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An Algebraic Version of Haag's Theorem

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Abstract: Under natural conditions (such as split property and geometric modular action of wedge algebras) it is shown that the unitary equivalence class of the net of local (von Neumann) algebras in the vacuum sector associated to double cones with bases on a fixed space-like hyperplane completely determines an algebraic QFT model. More precisely, if for two models there is a unitary connecting all of these algebras, then — *without* assuming that this unitary also connects their respective vacuum states or spacetime symmetry representations — it follows that the two models are equivalent. This result might be viewed as an algebraic version of the celebrated theorem of Rudolf Haag about problems regarding the so-called "interaction-picture" in QFT. Original motivation of the author for finding such an algebraic version came from conformal chiral QFT. Both the chiral case as well as a related conjecture about standard half-sided modular inclusions will be also discussed.

1. Introduction

1.1. Haag's theorem and its algebraic version. If we "freeze" a classical, nonrelativistic physical system — say a mechanical system of n point masses — at a certain time-instant, we do not see if the system was an "interactive" or a "free" one. A certain configuration with given velocities may correspond both to a free or to an interactive system. Interaction becomes visible only when one looks at how things *change*.

This is the basic idea behind the so-called "interaction-picture" in quantum field theory (QFT). Within the framework of *Wightman-axioms* [16], free models can be well-described in terms of Wightman-fields (i.e. operator-valued distributions on spacetime). Then to give an interactive model one should consider the restriction of the same free fields at a certain spacelike hyperplane but then extend it to spacetime with a

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different *time-evolution*. (So that the interactive and free fields will coincide at our fixed spacelike hyperplane but possible nowhere else.)

However, Haag's theorem (see the book [16] for a detailed account) has ruled out the existence of such a description. Suppose two QFT models are given: one in terms of the Wightman-fields Φ_r $(r = 1, \ldots, n)$ and another one in terms of the Wightman-fields $\tilde{\Phi}_r$ $(r = 1, \ldots, n)$. Assuming some relatively mild conditions (such as the existence of well-behaved restrictions for the fields along spacelike hyperplanes), if there exists a spacelike hyperplane H and a unitary operator V such that

$$V\Phi_r(x)V^* = \tilde{\Phi}_r(x) \qquad (x \in H, r = 1, \ldots n), \tag{1.1}$$

then it also follows that up to a possible phase-factor, $V\Omega = \tilde{\Omega}$ (where Ω and $\tilde{\Omega}$ are the respective vacuum-vectors), $V\Phi_r(x)V^* = \tilde{\Phi}_r(x)$ for all spacetime points x and $r = 1, \ldots, n$, and finally, that $VU(g)V^* = \tilde{U}(g)$ (where U and $\tilde{U}$ are the respective representations of spacetime symmetries) for all elements g of the connected Poincaré group. Thus V establishes an equivalence between the two models: if one was free, so is the other — we cannot make an interacting model out of a free one in this way.

So what would be an algebraic version of Haag's theorem? Fix a spacelike hyperplane H. We shall say that two nets of von Neumann algebras $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are **equivalent along** H, iff there is a unitary operator V such that

$$V\mathcal{A}(K^\diamond)V^* = \tilde{\mathcal{A}}(K^\diamond) \tag{1.2}$$

for every double-cone $K^\diamond$ with base $K \subset H$. (See Sect. 2 on the notions of double-cones and algebraic QFT.) In this paper — under certain additional assumptions of — it will be proved that if $(\mathcal{A}, U)$ and $(\tilde{\mathcal{A}}, \tilde{U})$ are two algebraic QFT models and $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are equivalent along a spacelike hyperplane, then there exists a unitary W such that $W\mathcal{A}(\mathcal{O})W^* = \tilde{\mathcal{A}}(\mathcal{O})$ for all double-cones $\mathcal{O}$ and $WU(g)W^* = \tilde{U}(g)$ for all elements g of the connected Poincaré group: thus the two models are equivalent.

