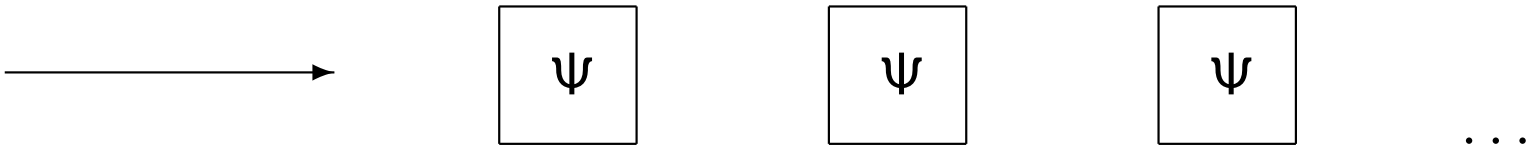

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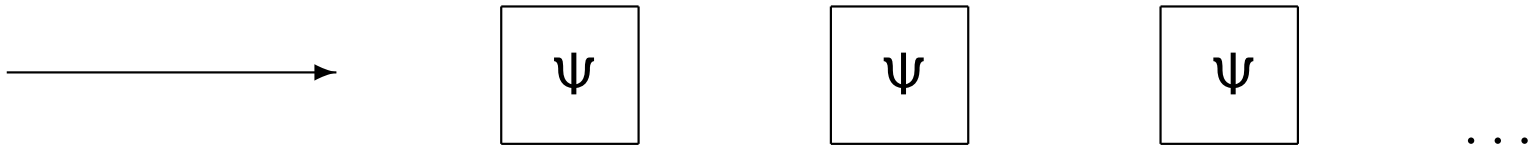

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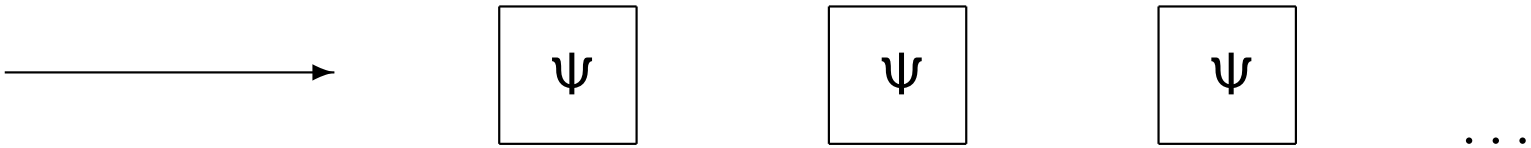

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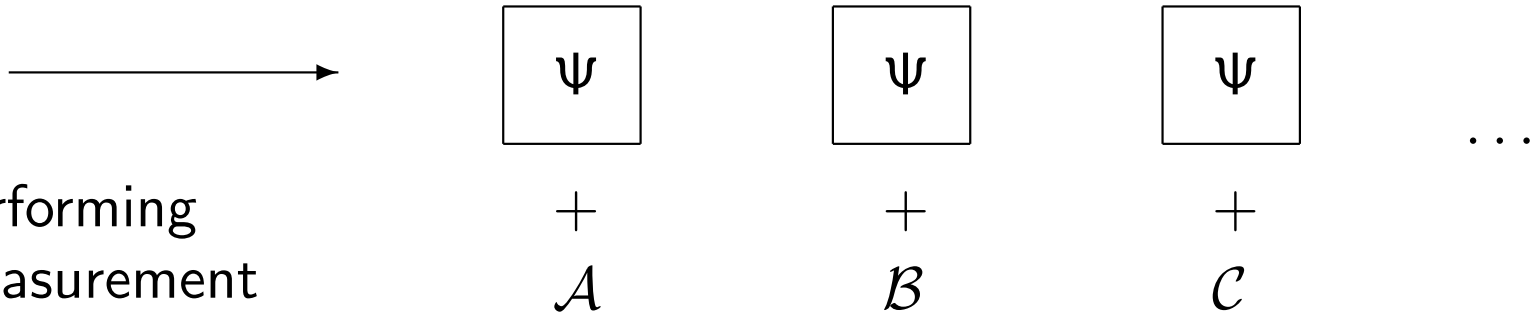

