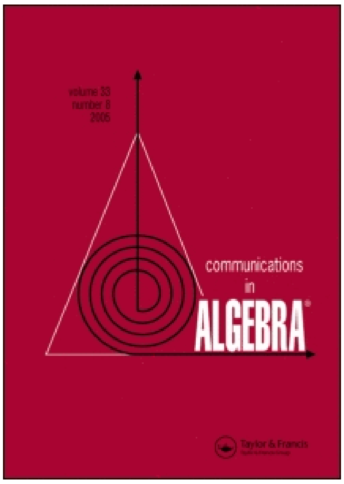

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The Semigroup of Values of a One-dimensional Local Ring with Two Minimal Primes

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1 Introduction

This paper deals with one-dimensional, local, Noetherian, reduced rings (R, m) such that the integral closure $\bar{R}$ is a finite R-module. The fact that $\bar{R}$ is a finite R-module is equivalent to the fact that R is analytically unramified, which means that the completion $\hat{R}$ (in the m-topology) is reduced, see [Ml 73, Theorem 10.2]. The number d of minimal primes in $\hat{R}$ equals the number of maximal ideals in $\bar{R}$ for a one-dimensional CM-ring R, see e. g. [Ka 86]. In particular, if R is analytically unramified, then R is analytically irreducible (i. e. $\hat{R}$ is a domain) if and only if $\bar{R}$ is local, i.e., $d = 1$. Since a local integrally closed one-dimensional domain is a DVR, we see that in an analytically irreducible domain every element has a value, and it is easy to see that the set of values $v(R)$ of nonzero elements constitute a numerical semigroup, i. e., a subset of the natural numbers $\mathbb{N}$ which contains 0, is closed under addition, and has finite complement to $\mathbb{N}$. In case (R, m) is analytically irreducible with integral

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closure (V, M) and furthermore is residually rational (i. e., $R/m \simeq V/M$), the semigroup of values tells a lot about the ring R. An example showing this is the classical theorem by Kunz saying that an analytically irreducible, residually rational, one-dimensional ring is Gorenstein if and only if its semigroup of values is symmetric, see [Ku 70]. We note that an important special case when (R, m) is residually rational is when R/m is algebraically closed. Furthermore we note that the condition residually rational is essential, since if R is not residually rational the semigroup of values tells almost nothing about the ring, indeed there are analytically irreducible Gorenstein rings with any prescribed semigroup of values, see [B-F 97', Proposition 8].

Now let (R, m) be a local, one-dimensional, analytically unramified, residually rational ring such that its integral closure has d maximal ideals. Then the set of values of nonzerodivisors in R constitute a subsemigroup $v(R)$ of $\mathbb{N}^d$. Also in this case the semigroup of values gives much information about the ring. Recall for instance that, if R/m is infinite, the Gorensteinness of R is characterized by the semigroup $v(R)$, see [C-D-K 94, Theorem 4.8] (in fact it suffices to assume that R/m has at least d elements, see [D'A 97, Corollary 4.3]). The semigroup of values of these kinds of rings have been studied in [G 82], [D 88], [C-D-K 94], and [G-L 95]. Our object is to continue to study this class of rings by means of their semigroup of values. Although some (but not all) our results could be stated and proved for any d, we restrict in this paper to the case $d = 2$.

We now make a closer description of the content of this paper. In Section 2 we recall some known results about the semigroup of values $S = v(R)$ of a ring R, that satisfies our hypotheses, and we prove several new results. Some numerical characters of S are introduced and put in relation with the numerical semigroups $S_i = v_i(R/P_i), i = 1, 2$. The semigroup ideal $v(P_1 + P_2)$ of S turns out to be important in our description. In fact, for $i = 1, 2$, $J_i = v_i((P_1 + P_2)/P_i)$ is a semigroup ideal of S_i and the numerical characters of J_i are used to give key results. In terms of J_i and S_i we define the "missing points" of S (and the one-to-one corresponding "sources" of S). The canonical ideal of R, which we denote by ω, is an important concept throughout the paper. We show that $l_R(\omega/R)$ equals the number of missing points (cf. Proposition 2.17) and, computing it in a different way, we get easy conditions for the ring to be Gorenstein or Kunz (cf. Corollaries 2.19 and 2.20). We give at the end of the section a list of examples of rings with their corresponding value semigroups and their pictures.

In Section 3 we make a closer study of $K = v(\omega)$ and compare it to $S = v(R)$. There is a kind of numerical duality between the maximal elements in K and in S (cf. Proposition 3.1). This duality is investigated more closely defining "processes", that starting from the "sources" of S touch all the non trivial (i.e., not already in S) maximal elements in K (cf. Proposition 3.5). This gives a method to find a set of generators for ω (cf. Proposition 3.8) and, since the cardinality of a minimal set of generators for ω equals the CM type of R, this permits also to get an inequality involving the CM types of R and of $R/P_i, i = 1, 2$ (cf. Proposition 3.10).

In Section 4 we solve the following problem. For which rings R does every strict overring T, $R \subset T \subseteq \bar{R}$, have a larger conductor than R (in the integral closure)? We call such rings R "Maximal with fixed conductor" (shortly MWFC), and we notice that they are exactly the rings of our class with a unique minimal overring (cf. Proposition 4.10). For analytically irreducible rings, the answer is known to be the Gorenstein and the Kunz rings. In Section 4 this is shown to be true also in the case of two minimal primes, provided that R/m is algebraically closed (cf. Corollary 4.7), but in general there are more examples and these are classified (cf. Theorem 4.6 and Corollary 4.9).

In Section 5 we consider the value semigroup for overrings to a given ring R, in particular the seminormalization of R. We study when an overring of a given ring is local, and this permits us to give a characterization of a ring (R, m) in order to be Gorenstein or Kunz, in terms of the ring $(m : m)/m$ (cf. Propositions 5.7 and 5.8).

1.1 Preliminaries

Throughout this paper we fix the following hypotheses and notation. Let (R, m) be a reduced, local, Noetherian, one-dimensional ring; let $\bar{R}$ be the integral closure of R in its total ring of quotients $Q(R)$. We assume that $\bar{R}$ is a finite R-module and that it has two maximal ideals.

We also assume that R has two minimal primes P_1 and P_2. Hence we have that, for $i = 1, 2$, each R/P_i is analytically irreducible and that $\bar{R} = \overline{R/P_1} \times \overline{R/P_2}$ (where $\overline{R/P_i}$ is the integral closure of R/P_i in its quotient field $Q(R/P_i)$). Since $\overline{R/P_i}$ is a local, one-dimensional, integrally closed domain, it is a DVR and we denote it by V_i.

Let $M_1 = (t)$ and $M_2 = (u)$ be the maximal ideals of V_1 and V_2, respectively; hence $n_1 \simeq M_1 \times V_2$ and $n_2 \simeq V_1 \times M_2$ are the two maximal ideals of $\bar{R}$. We furthermore assume that R is residually rational, i. e. that $R/m \simeq \bar{R}/n_1 \simeq \bar{R}/n_2$.

Let $v_i : Q(R/P_i) \longrightarrow \mathbb{Z}_\infty = \mathbb{Z} \cup \infty$ be the valuation associated to V_i. Since $Q(R) \cong Q(R/P_1) \times Q(R/P_2)$, for any $x = (x_1, x_2) \in Q(R)$ we denote by $v(x)$ the element $(v_1(x_1), v_2(x_2)) \in \mathbb{Z}^2_\infty$. If we restrict v to the nonzerodivisors $NZ(Q(R))$ of $Q(R)$, then $v(x) \in \mathbb{Z}^2$.

We denote $\{v(r) \mid r \in NZ(R)\}$ by $v(R)$ or by S. Then S is an additive subsemigroup of $\mathbb{N}^2$ which we call the value semigroup of R.

The assumption that R has two minimal primes makes notation much easier and makes it possible to compute some numerical characters of the semigroup $S = v(R)$ in terms of the value sets of the minimal primes P_i. However it is possible to define $v(R)$ also without this assumption (cf. [C-D-K 94]). On the other hand the completion $\hat{R}$ of R has two minimal primes and its value semigroup coincides with $v(R)$, cf. [N 56] or [D'A 97]. Because of this fact, for the general case, we can compute the numerical characters of $v(R)$ considering the completion of R and all the results of this paper hold (for more details, cf. [D'A 97]).

We always consider on $\mathbb{N}^2$ (and on S) the partial order given by $(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2)$ if and only if $\alpha_i \leq \beta_i$, for $i = 1, 2$. We call $E \subseteq \mathbb{Z}^2$ a relative semigroup ideal of S if $E + S \subseteq E$ and $\alpha + E \subseteq S$, for some $\alpha \in S$.

By a fractional ideal we always mean in this paper a regular fractional ideal, i.e., a fractional ideal containing a nonzero divisor. Since $\hat{R}$ is reduced, R has a canonical ideal, cf. [H-K 71, Satz 6.21], i.e., a fractional ideal ω, such that $\omega : (\omega : I) = I$ for each fractional ideal I of R. If ω is a canonical ideal, then $\omega : \omega = R$ and $l_R(I/J) = l_R(\omega : J/\omega : I)$, where $J \subseteq I$ are fractional ideals. We can in our situation assume that $R \subseteq \omega \subseteq \bar{R}$ (cf. Corollary 2.2). Note that ω is not uniquely defined by this property but, by [D'A 97, Theorem 4.1], the set of values $v(\omega)$ is. We denote $v(\omega) \subseteq \mathbb{N}^2$ by K. Then K is a relative semigroup ideal of S, and we call it the canonical ideal of S.

The type (or the CM-type) of a ring R is $l_R((R : m)/R)$. It is well known that the type of R equals $l_R(\omega/m\omega)$, the minimal number of generators for ω. Let $C = R : \bar{R}$ be the conductor of $\bar{R}$ in R. A Gorenstein ring is a ring of type 1. It is well known that R is Gorenstein if and only if $l_R(R/C) = l_R(\bar{R}/R)$ or if and only if $l_R(\omega/R) = 0$. We always have the inequality $l_R(\bar{R}/R) \leq l_R(R/C) + \text{type}(R) - 1$, cf. [H-K 71, Satz 3.6]. A ring R is defined to be Kunz if $l_R(\bar{R}/R) = l_R(R/C) + 1$ or equivalently $l_R(\omega/R) = 1$ (cf. [B-F 97, Proposition 21]). Kunz rings were introduced and studied in [B-D-F 97] in the analytically irreducible case, and in [B-F 97] and [B-F 97'] in the analytically unramified case.

2 The semigroup of values

We will now recall some results on the semigroup of values for the kind of rings we study, and prove some more. Some (but not all) of these results can be stated in the general case of a ring with d minimal primes (cf. [C-D-K 94], [G 82], and [D'A 97]). Many of them are already in the literature, although not always in the form or generality we will need them. So, for the convenience of the reader, we prove them directly, in the case $d = 2$.

If I is a fractional ideal of R, then $v(I) = \{v(r) \mid r \in NZ(I)\}$ is a relative semigroup ideal of S. The following is a slight extension of [G 82, Property A].

Lemma 2.1 *Let I be a fractional ideal of R. If (α_1, α_2) and (β_1, β_2) are elements in $v(I)$, then $(\min(\alpha_1, \beta_1), \min(\alpha_2, \beta_2)) \in v(I)$. In particular, any fractional ideal of R has an element of smallest value.*

Proof. Let $x \in R$ be such that $v(x) = (\alpha_1, \alpha_2)$ and $y \in R$ such that $v(y) = (\beta_1, \beta_2)$. If $\alpha_1 \neq \beta_1$ and $\alpha_2 \neq \beta_2$, then $v(x + y) = (\min(\alpha_1, \beta_1), \min(\alpha_2, \beta_2)) \in v(I)$. Otherwise, if e.g. $\alpha_1 = \beta_1$ and $\alpha_2 \leq \beta_2$, then $(\min(\alpha_1, \beta_1), \min(\alpha_2, \beta_2)) = (\alpha_1, \alpha_2) \in v(I)$.