Actually, it will suffice to assume equivalence along a "half-hyperplane" H^+; see the details in Sect. 4. Nevertheless, H^+ has still an infinite space-volume. In fact, it is well-known that two inequivalent models, when restricted to a compact region, may give rise to unitarily equivalent nets of algebras; see [6,10] for examples.

1.2. Algebraic vs. original version. Of course in a strict sense the two versions of Haag's theorem cannot be compared. They are statements made in two different frameworks and despite numerous attempts, the passage between the two frameworks — albeit clear in actual examples — in general is still unresolved.

Nonetheless, in some sense, as we shall see now, one may say that the algebraic version is stronger than the original one, and that the new version is not a simple reformulation of the old one. Let us see why.

In case we deal with algebraic QFT models associated to Wightman field theories, our additional assumptions — with the exception of the split property, which however probably could be avoided (see the comments at the end of Sect. 2) — are known to hold. To appreciate the differences, rather than at assumptions regarding frameworks, one must look at the respective notions of equivalence and the ways in which it is established.

The natural notion of equivalence of Wightman field theories (i.e. the existence of a unitary operator connecting the defining fields and representations) — and hence also the condition of equivalence along a spacelike hyperplane appearing in Haag's original version — is too narrow, and does not coincide with physical equivalence. (In a sense this was exactly the original motivation [13] for considering the local algebras generated by the fields rather than the fields themselves: they already contain all physical information — fields also depend on the choices made regarding our *description*.)

But there is more to this. In the original version, the unitary operator V appearing in Eq. (1.1) actually also turns out to be the unitary operator establishing the equivalence between the two models. This clearly does *not* hold in the algebraic case.

For example, let both models be the same scalar free field model. Since the adjoint action of a Weyl-operator $W(f)$ preserves every local algebra, $V := W(f)$ satisfies the requirement (1.2) made in the algebraic version. However, in general $W(f)\Omega \neq \lambda\Omega$ so $V = W(f)$ does not establish an equivalence between the model and itself. To put it another way: a unitary operator whose adjoint action leaves the fields along a hyperplane invariant must be a multiple of the identity and hence must preserve every local algebra. To the contrary, a unitary operator, whose adjoint action preserves every local algebra, does not necessarily preserve the vacuum and hence may not take a Poincaré-covariant field into a Poincaré-covariant field.

So even if the passage between Wightman field theory and algebraic QFT was clear, the introduced algebraic version of Haag's theorem would not become a simple consequence of the original one. Rather, it is the other way around.

1.3. Conformal QFT and half-sided modular inclusions. Though it is always nice to strengthen a theorem, this was not why the author considered an algebraic version of Haag's theorem. As it will be explained now, the original motivation came from conformal chiral QFT and in particular its relation to half-sided modular inclusions.

Möbius covariant nets on S^1 have remarkable properties. Many things that in "ordinary" algebraic QFT often appear as additional assumptions — like for example *additivity, Bisognano-Wichmann property* and *factoriality* of local algebras — can be in fact *derived*; see [4,8,9,11,12] on the general structure of such nets.

For simplicity of notations, let us consider such a net $\mathcal{A}$ with vacuum vector Ω on the real line $\mathbb{R}$ (see the last section on details of what it exactly means). Setting $\mathcal{M} := \mathcal{A}(0,\infty)$ and $\mathcal{N} := \mathcal{A}(1,\infty)$, by an application of the *Bisognano-Wichmann property* (which, as was mentioned, in the conformal case is automatic) we have that the $(\Omega, \mathcal{N} \subset \mathcal{M})$ is a **standard half-sided modular inclusion** of von Neumann factors. That is,

- Ω is a *standard vector* of the inclusion $\mathcal{N} \subset \mathcal{M}$: it is cyclic and separating for both $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}' \cap \mathcal{M}$,
- $\Delta^{it}_{\mathcal{M},\Omega} \mathcal{N} \Delta^{it}_{\mathcal{M},\Omega} \subset \mathcal{N}$ for all $t \leq 0$.