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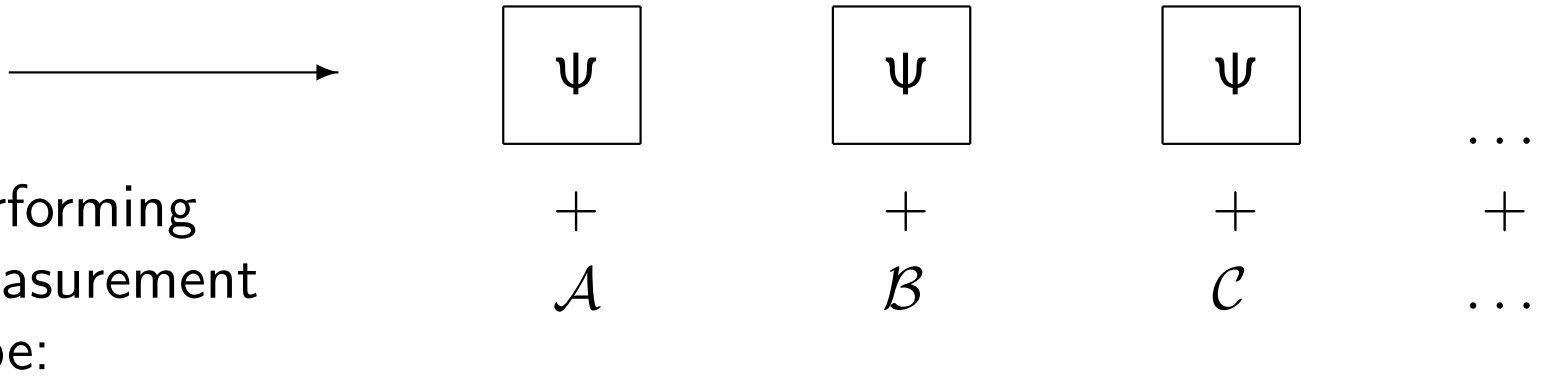

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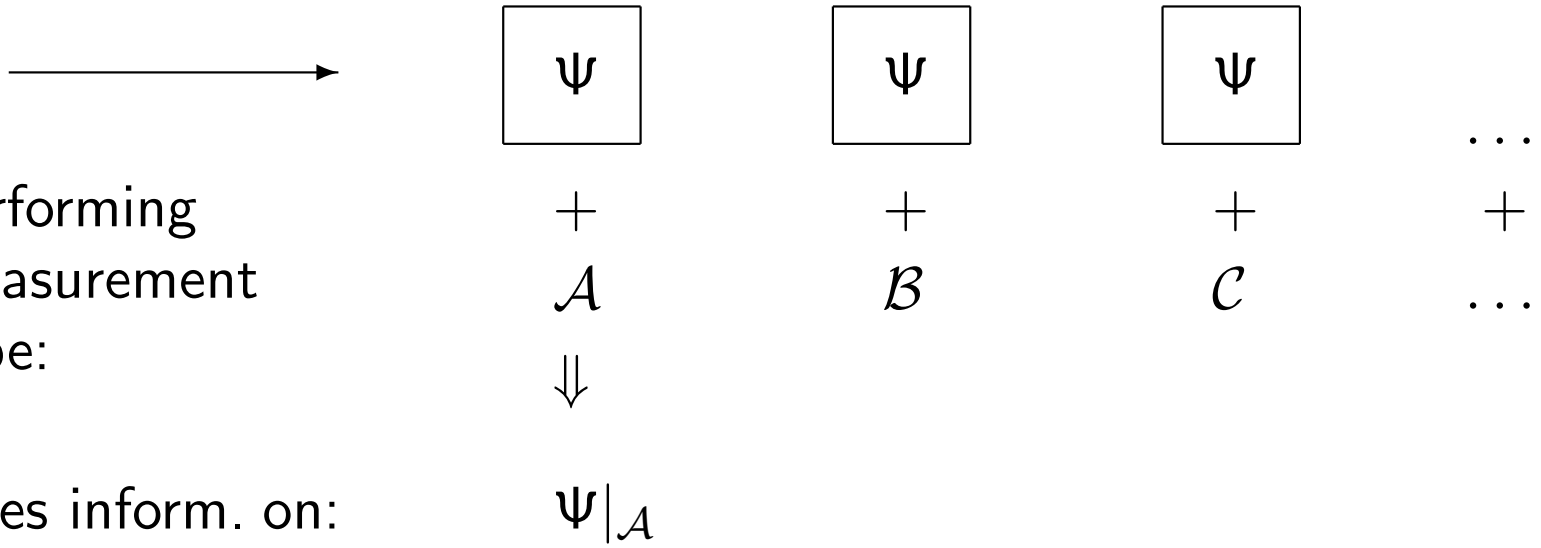