Corollary 2.2 *There is a canonical ideal ω with $R \subseteq \omega \subseteq \bar{R}$.*

Proof. Let I be a canonical ideal of R and let $x \in I$ be an element of minimal value. Then $\omega = x^{-1}I$ is a canonical ideal and $R \subseteq \omega$ since $1 \in \omega$ and since ω is an R-module. Since every element in ω has a value bigger or equal to $(0, 0)$, we have $\omega \subseteq \bar{R}$.

For any fractional ideal I and any $\alpha \in \mathbb{Z}^2$ we define $I(\alpha) = \{r \in I \mid v(r) \geq \alpha\}$. It is clear that $I(\alpha)$ is a fractional ideal of R. Let I be a fractional ideal of R and let $E = v(I)$. For any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ we will denote by $\Delta_i^E(\alpha)$ the set $\{\beta \in E \mid \beta_i = \alpha_i \text{ and } \beta_j > \alpha_j \text{ if } j \neq i\}$, $i = 1, 2$, and we set $\Delta^E(\alpha) = \Delta_1^E(\alpha) \cup \Delta_2^E(\alpha)$. An element $\alpha \in E$ is called a *maximal element in* E if $\Delta^E(\alpha) = \emptyset$.

The following is a slight generalization of [G 82, Property B].

Lemma 2.3 *Let I be a fractional ideal of R. If $\alpha = (\alpha_1, \alpha_2) \in v(I)$ and $\Delta_1^{v(I)}(\alpha) \neq \emptyset$, then $\Delta_2^{v(I)}(\alpha) \neq \emptyset$ and vice versa.*

Proof. Let $\alpha = (\alpha_1, \alpha_2) \in v(I)$ and suppose $(\beta_1, \alpha_2) \in v(I)$ with $\beta_1 > \alpha_1$. Take $r, s \in I$ such that $v(r) = \alpha$ and $v(s) = (\beta_1, \alpha_2)$. Since s and r have the same value in $Q(R/P_2)$ and since R is residually rational, there is a unit $x \in R$ such that $r - xs$ has value $\beta_2 > \alpha_2$ in $Q(R/P_2)$, so $v(r - xs) = (\alpha_1, \beta_2)$. The vice versa is proved similarly.

Let $I \subseteq J$ be fractional ideals of R. In order to compute the length $l_R(J/I)$ in terms of values, we need to generalize in a proper way [Ms 71, Proposition 1]. This proposition states, if (R, m) is an analytically irreducible, residually rational ring and if $I_1 \subseteq I_2$ are fractional R-ideals, then $l_R(I_2/I_1) = \mathrm{Card}(v(I_2) \setminus v(I_1))$ where $v(I_i)$ is the set (in fact the semigroup ideal) of values of nonzero elements in I_i, $i = 1, 2$. The key fact for the generalization to analytically unramified rings comes from [D 88, Proposition 1.11]. The following formulation suits our needs.

Lemma 2.4 *Let J be a fractional ideal of R. For each $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ we have $l_R(J(\alpha)/J(\alpha + (1,0))) \leq 1$ with equality if and only if $J(\alpha) \neq J(\alpha + (1,0))$. Similarly $l_R(J(\alpha)/J(\alpha + (0,1))) \leq 1$ with equality if and only if $J(\alpha) \neq J(\alpha + (0,1))$.*

Proof. Since $mJ(\alpha) \subseteq J(\alpha + (1,0))$, $V = J(\alpha)/J(\alpha + (1,0))$ is a vector space over R/m. Moreover, if $r, s \in J(\alpha) \setminus J(\alpha + (1,0))$, then, as in the proof of Lemma 2.3, there is a unit $x \in R$ such that $r - xs \in J(\alpha + (1,0))$, so r and s represent linearly dependent elements of V and $\dim V \leq 1$. Of course $\dim V = 0$ if and only if $J(\alpha) = J(\alpha + (1,0))$.

Let $C = R : \bar{R}$ be the conductor of $\bar{R}$ in R. Since $\bar{R}$ is finite over R, then C contains a non-zero divisor. Moreover any ideal of $\bar{R}$ is principal and so $C = c\bar{R}$, that is C consists of all elements in $\bar{R}$ of value $\geq v(c)$. Let J be a fractional ideal and let $\mu = (\mu_1, \mu_2)$ be its minimal value. Since $J\bar{R} \subseteq J$, then $JC \subseteq J$ and so J contains all elements of $\bar{R}$ with value $\geq \mu + v(c)$. For $E = v(J)$ and $\alpha \in \mathbb{Z}^2$ we set $E(\alpha) = \{x \in E \mid x \geq \alpha\}$. Let $(\mu_1, \mu_2) = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(n)} = \alpha = (\alpha_1, \alpha_2)$ be a chain which is maximal in the sense that for each $i = 1, \ldots, n$ we have that $\alpha^{(i)} - \alpha^{(i-1)}$ equals either $(1,0)$ or $(0,1)$ (so that $n = \alpha_1 + \alpha_2 - (\mu_1 + \mu_2)$). Set $n_i(E) = 1$ if $E(\alpha^{(i)}) \neq E(\alpha^{(i-1)})$ and $n_i(E) = 0$ otherwise. Using Lemma 2.3 and the definition of maximal element, it is not difficult to see that, for any such chain $(\mu_1, \mu_2) = \alpha^{(0)} < \alpha^{(1)} < \cdots < \alpha^{(n)} = \alpha$, there is a one to one correspondence between the set $\{i \mid n_i(E) = 1\}$ and the set

$$\{\text{maximal elements } \gamma \text{ in } E \mid \gamma < \alpha\} \cup \{y \in \mathbb{Z} \mid \Delta_2^E(\alpha_1, y) \neq \emptyset, y < \alpha_2\} \cup$$
$$\{x \in \mathbb{Z} \mid \Delta_1^E(x, \alpha_2) \neq \emptyset, x < \alpha_1\},$$

so the number $\sum_1^n n_i(E)$ only depends on E and α and not on the chain. Setting $L_E(\alpha) = \sum_1^n n_i(E)$ and using Lemma 2.4 we get the following two lemmas.

Lemma 2.5 *We have* $L_E(\alpha) = l_R(J/J(\alpha))$, *in particular the number* $L_E(\alpha)$ *is well defined.*

Lemma 2.6 *We have* $L_E(\alpha) = \mathrm{Card}\{\text{maximal elements } \gamma \in E \mid \gamma < \alpha\} + \mathrm{Card}\{y \mid \Delta_2^E(\alpha_1, y) \neq \emptyset, y < \alpha_2\} + \mathrm{Card}\{x \mid \Delta_1^E(x, \alpha_2) \neq \emptyset, x < \alpha_1\}$.

Proposition 2.7 *Let* $I \subseteq J$ *be fractional ideals of* R. *Then* $l_R(J/I) = L_{v(J)}(\alpha) - L_{v(I)}(\alpha)$ *for any sufficiently large* α. *In particular, if* $I \subseteq J$ *are fractional ideals and* $v(I) = v(J)$, *then* $I = J$.

Proof. Take $\alpha \in \mathbb{Z}^2$ so large that I contains every element of $Q(\bar{R})$ of value at least α, so $I(\alpha) = J(\alpha)$. Then, by Lemma 2.5, $l_R(J/I) = l_R(J/I(\alpha)) - l_R(I/I(\alpha)) = L_{v(J)}(\alpha) - L_{v(I)}(\alpha)$.

Remark A Notice that, as in the analytically irreducible case ($d = 1$), we get also in this case ($d = 2$) that a fractional ideal I of R, such that $v(I)$ contains all the values $\geq \alpha$, for some α, contains any element of $Q(\bar{R})$ of value $\geq \alpha$. In fact, we can consider the fractional ideal $I' = I \cup \{x \in Q(\bar{R}) \mid v(x) \geq \alpha\}$. Since $I \subseteq I'$ and $v(I) = v(I')$, by Proposition 2.7, we have $I = I'$.

Let T be a numerical semigroup, i.e., a subsemigroup of $\mathbb{N}$ with finite complement to $\mathbb{N}$, and let I be a relative semigroup ideal of T. Then $n \in T$ for all large $n \in \mathbb{N}$, and we will denote $\max\{n \in \mathbb{Z} \mid n \notin I\}$ by $g(I)$. (The number $g(T)$ is sometimes called the Frobenius number of T.) Set moreover $n(I) = \mathrm{Card}\{x \in I \mid x < g(I)\}$. In our situation, let $S_i = v_i(R/P_i), i = 1, 2$, be the two projections of $S = v(R)$, and let $J_i = v_i((P_1 + P_2)/P_i)$. For $i = 1, 2$, we denote the number $g(J_i)$ defined above by g_{i*} and we set $g_* = (g_{1*}, g_{2*})$. With this notation we get:

Lemma 2.8 *(Cf. [D'A 97, Lemma 1.4].) We have* $v(P_1 + P_2) = J_1 \times J_2$. *In particular* $J_1 \times J_2 \subseteq S$.

Proof. The inclusion $v(P_1 + P_2) \subseteq J_1 \times J_2$ follows from the definitions. Let us show the opposite inclusion. Let $\alpha_1 \in J_1$ and $\alpha_2 \in J_2$. Let $r = p_1 + p_2 \in P_1 + P_2$ (with $p_i \in P_i$) be such that $v_1(r) = \alpha_1$ and let $s = q_1 + q_2 \in P_1 + P_2$ (with $q_i \in P_i$) be such that $v_2(s) = \alpha_2$. Then the element $x = q_1 + p_2 \in P_1 + P_2$ is such that $v_1(x) = \alpha_1$ (since $x \equiv p_2 \pmod{P_1}$) and $v_2(x) = \alpha_2$ (since $x \equiv q_1 \pmod{P_2}$). Thus $(\alpha_1, \alpha_2) \in v(P_1 + P_2)$.

Lemma 2.9 *(Cf. [D'A 97, Corollary 1.5].) Let* $s_1 \in \mathbb{N}$. *Then the following are equivalent:*
i) $s_1 \in J_1$.
ii) $(s_1, g_{2*} + 1) \in S$.

iii) $(s_1, s_2) \in S$ *for every* $s_2 > g_{2*}$.
An analogous statement holds for J_2.

Proof. By the previous lemma we get that i)⇒ii), because $g_{2*} + 1 \in J_2$. To prove ii)⇒i), since there is an $r \in R$ such that $v(r) = (s_1, g_{2*} + 1)$, we have that $r = (r_1, r_2) \in R/P_1 \times R/P_2$ is such that $v_2(r) = v_2(r_2) = g_{2*} + 1$. So $r_2 \in (P_1 + P_2)/P_2$ (cf. Remark A applied to the fractional ideal $(P_1 + P_2)/P_2$ of the analytically irreducible ring R/P_2), i.e., $r \in P_1 + P_2$, and we get $v_1(r) = s_1 \in J_1$. The equivalence ii)⇔iii) follows from Lemma 2.1 and Lemma 2.3.

Lemma 2.10 *(Cf. [D'A 97, Proposition 1.3].) If* $C = R : \bar{R}$, *we have* $C = (t^{g_{1*}+1}, u^{g_{2*}+1})\bar{R}$, *where* $\bar{R} = V_1 \times V_2$ *and* t *and* u *are the uniformizing parameters in* V_1 *and* V_2, *respectively. Moreover* $\Delta^S(g_*) = \emptyset$.