This also works the other way around. Namely, it is shown [2,12,18,19] that if $(\Omega, \mathcal{N} \subset \mathcal{M})$ is a standard half-sided modular inclusion of factors, then one can construct a unique *strongly additive* Möbius covariant net $\mathcal{A}$ with vacuum vector Ω such that $\mathcal{A}(0,\infty) = \mathcal{M}$ and $\mathcal{A}(1,\infty) = \mathcal{N}$.

At first sight, this seems to give a great opportunity for constructing new conformal chiral QFT models. Indeed, instead of an entire net of algebras (together with a representation of the Möbius group), all we need is to present a certain standard inclusion of von Neumann factors.

Sadly, the reality is just the opposite way around. As far as the author knows, (nontrivial) standard half-sided modular inclusions have been constructed only with the help of Möbius covariant nets. However, there were hopes to find a more or less direct way to construct a new half-sided modular inclusion out of an existing one. R. Longo proposed[1] to consider the following "perturbation" of a half-sided modular inclusion of factors $(\Omega, \mathcal{N} \subset \mathcal{M})$.

For a vector Ψ which is cyclic and separating for $\mathcal{M}$, let us denote by J_Ψ and Δ_Ψ the modular objects associated to $(\Psi, \mathcal{M})$. By [1], for each $X \in \mathcal{M}$, $X^* = X$ there exists a vector Ω_X cyclic and separating for $\mathcal{M}$ such that

- Ω_X is in the natural cone of $(\Omega, \mathcal{M})$ and hence $J_{\Omega_X} = J_\Omega =: J$,
- $\ln(\Delta_{\Omega_X}) = \ln(\Delta_\Omega) + X + JXJ$.

In particular, if $X \in \mathcal{N}$ then by applying the *Trotter product formula* one can easily check that $(\Omega_X, \mathcal{N} \subset \mathcal{M})$ is still a half-sided modular inclusion. If Ω_X is also a standard vector for $\mathcal{N} \subset \mathcal{M}$, then we can go on and generate a new strongly additive net $\mathcal{A}_X$.

But are the original net $\mathcal{A}_0$ (from where we took our half-sided modular inclusion) and $\mathcal{A}_X$ really different? Using the mentioned product formula one can also easily show that with $X \in \mathcal{N}$ many local algebras will remain the same; not only that $\mathcal{A}_0(0,\infty) = \mathcal{A}_X(0,\infty) = \mathcal{M}$ and $\mathcal{A}_0(1,\infty) = \mathcal{A}_X(1,\infty) = \mathcal{N}$ but actually

$$\mathcal{A}_0(I) = \mathcal{A}_X(I) \quad \text{for all } I \subset (0,1). \tag{1.3}$$

On the other hand, by an easy reformulation (see Sect. 5) of the main result of the present paper, if $\mathcal{A}_0$ and hence also $\mathcal{A}_X$ satisfy the *split property*, then the above equality implies that $\mathcal{A}_0$ and $\mathcal{A}_X$, as Möbius covariant nets, are equivalent. Thus, in this way we cannot obtain new models.

Of course one may try to improve the situation. Instead of a self-adjoint $X \in \mathcal{N}$, more generally we could take any $X \in \mathcal{M}$, $X^* = X$ for which $e^{iXt}\mathcal{N}e^{-iXt} \subset \mathcal{N}$ for all $t \leq 0$. For example, X may be a self-adjoint of the form $X = X_1 + X_2$ with $X_1 \in \mathcal{N}$ and $X_2 \in \mathcal{M} \cap \mathcal{N}'$, and in concrete examples we may find further choices.

Nevertheless, in light of Haag's theorem, it seems unlikely for the author that retaining the same inclusion $\mathcal{N} \subset \mathcal{M}$ and changing only the "dynamics" one could obtain something really new. Actually in Sect. 5, regarding this question we shall observe two important facts. Let $(\Omega, \mathcal{N} \subset \mathcal{M})$ and $(\tilde{\Omega}, \tilde{\mathcal{N}} \subset \tilde{\mathcal{M}})$ be two standard half-sided modular inclusions of factors and denote the two corresponding strongly additive Möbius covariant nets by $(\mathcal{A}, U)$ and $(\tilde{\mathcal{A}}, \tilde{U})$, respectively.