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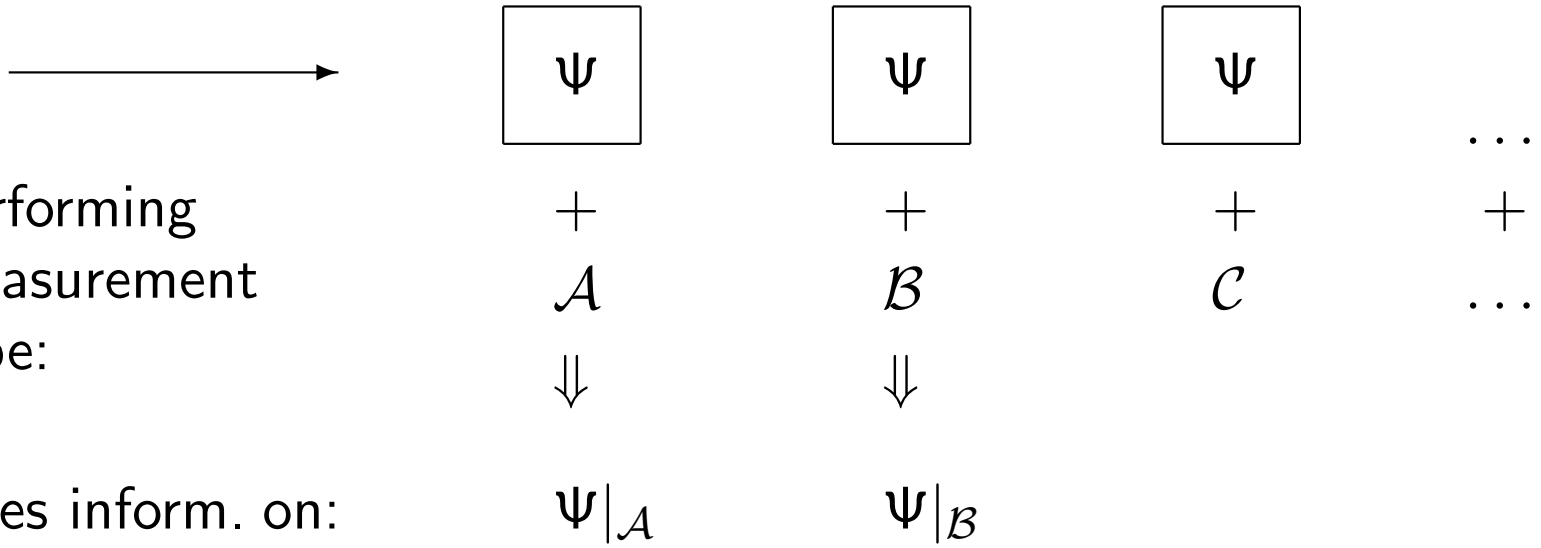