Proof. C is the biggest integral ideal that R and $\bar{R}$ share. Moreover any ideal of $\bar{R}$ is principal and consists of all elements in $\bar{R}$ of value $\geq \gamma$, where γ is the value of a generator of the ideal. So it is enough to show that $C = \{x \in \bar{R} \mid v(x) \geq g_* + (1,1)\}$. If $\alpha \in \mathbb{N}^2$ and $\alpha \geq g_* + (1,1)$, then $\alpha \in J_1 \times J_2 = v(P_1 + P_2)$ (cf. Lemma 2.8). So, by Remark A, the ideal $P_1 + P_2$ of R (and hence R) contains all elements of $\bar{R}$ of value $\geq g_* + (1,1)$. Thus $C \subseteq \{x \in \bar{R} \mid v(x) \geq g_* + (1,1)\}$. On the other hand, since $g_{i*} \notin J_i, i = 1, 2$, by Lemma 2.9, iii)⇔i), we have that $\Delta^S(g_*) = \emptyset$, and so $C = \{x \in \bar{R} \mid v(x) \geq g_* + (1,1)\}$.

Since $\Delta^S(g_*) = \emptyset$, if $\alpha \in S$, then $\Delta^S(g_* - \alpha) = \emptyset$. If also the converse holds, i.e., if

$$\alpha \in S \Leftrightarrow \Delta^S(g_* - \alpha) = \emptyset$$

the semigroup is said to be *symmetric* (cf. [D 88]). Set $K = \{\alpha \in \mathbb{Z}^2 \mid \Delta^S(g_* - \alpha) = \emptyset\}$. We have $S \subseteq K \subseteq \mathbb{N}^2$ and, by definition, $S = K$ if and only if S is symmetric. Moreover a fractional ideal J of R, $R \subseteq J \subseteq \bar{R}$, is a canonical ideal of R if and only if $v(J) = K$ (cf. [D'A 97, Theorem 4.1]). This characterization of $K = v(\omega)$ will be used several times in the sequel. It generalizes a result by Jäger (cf. [J 77, Satz 5]) who proved it in the analytically irreducible case.

Corollary 2.11 *We have:*
1) $l_R(R/C) = \text{Card}\{\text{maximal elements in } S\} + n(J_1) + n(J_2) =$
$n(S_1) + g_{1*} - g(S_1) + n(J_2) = n(S_2) + g_{2*} - g(S_2) + n(J_1)$.
2) $l_R(\bar{R}/C) = g_{1*} + g_{2*} + 2$. *In particular, if* R *is Gorenstein,* $g_{1*} + g_{2*}$ *is even and, if* R *is Kunz,* $g_{1*} + g_{2*}$ *is odd.*

Proof. Since, by Lemma 2.5 and Lemma 2.10, $l_R(R/C) = L_S(g_* + (1,1))$, the first equality in 1) follows from the Lemma 2.6 and Lemma 2.9. For the second and third equality in 1), it is enough to consider respectively the maximal chains $(0,0) < (1,0) < \cdots < (g_{1*} + 1, 0) < (g_{1*} + 1, 1) < \cdots < (g_{1*} + 1, g_{2*} + 1)$ and $(0,0) < (0,1) < \cdots < (0, g_{2*}+1) < (1, g_{2*}+1) < \cdots < (g_{1*}+1, g_{2*}+1)$ and apply Lemma 2.4. For the equality in 2), apply again Lemma 2.4. Finally, by definition of Gorenstein and Kunz rings, we have respectively $l_R(\bar{R}/C) = 2l_R(R/C)$ and $l_R(\bar{R}/C) = 2l_R(R/C) + 1$.

As in [G-L 95], we define also in our situation the *intersection degree* of R to be $i(R) = l_R(R/(P_1 + P_2)) = l_{R/P_j}(\frac{R/P_j}{(P_1+P_2)/P_j}), j = 1, 2$. By Lemma 2.8, Lemma 2.5, and Lemma 2.6 it is immediate that:

Proposition 2.12 *We have* $i(R) = \mathrm{Card}\{\text{maximal elements in } S\}$.

In the next proposition we denote R/P_i by R_i, $i = 1, 2$. It is proved in [G-L 95, Theorem 3.9] for a Gorenstein curve with $d \geq 2$ branches.

Proposition 2.13 *We have* $l_R(\overline{R}/R) = l_{R_1}(\overline{R_1}/R_1) + l_{R_2}(\overline{R_2}/R_2) + i(R)$.

Proof. The sequence of R-modules $0 \longrightarrow R \xrightarrow{\phi} R_1 \times R_2 \xrightarrow{\psi} R/(P_1 + P_2) \longrightarrow 0$, where $\phi(x) = (x + P_1, x + P_2)$ and $\psi(x + P_1, y + P_2) = x - y + (P_1 + P_2)$, is exact. Hence $l_R((R_1 \times R_2)/R) = i(R)$. Moreover $l_R(\bar{R}/R) = l_R(\bar{R}/R_1 \times R_2) + l_R(R_1 \times R_2/R)$ and $l_R(\bar{R}/R_1 \times R_2) = l_{R_1}(\overline{R}_1/R_1) + l_{R_2}(\overline{R}_2/R_2)$.

Lemma 2.14 *We have* $g_* \in K$ *and* $\Delta^K(g_*) = \emptyset$.

Proof. If $r \in \bar{R}$ and $v(r) = (\alpha_1, \alpha_2)$, then $\alpha_i > 0$ if and only if $r \in n_i$, where n_1 and n_2 are the maximal ideals of $\bar{R}$. Since $n_1 \cap R = n_2 \cap R = m$, we have $\Delta^S((0, 0)) = \emptyset$, thus $g_* \in K$. Furthermore both $\Delta^S((a, 0))$ and $\Delta^S((0, a))$ contain $(0, 0)$ if $a < 0$. Thus $\Delta^K(g_*) = \emptyset$.

Let T be a numerical semigroup and I a semigroup ideal of T. The map $\phi : I \longrightarrow \mathbb{Z} \setminus T$, $x \mapsto g(I) - x$, is a well defined injection (if $g(I) - x \in T$, then $x \notin I$, since otherwise $g(I) - x + x \in I$). Moreover $x > g(I)$ if and only if $\phi(x) < 0$, so we have $n(I) = \mathrm{Card}\{x \in I \mid x < g(I)\} \leq g(T) + 1 - n(T)$, where $n(T) = \mathrm{Card}\{x \in T \mid x < g(T)\}$. Recall that the (standard) canonical ideal of T is $K(T) = \{x \in \mathbb{Z} \mid g(T) - x \notin T\}$ (cf. [J 77]), and that a relative ideal I of T is a canonical ideal of T if and only if $I + a = K(T)$, for some $a \in \mathbb{Z}$. The following lemma is immediate.

Lemma 2.15 *Let I be any relative ideal of the numerical semigroup T. Then the following are equivalent:*
i) $I = K(T) + a$, *for some* $a \in \mathbb{Z}$.
ii) The map $I \longrightarrow \mathbb{Z} \setminus T$, $x \mapsto g(I) - x$, *is bijective.*
iii) $n(I) = g(T) + 1 - n(T)$, *where* $n(I) = \mathrm{Card}\{x \in I \mid x < g(I)\}$.

Let S_1 and S_2 be the two projections of $S \subseteq \mathbb{N}^2$ and $J_i = v_i((P_1 + P_2)/P_i)$, and let $\phi_i : J_i \longrightarrow \mathbb{Z} \setminus S_i, i = 1, 2$, be the maps defined as ϕ above with respect to the ideals J_i. We call $a \in \mathbb{Z}$ a *missing point for* ϕ_i, if $a \in (\mathbb{Z} \setminus S_i) \setminus \mathrm{Im}(\phi_i)$. Finally we call the *missing points of* S, the disjoint union of the missing points for ϕ_1 and for ϕ_2.

Lemma 2.16 *Let* $a \in \mathbb{Z}$. *Then the following are equivalent:*
i) a is a missing point for ϕ_1.
ii) $(g_{1*} - a, g_{2*} + 1) \in K \setminus S$.
iii) $(g_{1*} - a, s_2) \in K \setminus S$ *for every* $s_2 > g_{2*}$.
There is an analogous statement for ϕ_2.

Proof. i)⇒ii): Suppose that a is a missing point for ϕ_1. Then we have $\Delta^S(a,-1) = \emptyset$, because $a \notin S_1$. So $g_* - (a,-1) = (g_{1*} - a, g_{2*} + 1) \in K$. Moreover, by definition of missing point, $g_{1*} - a \notin J_1$ and so, by Lemma 2.9, $g_* - (a,-1) = (g_{1*} - a, g_{2*} + 1) \notin S$.
ii)⇒iii): Since $\Delta_1^S(a,-1) = \emptyset$, then $\Delta^S(a,h) = \emptyset$, for any $h < 0$. Hence we have $(g_{1*} - a, s_2) \in K$, for every $s_2 > g_{2*}$. Moreover $(g_{1*} - a, s_2) \notin S$, by Lemma 2.9.
iii)⇒i): Suppose that a is not a missing point for ϕ_1. If $a \in S_1$, then there exists a point $(a,b) \in S$. Since $(g_{1*} - a, s_2) \in K$, then $(a.b) + (g_{1*} - a, s_2) = (g_{1*}, s_2 + b) \in K$, hence $\Delta^K(g_*) \neq \emptyset$, a contradiction to Lemma 2.14. If $a \notin S_1$, then $g_{1*} - a \in J_1$, and so, by Lemma 2.9, $(g_{1*} - a, s_2) \in S$, for any $s_2 > g_{2*}$, that is also a contradiction. The second part of the lemma is proved similarly.

We will call the elements in $K \setminus S$ of the form $(s_1, g_{2*} + 1)$ or $(g_{1*} + 1, s_2)$ the *sources* of S. This terminology will be explained in the next section.

Proposition 2.17 *The number of missing points of S equals $l_R(\omega/R)$.*

Proof. The number of missing points of S is $g(S_1) + 1 - n(S_1) - n(J_1) + g(S_2) + 1 - n(S_2) - n(J_2)$. Since $l_R(\bar{R}/R) = l_R(\omega : R/\omega : \bar{R}) = l_R(\omega/C)$, we have $l_R(\omega/R) = l_R(\bar{R}/R) - l_R(R/C) = l_R(\bar{R}/C) - 2l_R(R/C)$. Moreover $l_R(\bar{R}/C) = g_{1*} + g_{2*} + 2$, $l_R(R/C) = n(S_1) + g_{1*} - g(S_1) + n(J_2) = n(S_2) + g_{2*} - g(S_2) + n(J_1)$ (cf. Corollary 2.11), hence we get the statement.

Corollary 2.18 *The number of sources of S equals $l_R(\omega/R)$.*

Proof. This follows from Lemma 2.16 and Proposition 2.17.

Corollary 2.19 *The following are equivalent:*
i) R is Gorenstein.
ii) There are no missing points of S, i.e., $l_R(\omega/R) = 0$.
iii) $J_i = K(S_i) + a, a \in \mathbb{Z}, i = 1,2$.
iv) There are no sources of S.

Corollary 2.20 *The following are equivalent:*
i) R is Kunz.
ii) There is exactly one missing point of S, i.e., $l_R(\omega/R) = 1$.
iii) $J_1 = K(S_1) + a, a \in \mathbb{Z}$ and there is one missing point for ϕ_2 or vice versa.
iv) There is exactly one source of S.

The following proposition generalizes (in case $d = 2$) [G-L 95, Theorem 3.13]. We denote again R/P_i by R_i, $i = 1,2$.

Proposition 2.21 *We have, for $i = 1,2$, that $g_{i*} + 1 = i(R) + 2l_{R_i}(\overline{R_i}/R_i) -$ Card{missing points for ϕ_i}.*

Proof. We have $i(R) = g_{i*} - g(S_i) + n(S_i) - n(J_i)$, $2l_{R_i}(\overline{R_i}/R_i) = 2(g(S_i) + 1 - n(S_i))$, and the number of missing points for ϕ_i is $g(S_i) + 1 - n(S_i) - n(J_i)$.

We conclude this section with some examples to illustrate the definitions we have introduced.