I. If there exists a unitary operator V such that $V\mathcal{N}V^* = \tilde{\mathcal{N}}$ and $V\mathcal{M}V^* = \tilde{\mathcal{M}}$, then for each $n \in \mathbb{N}$ there exists a unitary operator V_n such that

$$V_n \mathcal{A}(j,k) V_n^* = \tilde{\mathcal{A}}(j,k)$$

for every pair of integers $j, k \in \{0, 1, \ldots, n\}$.

In particular, this implies that if $\mathcal{A}$ is split, so is $\tilde{\mathcal{A}}$ and in fact $\mathcal{A}$ and $\tilde{\mathcal{A}}$ has unitarily equivalent 2-*interval inclusions*. Now this inclusion is a rich source of information; in the *completely rational* case essentially it contains [14] the entire representation theory of the net. So this already suggests that perhaps $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are equivalent. As a matter of fact, the conformal version of our algebraic Haag's theorem tells that just a slightly stronger condition indeed implies equivalence.

[1] This idea has never been published; the author learned about it through oral communication.

II. Let $(\mathcal{A}, U)$ and $(\tilde{\mathcal{A}}, \tilde{U})$ be two Möbius covariant nets and assume that at least one of them is split. If there exists a unitary operator V such that

$$V\mathcal{A}(j,k)V^* = \tilde{\mathcal{A}}(j,k)$$

for every pair of natural numbers $j, k \in \mathbb{N}$ then $(\mathcal{A}, U)$ and $(\tilde{\mathcal{A}}, \tilde{U})$ are equivalent.

Again, as was mentioned already and will be explained at the end of the next section, the author thinks that the split condition should be possible to remove. Now I + II + the remarks made after stating them — though it does not actually prove — seems to indicate the following:

Conjecture 1.1. *The unitary equivalence class of a standard half-sided modular inclusion of factors* $(\Omega, \mathcal{N} \subset \mathcal{M})$ *is completely determined (up to a possible normalization of* Ω*) by the unitary equivalence class of the inclusion* $\mathcal{N} \subset \mathcal{M}$*. That is, for another half-sided modular inclusion* $(\tilde{\Omega}, \tilde{\mathcal{N}} \subset \tilde{\mathcal{M}})$ *with equal normalization* $\|\tilde{\Omega}\| = \|\Omega\|$*, if there exists a unitary operator* V *such that* $V\mathcal{N}V^* = \tilde{\mathcal{N}}$ *and* $V\mathcal{M}V^* = \tilde{\mathcal{M}}$*, then there exists a unitary operator* W *such that not only* $W\mathcal{N}W^* = \tilde{\mathcal{N}}$ *and* $W\mathcal{M}W^* = \tilde{\mathcal{M}}$*, but also* $W\Omega = \tilde{\Omega}$*.*

2. Preliminaries: Axioms of Algebraic QFT

In this paper we shall consider an algebraic version of Haag's theorem. An algebraic QFT, rather than quantum fields, is given in terms of a net of local algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$. We shall work directly on the so-called "vacuum Hilbert space" and consider $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})'' = \mathcal{A}(\mathcal{O})$ to be a net of *von Neumann algebras*.

For a spacelike hyperplane H, and a bounded, connected and simply connected open subset K of H we set

$$K^\diamond := \mathrm{Int}(\overline{K^c}), \tag{2.1}$$

where K^c is the (closed) *causal completion* of K and "Int" stands for the (open) interior. We say that $K^\diamond$ is a **double-cone** with **base** on H; that is, with $K \subset H$.

For physical purposes (e.g. for determining the S-matrix or the structure of charged sectors) it is enough to work with special spacetime regions like double-cones. So considering only what is absolutely necessary, here we define an **algebraic QFT** to be a map associating to each double-cone $\mathcal{O}$ a von Neumann algebra $\mathcal{A}(\mathcal{O})$ on a fixed Hilbert space $\mathcal{H}$ together with a strongly continuous unitary representation U of the connected Poincaré group satisfying the following "minimal" conditions. (Note that some further additional properties will be later considered.)