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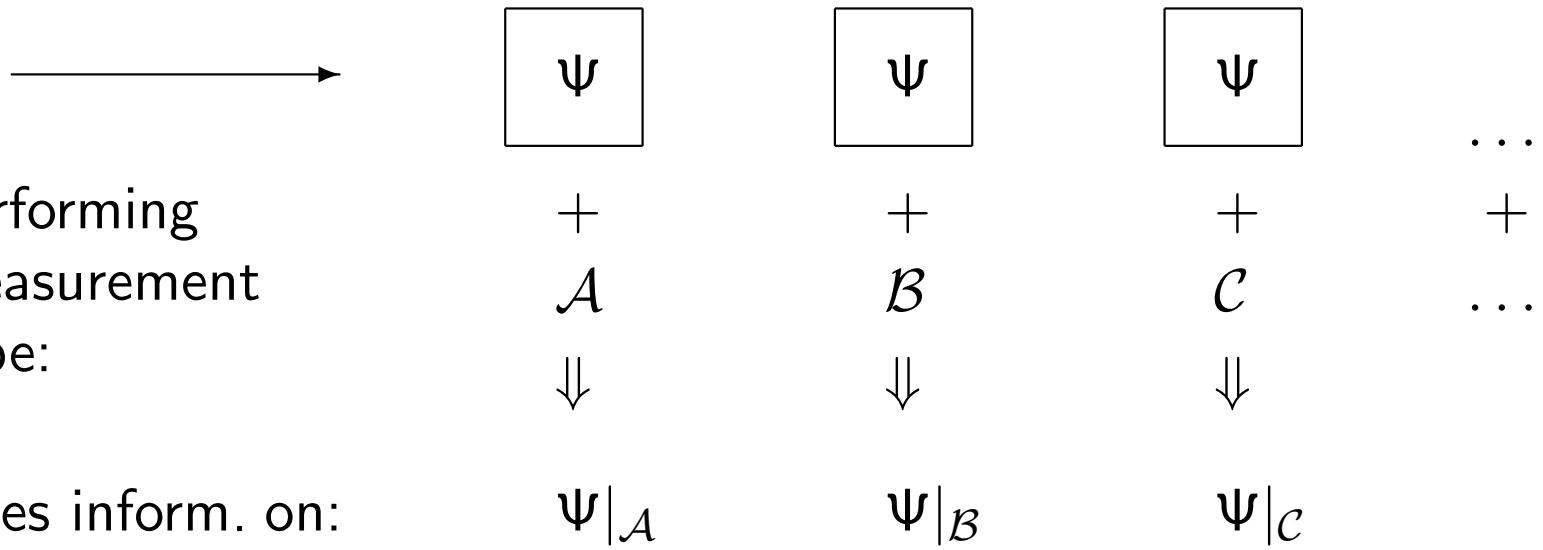