Example 1 Let $\psi_1 : k[[x,y]] \longrightarrow k[[t^2,t^3]], x \mapsto t^2, y \mapsto t^3$, and let $\psi_2 : k[[x,y]] \longrightarrow k[[u^2,u^3]], x \mapsto u^3, y \mapsto u^2$, and let $R = k[[x,y]]/\mathrm{Ker}(\psi_1) \cap \mathrm{Ker}(\psi_2) = k[[x,y]]/(x^3 - y^2) \cap (x^2 - y^3)$. Then R is a Gorenstein ring with $S = v(R)$ as in Figure 1 (it consists of the dots and $v(C) = \{\alpha \mid \alpha \geq (6,6)\}$). We have $l_R(R/C) = 6$ and $l_R(\bar{R}/C) = 12$. The two projections of S are $S_1 = S_2 = \{0, 2, \rightarrow\}$ where the symbol "$\rightarrow$" means that any $n \in \mathbb{N}, n \geq 2$ is in the set, and $J_1 = J_2 = \{4, 6, \rightarrow\}$. Moreover there are no missing points and $K = S$, i.e., $l_R(\omega/R) = 0$. There are four maximal elements in S, namely $(0,0), (2,3), (3,2)$, and $g_* = (5,5)$, so $i(R) = 4$.

Example 2 Let $\psi_1 : k[[x,y,z,v,w]] \longrightarrow k[[t^3,t^4]], x \mapsto t^3, y \mapsto t^4, z \mapsto t^8, v \mapsto t^8, w \mapsto t^7$, and let $\psi_2 : k[[x,y,z,v,w]] \longrightarrow k[[u^3,u^4]], x \mapsto u^4, y \mapsto u^3, z \mapsto u^8, v \mapsto u^7, w \mapsto u^8$, and let $R = k[[x,y,z,v,w]]/\mathrm{Ker}(\psi_1) \cap \mathrm{Ker}(\psi_2) =$ $k[[x,y,z,v,w]]/(xy - w, z - v, y^2 - v, xv - yw, x^4 - yv, v^2 - x^3w) \cap (xy - v, x^2 - z, z - w, xv - yw, y^4 - xw, v^2 - y^2w, y^3v - w^2)$. The value semigroup S and K is depicted in Figure 2. Here $K \setminus S$ consists of the circles. We have $g_* = (8,8). l_R(R/C) = 8, l_R(\bar{R}/C) = 18$, and $J_1 = J_2 = \{6, 7, 9, \rightarrow\}$. We have two missing points, namely $(0,5)$ and $(5,0)$, hence $l_R(\omega/R) = 2$ and R is neither Kunz nor Gorenstein. The sources of S are $(3,9)$ and $(9,3)$. There are four maximal elements in S, namely $(0,0), (3,4), (4,3)$, and $(8,8)$, so $i(R) = 4$.

Example 3 Let $\psi_1 : k[[x,y,z,w]] \longrightarrow k[[t]], x \mapsto 0, y \mapsto 0, z \mapsto 0, w \mapsto t$ and let $\psi_2 : k[[x,y,z,w]] \longrightarrow k[[u^3,u^4,u^5]], x \mapsto u^3, y \mapsto u^5, z \mapsto u^7, w \mapsto u^4$, and let $R = k[[x,y,z]]/\mathrm{Ker}(\psi_1) \cap \mathrm{Ker}(\psi_2) = k[[x,y,z]]/(x,y,z) \cap (z - xw, xy - w^2, x^3 - yw, y^2 - x^2w)$. The value semigroup is depicted in Figure 3 (as in the previous picture, $K \setminus S$ consists of the circles). We have $l_R(R/C) = 3, l_R(\bar{R}/C) = 7, S_1 = \mathbb{N}, S_2 = \{0, 3, \rightarrow\}, J_1 = \{2, \rightarrow\}, J_2 = \{3, 5, \rightarrow\}$. There is only one missing point $(0,2)$ (for ϕ_2). $l_R(\omega/R) = 1$ and R is a Kunz ring. The source of S is $(2,2)$. There are two maximal elements in S, namely $(0,0)$ and $(1,4)$, so $i(R) = 2$.

Example 4 Let $\psi_1 : k[[x,y,z,w]] \longrightarrow k[[t^2,t^7]], x \mapsto t^7, y \mapsto t^2, z \mapsto t^4, w \mapsto t^9$, and let $\psi_2 : k[[x,y,z,w]] \longrightarrow k[[u^4,u^5,u^7]], x \mapsto u^4, y \mapsto u^5, z \mapsto u^7, w \mapsto u^{10}$, and let $R = k[[x,y,z,w]]/\mathrm{Ker}(\psi_1) \cap \mathrm{Ker}(\psi_2) = k[[x,y,z,w]]/(z - y^2, x^2 - y^7, w - xy, xz - yw) \cap (w - y^2, x^3 - yz, z^2 - xy^2, y^3 - x^2z)$. In this example (cf. Fig. 4) we have $l_R(R/C) = 10, l_R(\bar{R}/C) = 21, S_1 = \{0, 2, 4, 6, \rightarrow\}, S_2 = \{0, 4, 5, 7, \rightarrow\}$, $J_1 = \{4, 6, 8, 10, \rightarrow\}, J_2 = \{7, 8, 9, 11, \rightarrow\}$. There is only one missing point $(0,6)$ (for ϕ_2). $l_R(\omega/R) = 1$ and R is a Kunz ring. The source of S is $(10,4)$. There are four maximal elements in S, namely $(0,0), (2,5), (7,4)$, and $(9,10)$, so $i(R) = 4$.

3 The canonical ideal

In this section we study the relation between the elements in $S = v(R)$ and in $K = v(\omega)$. We start with the following basic fact:

Proposition 3.1 *We have that $\alpha \in \mathbb{N}^2$ is a maximal element in S if and only if $g_* - \alpha$ is a maximal element in K.*

Proof. We have $l_R(\omega/R) = l_R(\omega/C) - l_R(R/C)$. By Lemma 2.6 and Lemma 2.5 we have $l_R(\omega/C) = L_K(g_* + (1,1)) =$

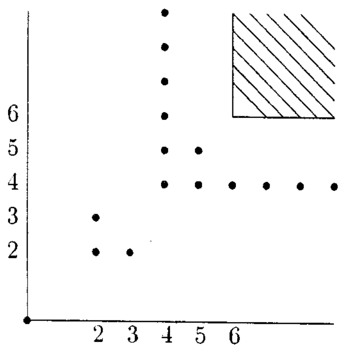

Fig. 1. The value semigroup of a Gorenstein ring, cf. Example 1.

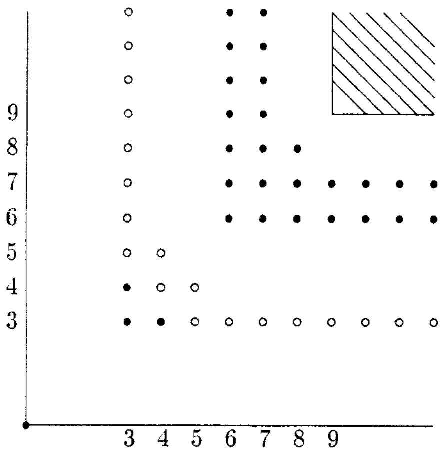

Fig. 2. The value semigroup of the ring in Example 2.

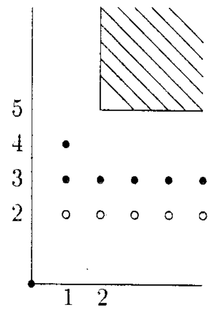

Fig. 3. The value semigroup of a Kunz ring, cf. Example 3.

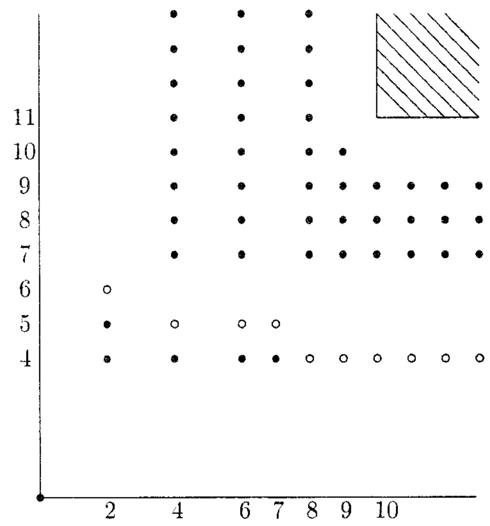

Fig. 4. The value semigroup of a Kunz ring, cf. Example 4.

$$\mathrm{Card}\{x \mid (x, g_{2*}+1) \in K, x < g_{1*}\} + \mathrm{Card}\{y \mid (g_{1*}+1, y) \in K, y < g_{2*}\} +$$

$$\mathrm{Card}\{\text{maximal elements in } K\}$$

and $l_R(R/C) =$

$$\mathrm{Card}\{x \mid (x, g_{2*}+1) \in S, x < g_{1*}\} +$$

$$\mathrm{Card}\{y \mid (g_{1*}+1, y) \in S, y < g_{2*}\} + \mathrm{Card}\{\text{maximal elements in } S\}.$$

By Corollary 2.18, $l_R(\omega/R) =$

$$\mathrm{Card}\{x \mid (x, g_{2*}+1) \in K \setminus S, x < g_{1*}\} +$$

$$\mathrm{Card}\{y \mid (g_{1*}+1, y) \in K \setminus S, y < g_{2*}\}.$$

hence there are just as many maximal elements in K as in S. Suppose α is a maximal element in S. Then $\Delta^S(\alpha) = \emptyset$ and so $g_* - \alpha \in K$. If $\Delta^K(g_* - \alpha) \neq \emptyset$, say that e.g. $g_* - \alpha + (c, 0) \in K$ for some $c > 0$, we would get $\Delta^S(g_* - (g_* - \alpha + (c, 0))) = \Delta^S(\alpha - (c, 0)) = \emptyset$, which is a contradiction. We have proved that if α is a maximal element in S, then $g_* - \alpha$ is a maximal element in K. Since K and S have the same number of maximal elements, we get also the converse.

We will next describe a process involving some elements of K and S for a non-Gorenstein ring. This technical part will be used in the next section to classify the rings R with the property that all integral overrings of R have larger conductor than R. By Proposition 2.17 we have a missing point a for ϕ_1 (or for ϕ_2), and thus, by Lemma 2.16, an element $\alpha^{(0)} = (g_{1*} - a, g_{2*} + 1)$ (or $\alpha^{(0)} = (g_{1*} + 1, g_{2*} - a)$, resp.) of $K \setminus S$, which we will call the *source* of the process. If $g_{1*} - a \notin S_1$, i.e., if $g_{1*} - a$ is also a missing point for ϕ_1 (if $g_{2*} - a \notin S_2$, i.e., if $g_{2*} - a$ is also a missing point for ϕ_2, resp.), we say that the process is *trivial* and we will call $\alpha^{(0)}$ also the *sink* of the process. Otherwise we construct $\alpha^{(1)} = (g_{1*} - a, \alpha_2)$ as the maximal element in S under $\alpha^{(0)}$ ($\alpha^{(1)} = (\alpha_1, g_{2*} - a)$ as the maximal element in S to the left of $\alpha^{(0)}$, resp.). Then the process is inductively described as follows:

$\alpha^{(2k+1)}$ is the maximal element in S under or to the left of $\alpha^{(2k)}$ (i.e., if $\alpha^{(2k)} = (\alpha_1, \alpha_2)$, then $\alpha^{(2k+1)} = (\alpha_1, \beta)$ with $\beta < \alpha_2$ or $\alpha^{(2k+1)} = (\beta, \alpha_2)$ with $\beta < \alpha_1$, resp.), and such that $\alpha^{(2k+1)} \neq \alpha^{(i)}$, for $i \leq 2k$. If such an element does not exist, we say that the process stops and call $\alpha^{(2k)}$ the *sink* of the process and $2k$ the *length* of the process. Otherwise we set $\alpha^{(2k+2)} = g_* - \alpha^{(2k+1)}$.