(1) **Isotony**: $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ whenever $\mathcal{O}_1 \subset \mathcal{O}_2$.
(2) **Locality**: $[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$ whenever $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated.
(3) **Covariance**: $U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(g(\mathcal{O}))$ for all regions $\mathcal{O}$ and elements g of the connected Poincaré group.
(4) **Positivity of energy**: $P_{\mathbf{x}} \geq 0$ whenever $\mathbf{x}$ is future like. $P_{\mathbf{x}}$ is defined by the equation $U(\tau_{t\mathbf{x}}) = e^{itP_{\mathbf{x}}}$ $(t \in \mathbb{R})$ in which $\tau_{\mathbf{z}}$ is a translation by $\mathbf{z}$.
(5) **Existence, uniqueness and cyclicity of vacuum**: up to phase there exists a unique unit vector Ω invariant for $U(\tau)$ for all spacetime translations τ. Moreover, Ω is cyclic for $\vee_{\mathcal{O}}\mathcal{A}(\mathcal{O})$.

Note that from a physical point of view one should assume U to be a *projective* representation rather than a true one. However, it is easy to see that if U is a projective representation of a group G and $N \subset G$ is a normal subgroup such that there exists a unique one-dimensional invariant subspace for $U(N)$, then actually this subspace is invariant for the action of the full group and hence one can arrange the "phase factors" in such a way that U becomes a true representation. So without loss of generality, for clarity we have stated the axioms with U being a true representation rather than just a projective one.

Although so far we have only associated algebras to double-cone like bounded regions, by setting

$$\mathcal{A}(\mathcal{O}) := \vee_{K^\diamond \subset \mathcal{O}} \mathcal{A}(K^\diamond), \tag{2.2}$$

we may talk about the algebra $\mathcal{A}(\mathcal{O})$ associated to any open region $\mathcal{O}$. Note that *isotony* implies that the new definition does not change the algebra associated to a double-cone and that properties (1, 2, 3, 4, 5) remain valid.

The standard *Reeh-Schlieder* argument combined with *locality* shows that Ω is cyclic and separating for $\mathcal{A}(\mathcal{W})$ whenever $\mathcal{W}$ is a *wedge region*. (See e.g. the book [13] for precise definition of wedge regions.) Actually, by [15, Thm. 3] it even follows that for a wedge region $\mathcal{W}$ the algebra $\mathcal{A}(\mathcal{W})$ is a type III_1 factor. Another well-known consequence of (1, 2, 3, 4, 5) is *irreducibility*:

$$\mathcal{A}(M)' \equiv \{\vee_{\mathcal{O}} \mathcal{A}(\mathcal{O})\}' = \cap_{\mathcal{O}} \mathcal{A}(\mathcal{O})' = \mathbb{C}\mathbb{1}. \tag{2.3}$$

Here M stands for the full spacetime.

Howeve, as was mentioned, (1, 2, 3, 4, 5) is only a "minimal set" of conditions; they still allow many pathological examples. In particular, while Ω turns out to be cyclic for $\mathcal{A}(\mathcal{W})$ whenever $\mathcal{W}$ is a wedge, it may not be so for a double-cone.[2]

Sometimes instead of *isotony* the stronger **additivity** property is required; namely, that $\mathcal{A}(\mathcal{O}) \subset \vee_{k=1}^n \mathcal{A}(\mathcal{O}_k)$ whenever $\mathcal{O} \subset \cup_{k=1}^n \mathcal{O}_k$. (Note that in the conformal case additivity is not needed as a further assumption since it can be actually *proved*, as will be discussed in Sect. 5.) Having additivity one can use the argument of Reeh and Schlieder and show that Ω is cyclic for every local algebra $\mathcal{A}(\mathcal{O})$ associated to a nonempty open region $\mathcal{O}$.