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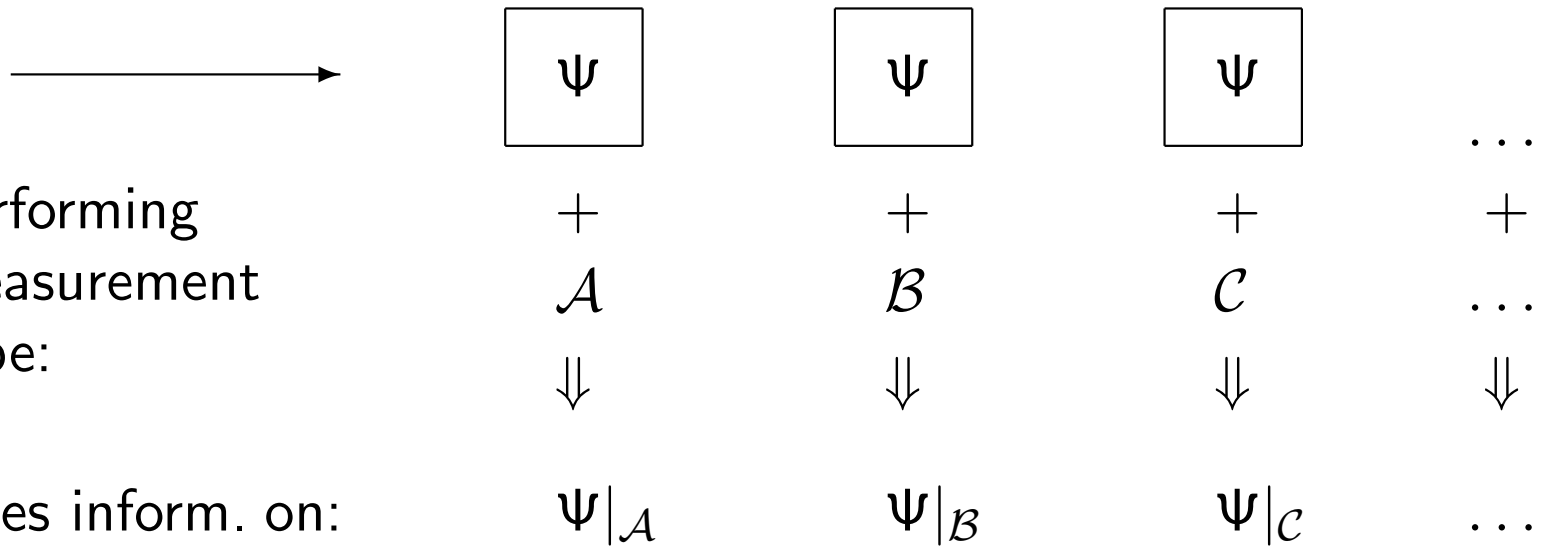

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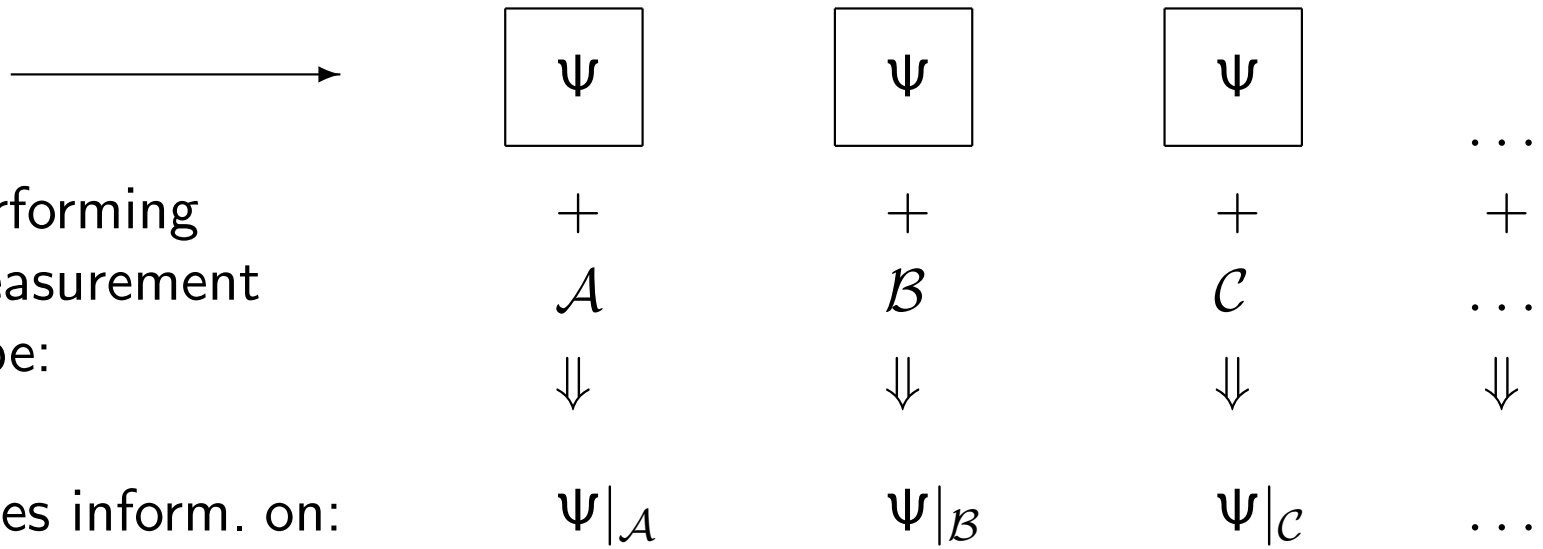