It follows from Proposition 3.1 that $\alpha^{(2k)}$ is maximal in K and $\alpha^{(2k)} \notin S$ if $k \geq 1$, and that $\alpha^{(2k+1)}$ is maximal in S but not in K. The process is univocally defined from a source, and it always stops because the number of maximal elements in K is finite. If $\alpha^{(2n)} = (\alpha_1, \alpha_2)$ is the sink of a process both of the following conditions are satisfied:

1) α_1 is a missing point for ϕ_1 or the maximal element in S under $\alpha^{(2n)}$ already appeared in the process (i.e., $\alpha^{(i)} = (\alpha_1, \alpha_2')$, for some $\alpha^{(i)}$ of the process with $i < 2n$).

2) α_2 is a missing point for ϕ_2 or the maximal element in S to the left of $\alpha^{(2n)}$ already appeared in the process (i.e., $\alpha^{(j)} = (\alpha_1', \alpha_2)$, for some $\alpha^{(j)}$ of the process with $j < 2n$).

Notice that all possibilities can occur as the following examples show:

Example 5 Let $R = k + k(t^5, u^3) + k(t^3, u^4) + t^6k[[t]] \times u^5k[[u]]$. Then S and K are as in Figure 5. There are processes starting from $(1,5), (4,5), (6,2), (3,5)$ and from $(6,3)$. The first three sources give rise to trivial processes, but $\alpha^{(0)} = (3,5)$ gives $\alpha^{(1)} = (3,4), \alpha^{(2)} = (2,0), \alpha^{(3)} = (0,0), \alpha^{(4)} = (5,4), \alpha^{(5)} = (5,3)$ and finally $\alpha^{(6)} = (0,1)$ (the sink). The maximal point $(0,0)$ under $\alpha^{(6)}$ already appeared in the process and the second coordinate 1 is a missing point for ϕ_2. The process starting from the source $(6,3)$ gives the same elements (in a different ordering), and has $(2,0)$ as sink. The maximal element $(0,0)$ to the left of $(2,0)$ already appeared in the process, and the first coordinate 2 is a missing point for ϕ_1.

Example 6 Let $R = k + k(t^3, 0) + k(0, u^2) + t^6k[[t]] \times u^4k[[u]]$. Then S and K are as in Figure 6. There are processes starting from $(1,4), (4,4), (6,0)$ and from $(0,4)$. The first two sources give rise to trivial processes, but $\alpha^{(0)} = (6,0)$ or $\alpha^{(0)} = (0,4)$ gives $\alpha^{(1)} = (0,0)$ and $\alpha^{(2)} = (5,3)$ (the sink). Here the first coordinate 5 is a missing point for ϕ_1 and the second coordinate 3 is a missing point for ϕ_2.

Example 7 Here we use the ring from Example 4. The (unique) source $\alpha^{(0)} = (10,4)$ gives $\alpha^{(1)} = (7,4), \alpha^{(2)} = (2,6), \alpha^{(3)} = (2,5)$, and $\alpha^{(4)} = (7,5)$ (the sink). Here both the maximal element $(7,4)$ under $\alpha^{(4)}$ and $(2,5)$ to the left of $\alpha^{(4)}$ are already used in the process.

Remark B A more precise description of the process could be done in terms of coordinates of the elements $\alpha^{(i)}$ of the process. Suppose $g_{1*} - x_1$ is a missing point for ϕ_1. Then: $\alpha^{(0)} = (x_1, g_{2*} + 1), \alpha^{(1)} = (x_1, x_2), \alpha^{(2)} = (g_{1*} - x_1, g_{2*} - x_2), \alpha^{(3)} = (g_{1*} - x_1 - \epsilon_1, g_{2*} - x_2), \alpha^{(4)} = (x_1 + \epsilon_1, x_2), \alpha^{(5)} = (x_1 + \epsilon_1, x_2 - \epsilon_2), \alpha^{(6)} = (g_{1*} - x_1 - \epsilon_1, g_{2*} - x_2 + \epsilon_2), \alpha^{(7)} = (g_{1*} - x_1 - \epsilon_1 - \epsilon_3, g_{2*} - x_2 + \epsilon_2), \alpha^{(8)} = (x_1 + \epsilon_1 + \epsilon_3, x_2 - \epsilon_2)$ a.s.o. (Here $x_i, \epsilon_i \in \mathbb{N}$ and $\epsilon_i > 0$.) Notice that any natural number may not occur more than twice as the first (second, resp.) coordinate of the elements $\alpha^{(i)}$ of a process. This follows from the fact that the elements $\alpha^{(i)}$ are maximal elements in S or K. Moreover, if for any $l \in \mathbb{N}$, $\alpha^{(l)} = (\alpha_1^{(l)}, \alpha_2^{(l)})$, for any $k \geq 1$ we have the following equalities:

$$(A) \begin{cases} \alpha_2^{(4k)} = \alpha_2^{(4k-3)} \\ \alpha_2^{(4k-1)} = \alpha_2^{(4k-2)} \end{cases}$$

$$(B) \begin{cases} \alpha_1^{(4k+2)} = \alpha_1^{(4k-1)} \\ \alpha_1^{(4k+1)} = \alpha_1^{(4k)} \end{cases}$$

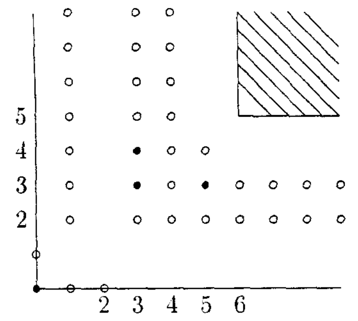

Fig. 5. The value semigroup of the ring in Example 5.

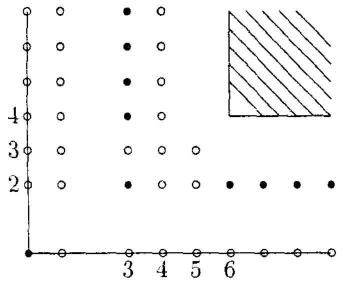

Fig. 6. The value semigroup of the ring in Example 6.

Finally notice that $\alpha_1^{(1)} = \alpha_1^{(0)}$ and $\alpha_1^{(2)} = g_{1*} - x_1$ is different from $\alpha_1^{(n)}$, for any $n \neq 2$, since $g_{1*} - x_1$ is a missing point for ϕ_1. If the process starts from a missing point for ϕ_2, we get similarly:

$$(A')\begin{cases} \alpha_1^{(4k)} = \alpha_1^{(4k-3)} \\ \alpha_1^{(4k-1)} = \alpha_1^{(4k-2)} \end{cases}$$

$$(B')\begin{cases} \alpha_2^{(4k+2)} = \alpha_2^{(4k-1)} \\ \alpha_2^{(4k+1)} = \alpha_2^{(4k)} \end{cases}$$

Lemma 3.2 *If a process stops at the sink* $\alpha^{(2n)} = (\alpha_1, \alpha_2)$ *and if* $\alpha^{(i)} = (\alpha_1, \alpha_2')$ *and* $\alpha^{(j)} = (\alpha_1', \alpha_2)$, *for some* $i, j < 2n$, *then* $\alpha_1 = g_{1*}/2$ *or* $\alpha_2 = g_{2*}/2$.

Proof. We can assume that the process starts from a missing point for ϕ_1. Given a process $\{\alpha^{(1)}, \dots, \alpha^{(2n)}\}$, for any $\alpha^{(2k)}, 1 \leq k \leq n$, the first or second coordinate of $\alpha^{(2k)}$ appeared as the first (second, resp.) coordinate of $\alpha^{(2k-3)}$ (cf. Remark B). Suppose that $2n = 4k+2$, for some $k \geq 1$. Then $\alpha_1^{(4k+2)} = \alpha_1^{(4k-1)}$. By hypothesis we have that there exists in the process $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)})$, with $j < 4k+2 = 2n$ and $\alpha_2^{(j)} = \alpha_2^{(4k+2)}$. Since any natural number may not occur more than twice as the second coordinate of the elements in a process and since, for any $k \geq 1$, the equalities (A) in Remark B hold, necessarily $j = 4k+1$. By construction, $\alpha^{(2n)} = g_* - \alpha^{(2n-1)} = g_* - \alpha^{(4k+1)}$ and so $\alpha_2^{(2n)} = \alpha_2 = g_{2*} - \alpha_2$, i.e., $\alpha_2 = g_{2*}/2$. Arguing in a similar way and using equalities (B) of Remark B, we get $\alpha_1^{(2n)} = \alpha_1 = g_{1*}/2$, if $2n = 4k$, for some $k \geq 1$.

Notice that, if $\alpha = (\alpha_1, \alpha_2)$ is a maximal element in S which is not maximal in K, then the following conditions hold:

1′) $(\alpha_1, g_{2*}+1)$ is a source or there exists one (and only one) maximal element in K (that is not in S) $\beta = ((\alpha_1, \alpha_2')$ with $\alpha_2' > \alpha_2$ ("above α").

2′) $(g_{1*}+1, \alpha_2)$ is a source or there exists one (and only one) maximal element in K (that is not in S) $\beta = ((\alpha_1', \alpha_2)$ with $\alpha_1' > \alpha_1$ ("to the right of α").

Suppose that $\alpha^{(2n)} = \beta^{(0)}$ is the sink of a nontrivial process. Recalling Proposition 3.1, we can inductively define a backwards process setting:

$\beta^{(2k+1)} = g_* - \beta^{(2k)}$

$\beta^{(2k+2)}$ is a maximal element in K (that is not in S) above or to the right of $\beta^{(2k+1)}$ and such that $\beta^{(2k+2)} \neq \beta^{(i)}$, for $i \leq 2k+1$. If such a $\beta^{(2k+2)}$ does not exist, we say that the backwards process stops.

If (α_1, α_2) is a sink, then at least one α_i is either a missing point for ϕ_i or equal to $g_{i*}/2$. In both cases it follows that the backwards process is univocally defined, and it gives the elements of the process starting in $\beta^{(2n-1)}$ in the reverse order, i.e., $\beta^{(0)} = \alpha^{(2n)}, \dots, \beta^{(2n-1)} = \alpha^{(1)}$. It stops at $\alpha^{(1)} = (\alpha_1, \alpha_2)$ under the source $(\alpha_1, g_{2*}+1)$ or to the left of the source $(g_{1*}+1, \alpha_2)$.

Lemma 3.3 *If two processes starting from different sources* $\alpha^{(0)}$ *and* $\alpha'^{(0)}$ *have the same sink, then the length of both processes is 2 and* $\alpha^{(1)} = \alpha'^{(1)}, \alpha^{(2)} = \alpha'^{(2)}$.

Proof. If the length of a process is $2n \geq 4$, then the backwards process starting from the sink $\beta^{(0)} = \alpha^{(2n)}$ will stop at $\alpha^{(1)} = \beta^{(2n-1)}$ that either has a source above and the element $\alpha^{(4)} = \beta^{(2n-4)}$ to the right or has a source to the right and the element $\alpha^{(4)} = \beta^{(2n-4)}$ above. In any case the source we get is unique.

Lemma 3.4 *If the sink of a process is* $\alpha^{(2n)} = (\alpha_1, \alpha_2)$ *is such that* $\alpha_1 = g_{1*}/2$ *or* $\alpha_2 = g_{2*}/2$, *then its source is unique.*

Proof. Suppose the sink is $\alpha^{(2n)} = (g_{1*}/2, \alpha_2)$. If the length of the process is 2 we get the unique source $(g_{1*}+1, g_{2*}-\alpha_2)$. If the length of the process is > 2, we can apply Lemma 3.3. The argument is analogous if the sink is $\alpha^{(2n)} = (\alpha_1, g_{2*}/2)$.