Local von Neumann algebras were originally introduced to replace the unbounded polynomial algebra of local fields. From a physical point of view it seems reasonable to assume that our local von Neumann algebras are in fact generated by unbounded (Wightman) fields (i.e. that there is an "underlying" Wightman field theory and $\mathcal{A}(\mathcal{O})$ is the smallest von Neumann algebra to which the closure of all fields smeared with testfunctions with support in $\mathcal{O}$ are affiliated). Now for the algebra of fields additivity is evident. However, the passage from unbounded operators to von Neumann algebras is a delicate issue. In particular — at least, to the author's knowledge — even assuming an underlying Wightman field theory, so far additivity could not be proved. On the other hand, it is easy to see that the cyclicity guaranteed (at the level of Wightman fields) by the Reeh-Schlieder theorem passes without problems to the level of local von Neumann algebras. For this reason here we shall assume directly this cyclicity rather than making the stronger assumption of additivity.

[2] For an example, let us fix a bounded open set of spacetime and call a region "small" if it can be moved into this set by a Poincaré tranformation. Now take a "nice" model and reset all local algebras that are associated to the "small region" to be equal to the trivial algebra $\mathbb{C}\mathbb{1}$. It is easy to see that all listed properties remain valid, but now, starting from a "nice" model we have produced one with the mentioned pathological property.

(6) **Reeh-Schlieder property**: Ω is cyclic for $\mathcal{A}(\mathcal{O})$ for every nonempty open region $\mathcal{O}$.

Let $\mathcal{W}$ be a wedge region and consider the modular operator $\Delta_{\mathcal{A}(\mathcal{W}),\Omega}$ and modular conjugation $J_{\mathcal{A}(\mathcal{W}),\Omega}$ associated to $(\mathcal{A}(\mathcal{W}), \Omega)$. (As was mentioned, Ω is cyclic and separating for $\mathcal{A}(\mathcal{W})$, so these objects are well-defined). Assuming the existence of an "underlying" Wightman field theory, it is known [3] that these objects have a "geometrical meaning". Though attempts were made, so far it has not been proved that in general, the geometrical nature of these modular objects is a consequence of (1, 2, 3, 4, 5). So we shall simply assume it.

(7) **Bisognano-Wichmann property**. If $\mathcal{W}$ is a *wedge-region*, then $\Delta^{it}_{\mathcal{A}(\mathcal{W}),\Omega} = U(\beta_t)(t \in \mathbb{R})$, where $t \mapsto \beta$ is the one-parameter group of *boosts* associated to $\mathcal{W}$ with a certain parametrization.

For definition of the one-parameter group of boosts associated to a wedge and details on the parametrization we refer to the book [13]. Note that as will be explained in Sect. 5, in the conformal case not only (6), but also this property can be derived eliminating the need to additionally assume it.

The discussed properties (1, 2, 3, 4, 5, 6, 7) are essential for the proof of our argument. However — though for a somewhat technical reason — we shall actually need one more property:

(8) **Split property**. The inclusion $\mathcal{A}(K_1^\diamond) \subset \mathcal{A}(K_2^\diamond)$ is split whenever $\overline{K_1^\diamond} \subset K_2^\diamond$.

(Actually *distant split property* would suffice for us, but for simplicity here we only talk about split property.) For physical significance of the split property we again refer to the book [13]. Here we briefly comment only on the *difference* between how (8) and the other properties will be used.

In the course of the proof of our main theorem, we shall construct a sequence of unitary operators $n \mapsto W_n$. The equivalence between the two models is then to be established by the strong limit of this sequence. But though the author is convinced that this limit exists, he could not show this. So instead, split property is used to obtain a compactness condition by which at least the existence of a convergent subsequence can be established.

Now the way in which split property can be turned into the right compactness condition is not simple; in fact the whole next section will be dedicated to this question. Nevertheless, the author feels that the split property should not play an essential role in the algebraic Haag's theorem.