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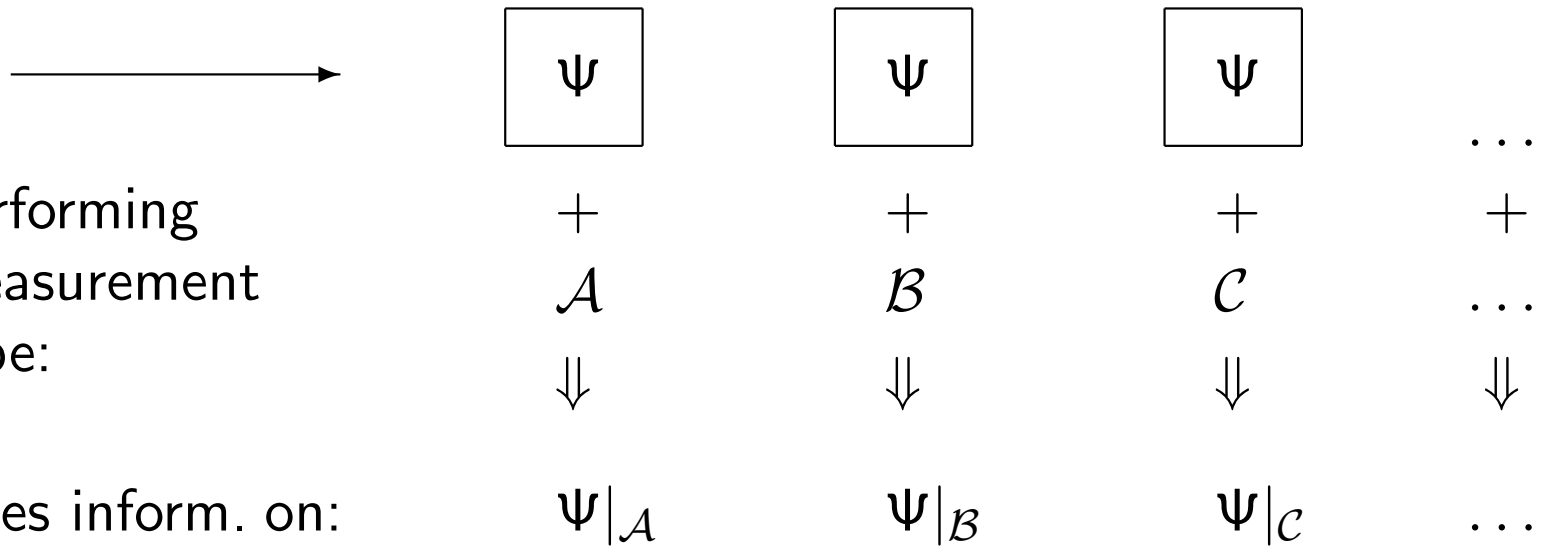

The best is to choose $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ so that they are statistically independent. Moreover, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ is a **complete set of measurables** $\Leftrightarrow$ they span $\mathcal{B}(\mathcal{H})$.

Historical remarks

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- there are indications that they are the *same* problem, e.g. the construction of Wocjan and Beth (2004)

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- the "king's problem" of Vaidman, Aharonov and Albert has a simple generalisation for an n-level q-system, given that there is a complete set of MUB.. but solution of this problem exists if and only if there exists a complete sets of orthogonal Latin squares of order n (Hayashi, Horibe, Hashimoto, 2005)

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Theorem (M. Weiner)

Suppose $\mathcal{E}_1, \dots, \mathcal{E}_n$ is a collection of MUB in $\mathbb{C}^n$. Then there exists an ONB $\mathcal{E}_{n+1}$ such that $\mathcal{E}_1, \dots, \mathcal{E}_n, \mathcal{E}_{n+1}$ is a complete collection of MUB.