Proposition 3.5 *Every maximal element in K, that is not in S, occurs as some $\alpha^{(2k)}$, for some $k \geq 1$, in some process. Every maximal element in S, that is not a maximal element in K occurs as some $\alpha^{(2k-1)}$, for some $k \geq 1$, in some process.*

Proof. Let $\alpha = (\alpha_1, \alpha_2)$ be a maximal element in K, $\alpha \notin S$. Suppose first that $\alpha_1 > g_{1*}/2$ and $\alpha_2 \leq g_{2*}/2$. Apply the backwards process starting from $\beta^{(0)} = \alpha$. In this case at $\beta^{(1)} = g_* - \beta^{(0)}$ the backwards process is not necessarily univocally defined. Let's choose for $\beta^{(2)}$ the maximal element in K (not in S) above $\beta^{(1)}$. (If above $\beta^{(1)}$ there is no maximal element of K, but a source, we are done.) With these choices of $\beta^{(0)}$ and $\beta^{(2)}$, as is easily seen, the backwards process is univocally defined. Since the second coordinate of $\beta^{(2)}, \beta^{(6)}, \dots, \beta^{(4k+2)}, \dots$ and the first coordinate of $\beta^{(0)}, \beta^{(4)}, \dots, \beta^{(4k)}, \dots$ give two strictly increasing sequences of natural numbers, after a finite number of steps we must reach a source. So in this case α will appear in the process starting from that source. If $\alpha_1 \leq g_{1*}/2$ and $\alpha_2 > g_{2*}/2$, a similar argument works, choosing $\beta^{(2)}$ to the right of $\beta^{(1)}$. Now suppose that α is a maximal element in K (not maximal in S) without any other assumption. Apply the backwards process starting from $\beta^{(0)} = \alpha$. As before there are possibly two choices for $\beta^{(2)}$. Let's choose a $\beta^{(2)}$ and go ahead with the backwards process that is now univocally defined. If the backwards process stops finding a source, our element α will occur in the process starting from that source. Otherwise it will stop at $\beta^{(2k+1)} = (\beta_1, \beta_2)$ such that, for some $i, j \leq 2k$, $\beta^{(i)} = (\beta'_1, \beta_2)$ and $\beta^{(j)} = (\beta_1, \beta'_2)$. Arguing as in the proof of Lemma 3.2, we easily get that $\beta_1 = g_{1*}/2$ or $\beta_2 = g_{2*}/2$. Now apply the process starting from $\alpha^{(1)} = \beta^{(2k+1)}$, i.e. consider $\alpha^{(2)} = g_* - \alpha^{(1)}$, $\alpha^{(3)}$ as the maximal element in S to the left of $\alpha^{(2)}$, if $\beta_1 = g_{1*}/2$ (resp. $\alpha^{(3)}$ as the maximal element in S under $\alpha^{(2)}$ if $\beta_2 = g_{2*}/2$) and so on. We have $\alpha^{(2k+2)} = \alpha$. since with this process we get in reverse order the elements of the backwards process $\{\alpha = \beta^{(0)}, \dots, \beta^{(2k+1)}\}$. Moreover any $\alpha^{(2k)} = (\alpha_1, \alpha_2)$ of this process satisfies one of the following conditions:
a) $\alpha_1 > g_{1*}$ and $\alpha_2 \leq g_{2*}/2$
b) $\alpha_1 \leq g_{1*}$ and $\alpha_2 > g_{2*}/2$.
So α will occur in some process, by the first part of the proof.

Corollary 3.6 *Let α be a maximal element in K. Then the following are equivalent:*
i) $\alpha \notin S$.
ii) $\alpha = \alpha^{(2k)}$ for some process $\{\alpha^{(0)}, \dots, \alpha^{(2n)}\}$ of S and some k, $1 \leq k \leq n$.
iii) $g_* - \alpha$ is not a maximal element in K.

Corollary 3.7 *Let α be a maximal element in S. Then the following are equivalent:*

i) α is not a maximal element in K.
ii) $\alpha = \alpha^{(2k-1)}$ for some process $\{\alpha^{(0)}, \dots, \alpha^{(2n)}\}$ of S and some k, $1 \leq k \leq n$.
iii) $g_* - \alpha \notin S$.

We call the maximal elements in K (in S, resp.) that satisfy the equivalent conditions in Corollary 3.6 (Corollary 3.7, resp.) the *asymmetric* maximal elements in K (in S, resp.). It follows from the corollaries above, that the maximal elements in K which are not asymmetric coincides with the maximal elements in S which are not asymmetric. We call these maximal elements *symmetric*.

Example 8 In the ring R of Example 4 we have four maximal elements in S, two are symmetric and two asymmetric. The same happens for the maximal elements in K. In the ring of Example 5 we have three maximal elements in S and they are all asymmetric. The same happens for the maximal elements of K.

Proposition 3.8 *Let $\gamma_1, \dots, \gamma_n$ be the asymmetric maximal elements in K, and let $\delta_1, \dots, \delta_m$ be the sources of S. Let $x_i \in \omega, v(x_i) = \gamma_i, i = 1, \dots, n$, and let $y_j \in \omega, v(y_j) = \delta_j, j = 1, \dots, m$. Then $\omega = R + Rx_1 + \cdots + Rx_n + Ry_1 + \cdots + Ry_m$.*

Proof. Let $J = R + Rx_1 + \cdots + Rx_n + Ry_1 + \cdots + Ry_m$. It follows from Lemma 2.1 that $v(J) = K$. Since $J \subseteq \omega$, we get equality from Proposition 2.7.

Lemma 3.9 *(Cf. [D'A 97, Corollary 4.4].) Let $z \in \mathbb{Z}$ and $S_i = v_i(R/P_i)$. Then:*
a) $z - g_{1*} + g_1 \in K(S_1)$ if and only if $(z, g_{2*} + 1) \in K$.
b) $z - g_{2*} + g_2 \in K(S_2)$ if and only if $(g_{1*} + 1, z) \in K$.

Proof. Let's prove a). Since $K(S_1) = \{z \in \mathbb{Z} \mid g_1 - z \notin S_1\}$, we have that $z - g_{1*} + g_1 \in K(S_1)$ if and only if $g_{1*} - z \notin S_1$. This is equivalent to $\Delta^S(g_{1*} - z, -1) = \emptyset$, i.e.$(z, g_{2*} + 1) \in K$. Point b) is proved similarly.

Proposition 3.10 *Let $t(R)$ ($t(R/P_i)$, resp.) be the CM type of R (of R/P_i, resp.). Then:*
$t(R) \leq 1 + t(R/P_1) + t(R/P_2) + \mathrm{Card}\{\text{asymmetric maximal elements in } K\}$.

Proof. Let $t_1 = t(R/P_1)$ and let $z_1, \dots, z_{t_1}$ be a minimal set of generators for ω_1, the canonical ideal of R/P_1, $R/P_1 \subseteq \omega_1 \subseteq \overline{R/P_1}$, and let $v_1(z_j) = \delta_j$, $j = 1, \dots, t_1$. Let $x_1, \dots, x_{t_1}$ be elements of ω such that $v(x_j) = (\delta_j + g_{1*} - g_1, g_{2*} + 1)$, $j = 1, \dots, t_1$. (By Lemma 3.9 these are values in K.) In a similar way let $x'_1, \dots, x'_{t_2}$, with $t_2 = t(R/P_2)$, be elements of ω such that, for $j = 1, \dots, t_2$, $v(x'_j) = (g_{1*} + 1, \delta'_j)$, where δ'_j are the v_2-values of a minimal set of generators for ω_2, the canonical ideal of R/P_2, $R/P_2 \subseteq \omega_2 \subseteq \overline{R/P_2}$. Let finally $y_1, \dots, y_m$ be elements of ω of values $\gamma_1, \dots, \gamma_m$, where $\gamma_1, \dots, \gamma_m$ are the asymmetric maximal elements in K. Consider the fractional ideal J of R generated by $1, x_1, \dots, x_{t_1}, x'_1, \dots, x'_{t_2}, y_1, \dots, y_m$. By Lemma 3.9, for any source $(s_1, g_{2*} + 1)$ of $S = v(R)$ ($(g_{1*} + 1, s_2)$, resp.) there is an element $x \in J$ with $v_1(x) = s_1$ ($v_2(x) = s_2$, resp.), so using Lemma 2.1 we get in $v(J)$ all the sources of S. By Proposition 3.8 we get $J = \omega$. Since $t(R)$ is the cardinality of a minimal set of generators for ω, we get the inequality of the statement.

Example 9 The inequality in Proposition 3.10 can be an equality as the following example shows. Let $R = k + k(t^2, u^2) + t^3k[[t]] \times u^3k[[u]]$. Then $S = \{(0,0)\} \cup \{(2,2)\} \cup \{x \in \mathbb{N}^2 \mid x \geq (3,3)\}$. We have $t(R/P_1) = t(R/P_2) = 1$ and there are no asymmetric maximal elements in K, thus $1 + t(R/P_1) + t(R/P_2) + \text{Card}\{\text{asymmetric maximal elements in } K\} = 1 + 1 + 1 + 0 = 3$. It is easily checked that $m : m = k + (t,u)k[[t]] \times k[[u]]$, so $v(m : m) = \{0,0)\} \cup \{x \in \mathbb{N}^2 \mid x \geq (1,1)\}$. By Proposition 2.7 we get $t(R) = l_R(m : m/R) = L_{v(m:m)}(3,3) - L_{v(R)}(3,3) = 5 - 2 = 3$.

4 Maximality

If for any strict overring T of R, $R \subset T \subseteq \bar{R}$, the conductor $T : \bar{R}$ is strictly larger than $C = R : \bar{R}$, we say that R is *maximal with fixed conductor* (MWFC). It was shown in [B-F 97', Propositions 2 and 4] that Gorenstein and Kunz rings are MWFC. In case R is analytically irreducible (i.e., $d = 1$), these are the only rings which are MWFC. cf. [B-F 97', Theorem 1]. We will now determine which rings with two minimal primes that are MWFC. We will use the construction of the processes of last section. We need some preliminaries.

Lemma 4.1 *Let T be an overring of R, $R \subseteq T \subseteq \bar{R}$, and let K be the canonical ideal of $S = v(R)$. Then $v(T) \subseteq K$ if and only if $R : \bar{R} = T : \bar{R}$.*

Proof. It is clear that $R : \bar{R} \subseteq T : \bar{R}$. If $v(T) \subseteq K$, then, since by Lemma 2.14 $\Delta^K(g_*) = \emptyset$, we have $v(T) \cap \Delta^K(g_*) = \Delta^{v(T)}(g_*) = \emptyset$. Conversely let $R : \bar{R} = T : \bar{R}$. If $v(t) \notin K_R$, for some $t \in T$, then there is an element $r \in R$ with $v(r) \in \Delta^{v(R)}(g_* - v(t))$, hence $v(rt) \in \Delta^{v(T)}(g_*)$, so T has larger conductor than R.

We will use this lemma in several ways.

Lemma 4.1.1 *If $g_* \notin S$, then R is not MWFC.*

Proof. In fact if we extend R to $R[x]$, where x is any element of $\bar{R}$ of value g_*, then $v(R[x]) = S \cup \{g_*\} \subseteq K$.

Lemma 4.1.2 *If $x \in K \setminus S$ and $2x \in v(C)$, then R is not MWFC.*

Proof. If $v(r) = x$, then $R[x] = \{r_0 + r_1x \mid r_i \in R, i = 1,2\}$. Thus $v(R[x]) \subseteq K$.

Lemma 4.1.3 *If a is a missing point for ϕ_1, $a < g_{1*}/2$ (or b is a missing point for ϕ_2, $b < g_{2*}/2$), then R is not MWFC.*

Proof. By Lemma 2.16, $g_* - (a,-1) \in K \setminus S$ and furthermore $2(g_* - (a,-1)) \in v(C)$. So we can apply Lemma 4.1.2.

Lemma 4.1.4 *Suppose α is a maximal element of S but not maximal in K. If $\alpha < g_*/2$, then R is not MWFC.*

Proof. Suppose $\alpha < g_*/2$. We have $g_* - \alpha \in K \setminus S$ and furthermore $2(g_* - \alpha) \in v(C)$. So we can apply Lemma 4.1.2.

Lemma 4.1.5 *If both g_{1*} and g_{2*} are even and $g_*/2 \notin S$, and furthermore some element $r \in R$ with $v(r) = g_*$ is a square (mod C) of an element in $\bar{R}$, then R is not MWFC.*

Proof. Suppose $r \in R$, $v(r) = g_*$, and $r + c = z^2, c \in C$, for some element $z \in \bar{R}$. Then $R[z] = \{r_0 + r_1 z \mid r_i \in R, i = 1, 2\}$. Hence, since $g_*/2 \in K$, $v(R[z]) \subseteq K$.

Lemma 4.2 *Suppose that R is MWFC and that $\alpha = (\alpha_1, \alpha_2)$ is a sink. Then either $\alpha_1 = g_{1*}/2$ or $\alpha_2 = g_{2*}/2$.*

Proof. By Lemma 4.1.3, Lemma 2.16 (and symmetry) we can assume that a source has coordinates $\alpha^{(0)} = (x_1, g_{2*} + 1)$ with $x_1 \leq g_{1*}/2$. If $x_1 = g_{1*}/2$ the process starting from that source is trivial and we are done. Otherwise if $\alpha^{(1)} = (x_1, x_2)$, then, by Lemma 4.1.4, $x_2 \geq g_{2*}/2$. If $x_2 = g_{2*}/2$, then $\alpha^{(2)} = (g_{1*} - x_1, g_{2*}/2)$ is the sink and we are done. Otherwise $\alpha^{(2)} = (g_{1*} - x_1, g_{2*} - x_2)$ with $g_{1*} - x_1 > g_{1*}/2$ and $g_{2*} - x_2 < g_{2*}/2$. As above, by Lemma 4.1.3, $g_{2*} - x_2$ is not a missing point for ϕ_2, and $\alpha^{(3)} = (g_{1*} - x_1 - \epsilon_1, g_{2*} - x_2)$. By Lemma 4.1.4 we have $g_{1*} - x_1 - \epsilon_1 \geq g_{1*}/2$ a.s.o. Since the sequence of first coordinates in $\alpha^{(0)}, \alpha^{(4)}, \alpha^{(8)}, \ldots$ strictly increases and its terms are all $\leq g_{1*}/2$ and the sequence of second coordinates in $\alpha^{(2)}, \alpha^{(6)}, \alpha^{(10)}, \ldots$ also strictly increases and its terms are all $\leq g_{2*}/2$, we must end up on $\alpha = (\alpha_1, \alpha_2)$, with $\alpha_1 = g_{1*}/2$ or $\alpha_2 = g_{2*}/2$.

Lemma 4.3 *If R is MWFC, then $l_R(\omega/R) \leq 2$.*

Proof. Suppose that R is MWFC and that $l_R(\omega/R) > 2$. By Lemma 2.16 and Proposition 2.17, there are at least three sources, and by Lemma 4.2 and Lemma 3.4 they have different sinks (α_1, α_2) with $\alpha_1 = g_{1*}/2$ or $\alpha_2 = g_{2*}/2$. We can not have two different sinks on the same horizontal or vertical line, so we get a contradiction.

If R is Gorenstein, then $K = S$ and hence all maximal elements in S (or in K) are symmetric. This fact was already shown in [G 82, Theorem 12]. We can extend this to:

Corollary 4.4 *Suppose that R satisfies one of the following conditions:*
a) R is Gorenstein.
b) R is Kunz with $g_{1*}/2$ a missing point for ϕ_1 (or $g_{2*}/2$ a missing point for ϕ_2).
c) R is MWFC with both $g_{1*}/2$ and $g_{2*}/2$ as missing points.
Then every maximal element in K is symmetric, hence belongs to S.

Proof. There are no nontrivial processes, so we can apply Corollary 3.6.

Corollary 4.5 *Suppose that R is MWFC and that α is a maximal element in*

K. If $\alpha > g_*/2$ or if $\alpha < g_*/2$, then α is a symmetric maximal element, hence $\alpha \in S$.

Proof. By Proposition 3.5, every maximal element in $K \setminus S$ appears in some process. If R is MWFC, then no element α with $\alpha > g_*/2$ or $\alpha < g_*/2$ appears in some process (cf. the proof of Lemma 4.2).

Theorem 4.6 *R is MWFC if and only if one (and only one) of the following possibilities happens:*
1) R is Gorenstein.
2) R is Kunz.
3) g_{1*} and g_{2*} are both even, $g_* \in S$, but no element $r \in R$ with $v(r) = g_*$ is a square (mod C) of an element in $\bar{R}$, and $l_R(\omega/R) = 2$.

Proof. If R is Gorenstein or Kunz, then R is MWFC, cf. [B-F 97', Theorem 1]. Suppose g_{1*} and g_{2*} are both even, that no element $r \in R$ with $v(r) = g_*$ is a square (mod C) of an element in $\bar{R}$, and $l_R(\omega/R) = 2$. Now suppose T is a strict overring of R with $R : \bar{R} = T : \bar{T} = C$. Then $l_R(R/C) = l_{R/C}(R/C) < l_{T/C}(T/C) = l_T(T/C)$, cf. [B-F 97', Lemma 3]. We also have $l_T(\bar{R}/T) \leq l_R(\bar{R}/T) \leq l_R(\bar{R}/R) - 1$. Since $l_R(\omega/R) = l_R(\bar{R}/R) - l_R(R/C) = 2$ we get $l_T(\bar{R}/T) - l_T(T/C) \leq 0$. For any ring A, we have $l_A(\bar{A}/A) - l_A(A/A : \bar{A}) \geq 0$, with equality if and only if A is Gorenstein. Hence all inequalities above are equalities and T is Gorenstein. Since T is Gorenstein, $v(T)$ is symmetric. Since $\Delta^{v(T)}(g_*) = \emptyset$ (cf. Lemma 2.10), then also $\Delta^{v(T)}(g_*/2) = \emptyset$, hence the symmetry of $v(T)$ implies that $g_*/2 \in v(T)$. Take $t \in T$ with $v(t) = g_*/2$, and let $r \in R$ be of value g_*. Then, for some unit $u \in R/P_1$, we get $r - ut^2 \in \Delta^{v(T)}(g_*)$, which is a contradiction to $\Delta^{v(T)}(g_*) = \emptyset$. Thus R is MWFC also in this case.

For the converse we assume that R is MWFC. First assume that $g_{1*} + g_{2*}$ is odd. Suppose that g_{1*} is even (the other case follows similarly). Lemma 4.2 gives that the sinks are on the line $x = g_{1*}/2$, so there is only one sink, hence, by Lemma 4.2 and Lemma 3.4, $S = v(R)$ has a unique source, i.e., by Corollary 2.20, R is Kunz. Next assume that g_{1*} and g_{2*} are both odd. Lemma 4.2 gives that there are no sinks, hence no sources, and so, by Corollary 2.19, R is Gorenstein. Now assume that g_{1*} and g_{2*} are both even. We claim that, if R is not Gorenstein, then $g_*/2 \in K \setminus S$. By Proposition 2.17 and Corollary 2.19, if R is not Gorenstein, we have at least two missing points (R is not Kunz since $g_{1*} + g_{2*}$ is even). If $g_{1*}/2$ is a missing point for ϕ_1 and $g_{2*}/2$ is a missing point for ϕ_2, we get $g_*/2 \in K \setminus S$. If $g_{1*}/2$ is a missing point for ϕ_1 and $g_{2*}/2$ is not a missing point for ϕ_2, then Lemma 4.2 gives a sink $(a, g_{2*}/2)$ with $a > g_{1*}/2$, which again gives $g_*/2 \in K \setminus S$. If neither $g_{1*}/2$ is a missing point for ϕ_1, nor $g_{2*}/2$ is a missing point for ϕ_2, then Lemma 4.2 gives two sinks $(a, g_{2*}/2)$ with $a > g_{1*}/2$ and $(g_{1*}/2, b)$ with $b > g_{2*}/2$. This gives, by Lemma 2.1, that $g_*/2 \in K$. Of course $g_*/2 \notin S$ also in this case. Now assume that R is MWFC, that g_{1*}, g_{2*} are both even, and that R is not Gorenstein. We must show that R satisfies the conditions in 3) in the statement of the theorem. By Lemma 4.1.1 we have $g_* \in S$. If some element $r \in R$ of value g_* is a square (mod C) of an element in $\bar{R}$, then, since (as we showed above) $g_*/2 \in K \setminus S$, by Lemma 4.1.5

we get a contradiction. Thus no element in R of value g_* is a square (mod C) of an element in $\bar{R}$. We noticed already that there are at least two missing points, i.e., $l_R(\omega/R) \geq 2$ (cf. Proposition 2.17). By Lemma 4.3, we get $l_R(\omega/R) = 2$.

We note that our condition that an element $r \in R$ of value g_* has a square root (mod C) is always satisfied in the important special case when R/m is algebraically closed. Hence we have:

Corollary 4.7 *If R/m is algebraically closed, then R is MWFC if and only if R is Gorenstein or Kunz.*

We now give an example of a ring which is MWFC but not Gorenstein or Kunz.

Example 10 Let $\psi_1 : \mathbb{Q}[[x,y,z,v,w]] \longrightarrow \mathbb{Q}[[t^3,t^4,t^5]], x \mapsto t^3, y \mapsto 0, z \mapsto 2t^4, v \mapsto t^5, w \mapsto t^6$, and let $\psi_2 : \mathbb{Q}[[x,y,z,v,w]] \longrightarrow \mathbb{Q}[[u^3,u^4,u^5]], x \mapsto 0, y \mapsto u^3, z \mapsto u^4, v \mapsto u^6, w \mapsto u^5$, and let $R = \mathbb{Q}[[x,y,z,v,w]]/\mathrm{Ker}(\psi_1) \cap \mathrm{Ker}(\psi_2) = \mathbb{Q}[[x,y,z,v,w]]/(x^2-w, z^2-4xv, zv-2xw, 2v^2-zw, y) \cap (x, y^2-v, z^2-yw, yv-zw, zv-w^2)$. The semigroup of values is depicted in Figure 7. According to Theorem 4.6, R is MWFC. In fact, for any element $z \in \bar{R}$ with $v(z) = g_*/2 = (2,2)$, we have $z^2 \notin R$. If, in this example, we replace $\mathbb{Q}$ with $\mathbb{C}$, according to Corollary 4.7, the corresponding ring is not MWFC.

If R is Gorenstein, then $\omega = R$. If R is Kunz, then $\omega = R + Rx$ for any element $x \in \omega \setminus R$, since $l_R(\omega/R) = 1$. We now give a minimal generating set for ω if R is MWFC with $l_R(\omega/R) = 2$.

Proposition 4.8 *Suppose that R is MWFC and $l_R(\omega/R) = 2$. If $x_1, x_2 \in \omega$ and $v(x_1), v(x_2)$ are the two sinks of $S = v(R)$, then $\omega = R + Rx_1 + Rx_2$.*

Proof. By Proposition 3.8, we have to show that every asymmetric maximal element in K and every source of S belong to $E = v(R + Rx_1 + Rx_2)$. Let $\alpha = (\alpha_1, \alpha_2)$ be an asymmetric maximal element of K or a source. Suppose that $\alpha_2 \geq g_{2*}/2$. Let's argue by induction on α_2. If $\alpha_2 = g_{2*}/2$, then $\alpha = (\alpha_1, \alpha_2)$ is a sink, so $v(x_1) = \alpha$ or $v(x_2) = \alpha$, and so $\alpha \in E$. The inductive step: If $\alpha = (g_{1*}/2, g_{2*} + 1)$, then α is a sink so $\alpha \in E$. If $\alpha \neq (g_{1*}/2, g_{2*} + 1)$, by construction $\alpha = \alpha^{(2n)}$ is above the maximal element $\alpha^{(2n+1)}$ in S. Since $\alpha^{(2n+1)}$ is not a maximal element in K, there is, to the right of $\alpha^{(2n+1)}$, an asymmetric maximal element in K that, by the inductive hypothesis, is in E. So, by Lemma 2.3, there exists an element $\gamma \in E$, $\alpha^{(2n+1)} < \gamma \leq \alpha^{(2n)}$. If $\gamma \neq \alpha^{(2n)}$, then γ is not a maximal element in K, so to the right of γ there is an asymmetric maximal element in K or an element of S, in any case an element in E. Again, by Lemma 2.3, there exists an element $\gamma' \in E, \gamma < \gamma' \leq \alpha^{(2n)}$. In this way, after a finite number of steps, we get $\alpha^{(2n)} \in E$. If, on the other hand, $\alpha = (\alpha_1, \alpha_2)$ and $\alpha_2 < g_{2*}/2$, then $\alpha_1 \geq g_{1*}/2$ (cf. Lemma 4.1.4) and we can argue in a similar way on α_1. So the proof is complete.