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(Occasionaly mathematicians even resolve some problems.)

Quasi-orthogonal decompositions

Def. A collection of pairwise quasi-orthogonal *-subalgebras $\mathcal{A}, \mathcal{B}, \mathcal{C} \ldots M_n(\mathbb{C})$ such that $\mathcal{A} + \mathcal{B} + \mathcal{C} \ldots = M_n(\mathbb{C})$ is called a **quasi-orthogonal decomposition** of $M_n(\mathbb{C})$.

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- New question (D. Petz): what kind of decompositions exist in general?

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$\Rightarrow$ study quasi-orthogonal copies of $M_2(\mathbb{C})$ in $\otimes_k M_2(\mathbb{C}) \equiv M_{2^k}(\mathbb{C})$

Trivial necessary conditions

To find a quasi-orthogonal decomposition $\mathcal{B}_1, \mathcal{B}_2, \ldots$ of $M_n(\mathbb{C})$ in which $\mathcal{B}_k \simeq \mathcal{A}_k$, where $\mathcal{A}_1, \mathcal{A}_2, \ldots$ are given algebras,

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- $\dim A_k \dim A_l \leq n^2$ for any $k \neq l$.

Quasi-orthogonal copies of M_n in M_{n^k}

Can $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \ldots \otimes M_2(\mathbb{C}) \equiv M_{2^k}(\mathbb{C})$ be decomposed into a sum of quasi-orthogonal copies of $M_2(\mathbb{C})$?

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Regardless of n and k, the decomposition of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes \ldots \otimes M_n(\mathbb{C}) \equiv M_{n^k}(\mathbb{C})$ into quasi-orthogonal copies of $M_n(\mathbb{C})$ cannot be excluded by the trivial necessary conditions.

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$k := 2 \Rightarrow$ dimensional similarity to the MUB case (but from the point of view of commutations it is exactly the *opposite* case)

Quasi-orthogonal copies of M_n in M_{n^k}

Theorem (M. Weiner)

For $n = p$ prime it is possible to give $\frac{n^{2k}-1}{n^2-1} - 1$ quasi-orthogonal copies of $M_n(\mathbb{C})$ in $M_{n^2}(\mathbb{C})$

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For $n = p^\alpha$ where $p > 2$ it is possible to give $\frac{n^{2k}-1}{n^2-1}$ quasi-orthogonal copies of $M_n(\mathbb{C})$ in $M_{n^k}(\mathbb{C})$; that is, $M_{n^k}(\mathbb{C})$ can be decomposed into a sum of quasi-orthogonal copies of $M_n(\mathbb{C})$.

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Theorem (D. Petz, J. Kahn, 2006)

$M_4(\mathbb{C})$ cannot be decomposed into a sum of quasi-orthogonal copies of $M_2(\mathbb{C})$. The maximum number of quasi-orthogonal copies of $M_2(\mathbb{C})$ in $M_4(\mathbb{C})$ is 4.

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Theorem (D. Petz, A. Szántó, M. Weiner, 2008)

$M_4(\mathbb{C})$ can be decomposed into a quas-orthogonal sum of k copies of $M_4(\mathbb{C})$ and $5-k$ copies of $\mathbb{C}^4$ iff $k = 0, 2$ or 4.

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$\mathcal{A}, \mathcal{B}$ quasi-orthogonal $\not\Rightarrow$ $\mathcal{A}', \mathcal{B}'$ quasi-orthogonal.

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$E_{\mathcal{A}}, E_{\mathcal{B}}$: ortho-projections onto $\mathcal{A}$ and $\mathcal{B}$. Then $1 = \mathrm{Tr}(E_{\mathcal{A}} E_{\mathcal{B}})$ iff $\mathcal{A}$ and $\mathcal{B}$ are quasi-orthogonal and in general

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Applying the above theorem $\Longrightarrow$ various nontrivial obstruction. Can be considered a new general method.