In [B-F 97] the concept of *almost Gorenstein ring* was introduced. A ring R is called almost Gorenstein if $l_R(\bar{R}/R) = l_R(R/C) + \mathrm{type}(R) - 1$, and it was shown in [B-F 97, Proposition 20] that R is almost Gorenstein if and only if $m\omega = m$, and also if and only if $\mathrm{type}(R) = l_R(\omega/R) + 1$.

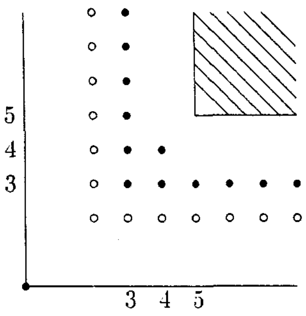

Fig. 7. The value semigroup of the ring in Example 10.

Corollary 4.9 *If R is MWFC with* $l_R(\omega/R) = 2$, *then R is almost Gorenstein of type 3.*

Proof. Let x be one of the three generators of ω given in Proposition 4.8. In each case it is easy to see that $mx \subseteq m$. Hence $m\omega = m$, so R is almost Gorenstein by [B-F 97, Proposition 20]. By the same proposition in [B-F 97] we get $\text{type}(R) = l_R(\omega/R) + 1 = 3$.

Proposition 4.10 *R is MWFC if and only if R has a unique minimal overring.*

Proof. Let R be MWFC. Thus, if T is an overring of R, $T : \bar{R}$ strictly contains $R : \bar{R}$ and so $\Delta^S(g_*) \subseteq v(T)$. On the other hand, if $x \in \bar{R}$ with $v(x) \in \Delta^S(g_*)$, then $v(R[x]) \supseteq \Delta^S(g_*)$ and $R[x] = R[y]$, if y is another element of $\bar{R}$, with $v(y) \in \Delta^S(g_*)$. It follows that $R[x] = R + Rx$ is the unique minimal overring of R. Conversely, suppose R has a unique minimal overring. If R is not MWFC, we can enlarge R at least in two different ways. Either by adjoining an element $x \in \bar{R}$, with $v(x) \in \Delta^S(g_*)$ (and in this way the conductor of the new ring is strictly larger than the conductor of R), or by taking a minimal overring with the same conductor.

5 Overrings

As before we denote the maximal ideals in the integral closure of R by n_1, n_2. We have that $v(\bar{R}) = \mathbb{N}^2$ and that, if T is an intermediate ring between R and $\bar{R}$, then $v(R) \subseteq v(T) \subseteq \mathbb{N}^2$. Moreover T is local if and only if $n_1 \cap T = n_2 \cap T$. Notice that, if $r \in \bar{R}$ and $v(r) = (\alpha_1, \alpha_2)$, then $\alpha_i > 0$ if and only if $r \in n_i$. So the ring T is local if and only if $\Delta^{v(T)}(0,0) = \emptyset$. Recall that the seminormalization R^+ of R (in $\bar{R}$) is the largest overring T of R (in $\bar{R}$) such that:

i) For each $P \in \text{Spec}(R)$, there is exactly one $Q \in \text{Spec}(T)$ above P.

ii) The canonical homomorphism $k(P) \longrightarrow k(Q)$ is an isomorphism (where $k(P) = R_P/PR_P$ and $k(Q) = T_Q/QT_Q$),
(cf. [T 70, (1.1)]). In our hypotheses, for each minimal prime P_i, $i = 1, 2$, of R, there is exactly one prime p_i of $\bar{R}$ above P_i, they are in fact $p_1 = 0 \times V_1$ and $p_2 = V_2 \times 0$, resp. Since $\bar{R}/p_i \simeq V_i$ and the quotient field of $\bar{R}/p_i$ is the quotient field of of V_i, we get that conditions i) and ii) are satisfied for the minimal primes of R. It follows that the overring T is the seminormalization of R if and only if conditions i) and ii) are satified for the maximal ideal m of R. Thus, since R is residually rational, the overring T of R satisfies conditions i) and ii) if and only if T is local. We collect this in a lemma.

Lemma 5.1 *Let T be an overring of R, $T \subseteq \bar{R}$.*
a) The seminormalization R^+ of R is the maximal local overring of R in $\bar{R}$ and $R^+ = R + (n_1 \cap n_2) = R + (M_1 \times M_2)$.

b) The following are equivalent:
i) T is local.
ii) $T \subseteq R^+$.
iii) $\Delta^{v(T)}(0,0) = \emptyset$.

c) The following are equivalent:
i) $R = R^+$, i.e., R is seminormal.
ii) $v(R) = \{(0,0)\} \cup \{x \in \mathbb{Z}^2 \mid x \geq (1,1)\}$.
iii) $g_*(R) = (0,0)$.

Corollary 5.2 *In our hypotheses, R^+ is a Gorenstein ring with $i(R^+) = 1$ and $l_{R^+}(R^+/C_{R^+}) = 1$, where $C_{R^+} = R^+ : \bar{R}$.*

Proof. Since $v(R^+) = \{(0,0)\} \cup \{x \mid x \geq (1,1)\}$, we have that $l_{R^+}(\bar{R}/R^+) = l_{R^+}(R^+/C) = 1$, cf. Corollary 2.11. That $i(R^+) = 1$ follows from Proposition 2.12.

Let $\alpha \in \mathbb{N}^2$ and $R(\alpha) = \{r \in R \mid v(r) \geq \alpha\}$ as before. As usual we denote $R : R(\alpha)$ by $(R(\alpha))^{-1}$ and $((R(\alpha))^{-1})^{-1}$ by $(R(\alpha))_v$. It is easy to check that $(R(\alpha))^{-1} = R(\alpha) : R(\alpha)$ (and hence $R(\alpha)$ is an overring) and that $R(\alpha)_v = R(\alpha)$, i.e., $R(\alpha)$ is divisorial. In particular we have that $R(\alpha)^{-1}$ is strictly larger than $R(\beta)^{-1}$, if $R(\alpha)$ is strictly smaller than $R(\beta)$. With this notation we can observe that:

Proposition 5.3 *The overring $(R(\alpha))^{-1}$ is local if $v(R(\alpha))$ contains a maximal element. If $\alpha \in v(C)$, then $(R(\alpha))^{-1}$ is not local.*

Proof. Suppose $v(R(\alpha))$ contains a maximal element $\beta = (\beta_1, \beta_2)$. So there exists $x \in R$, with $v(x) = \beta \geq \alpha$ and $\Delta^S(\beta) = \emptyset$. If $(R(\alpha))^{-1}$ is not local, by Lemma 5.1b), there is an element $y \in (R(\alpha))^{-1}$, with $v(y) = (\gamma_1, 0), \gamma_1 > 0$ (or $v(y) = (0, \gamma_2), \gamma_2 > 0$). Therefore $v(xy) \in S$ and $v(xy) = v(x) + v(y) = (\beta_1 + \gamma_1, \beta_2)$ (or $v(xy) = (\beta_1, \beta_2 + \gamma_2)$), a contradiction because $\Delta^S(\beta) = \emptyset$. If $\alpha \in v(C)$, then $(R(\alpha))^{-1} = \bar{R}$, which is not local.

Corollary 5.4 *If $g_* \in S$, then $(R(\alpha))^{-1}$ is local if and only if $\alpha \leq g_*$.*

Proposition 5.5 *The following are equivalent:*
i) $g_* \in S$.
ii) $R^+ = R(g_*)^{-1}$.
iii) $R^+ = R(\alpha)^{-1}$*, for some* $\alpha \in \mathbb{N}^2$.
iv) R^+ *is a fractional divisorial ideal of* R.

Proof. i)⇒ii): By Corollary 5.4, $T = (R(g_*))^{-1}$ is local. Moreover T satisfies conditions i) and ii) of the definition of seminormality, thus $T \subseteq R^+$. Since $v(T) = v(R^+)$, we get by Proposition 2.7 that $T = R^+$.
ii)⇒iii): Immediate.
iii)⇒iv): Any ideal of the form $R : I$ is divisorial.
iv)⇒i): If $g_* \notin S$, then $R : R^+ = R : \bar{R} = C$. Since $(R^+)_v = R : C = \bar{R}$ and R^+ is strictly contained in $\bar{R}$, the seminormalization R^+ is not divisorial in this case.

Lemma 5.6 *If* $g_* \in S$ *and* $g_* \neq (0,0)$*, then* $m : m$ *is local. In particular, if* (R, m) *is Gorenstein and not seminormal or if* (R, m) *is Kunz, then* $m : m$ *is local.*

Proof. If $g_* \in S$, then g_* is a maximal element. We have $g_* \in v(m)$ since $g_* \neq (0,0)$, hence, by Proposition 5.3, $m : m = R : m$ is local. We know that, if R is Gorenstein or Kunz, then R is MWFC. So, by Lemma 4.1.1, $g_* \in S$. Moreover, if R is Kunz, certainly $g_* \neq (0,0)$ and, if R is Gorenstein and not seminormal, also $g_* \neq (0,0)$ (cf. Lemma 5.1c)).

Proposition 5.7 *The ring* (R, m) *with residue field* k *is Gorenstein and not seminormal if and only if* $m : m/m \simeq k[X]/(X^2)$.

Proof. If R is Gorenstein and not seminormal then, by Lemma 5.6, $m : m$ is local. With the same argument as in the proof of Proposition 6, (2)⇒(4), in [B-D-F 90], we get that either $m : m/m \simeq k[X]/(X^2)$ or $m : m/m$ is a two-dimensional field extension of k. But in the second case the ring R is not residually rational, in contradiction to our hypotheses. Conversely, if $m : m/m \simeq k[X]/(X^2)$ then $l_R(m : m/R) = 1$, which gives that R is Gorenstein. Moreover if R is not seminormal, then, for any $x \in \bar{R}$ and $r \in m$, $v(xr) \geq (1,1)$, so, by Remark A and Lemma 5.1c), $xr \in m$. Thus $m : m = \bar{R}$ has two maximal ideals and $m : m/m \simeq k[X]/(X^2)$ is local, a contradiction. It follows that R is not seminormal.

Proposition 5.8 *The ring* (R, m) *with residue field* k *is Kunz if and only if* $m : m/m \simeq k[X]/(X^3)$.

Proof. By Lemma 5.6, $m : m$ is local. With the same argument as of the proof of Proposition 27 in [B-F 97], we get that $m : m/m$ is isomorphic to one of $k[X]/(X^3)$, $k[X,Y]/(X,Y)^2$, or a three-dimensional field extension of k. The second case is not possible, if R is Kunz (cf. again the proof of Proposition 27 in [B-F 97], at the end), the third is not possible since R is residually rational, hence $R \simeq k[X]/(X^3)$. For the other implication we note that type$(R) = l_R(m :$

$m/R) = 2$. If R is not Kunz, since typev$(R) = 2$, by Theorem 4.6, we get that R is not MWFC. By Proposition 4.10, R has at least two minimal overrings T_1 and T_2. Since $T_1, T_2 \subseteq m : m$, we get, modulo m, two minimal overrings of k in $k[X]/(X^3)$, which is a contradiction, since k has a unique minimal overring in $k[X]/(X^3)$, namely $k + kx^2$.

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