3. On Split Inclusions

An inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ for which there exists a type I factor $\mathcal{R}$ "in between": $\mathcal{N} \subset \mathcal{R} \subset \mathcal{M}$, is said to be a **split** inclusion. Let $\mathcal{N} \subset \mathcal{M}$ be a split inclusion and Ω a **standard vector** for the inclusion in question; i.e. we suppose that Ω is cyclic and separating for both $\mathcal{N}$, $\mathcal{M}$ and the relative commutant $\mathcal{N}' \cap \mathcal{M}$. Denoting the modular conjugation associated to $\mathcal{N}' \cap \mathcal{M}$ and the vector Ω by J_Ω, if $\mathcal{N}$ is a factor, we shall set

$$\mathcal{R}_\Omega := \mathcal{N} \vee J_\Omega \mathcal{N} J_\Omega. \tag{3.1}$$

Alternatively, if $\mathcal{M}$ is a factor, we shall set

$$\mathcal{R}_\Omega := \mathcal{M} \cap J_\Omega \mathcal{M} J_\Omega. \tag{3.2}$$

By [7], under the assumptions made our notation is unambiguous: if both $\mathcal{N}$ and $\mathcal{M}$ are factors, then $\mathcal{N} \vee J_\Omega \mathcal{N} J_\Omega = \mathcal{M} \cap J_\Omega \mathcal{M} J_\Omega$. Moreover, the thus defined von Neumann algebra $\mathcal{R}_\Omega$ is a type I factor between $\mathcal{N}$ and $\mathcal{M}$; we shall say that that $\mathcal{R}_\Omega$ is the **canonical type I factor** of the inclusion.

If $(\Omega, \mathcal{N} \subset \mathcal{M})$ is a standard split inclusion in which one of the algebras is a factor, and W is a unitary operator such that it preserves the vector Ω and its adjoint action preserves the algebras $\mathcal{N}$ and $\mathcal{M}$, then the adjoint action of W must also preserve the canonical type I factor $\mathcal{R}_\Omega$ of the inclusion. Using this fact in [7] it was proved that the group of such unitary operators is compact and metrizable (with respect to the strong operator topology). In particular, if $n \mapsto W_n$ is a sequence of such unitary operators, then one can always find a subsequence s such that $n \mapsto W_{s(n)}$ will strongly converge to a unitary operator.

In this section we assume that $(\Omega, \mathcal{N} \subset \mathcal{M})$ is a standard split inclusion in which at least one of the algebras is a factor, and $n \mapsto W_n$ is a sequence of unitaries such that the adjoint action of W_n preserves the algebras $\mathcal{N}$ and $\mathcal{M}$ for all $n \in \mathbb{N}$. We set $\Omega_n := W_n\Omega$ and assume that $n \mapsto \Omega_n$ is convergent; more precisely, that there is a standard vector Ψ for our inclusion such that $\|\Omega_n - \Psi\| \to 0$ as $n \to \infty$. Our aim is to find a suitable modification of the proof of [7] in order to show the existence of a subsequence of $n \mapsto W_n$ converging strongly to a unitary operator.

We shall proceed in several intermediate steps. We shall begin with an important observation which generalizes [7, Lemma 3.2].

Lemma 3.1. *Let $\mathcal{H}$ be a Hilbert space, $n \mapsto U_n$ a sequence of unitary operators on $\mathcal{H}$ and φ a faithful normal state on $\mathcal{B}(\mathcal{H})$. If $n \mapsto \varphi_n := \varphi \circ \mathrm{Ad}(U_n)$ converges in norm, then there exists a subsequence s such that $n \mapsto U_{s(n)}$ converges strongly. Moreover, if the norm limit of $n \mapsto \varphi_n$ is faithful, then the strong limit of $n \mapsto U_{s(n)}$ is unitary.*

Proof. If $\dim(\mathcal{H}) < \infty$, then the statement is trivially true. On the other hand, as $\mathcal{B}(\mathcal{H})$ was assumed to have a faithful normal state, $\mathcal{H}$ must be separable. So we may assume that $\mathcal{H}$ is the (up to unitary equivalence) unique infinite dimensional separable Hilbert space.

For each normal state η there exists a unique positive trace-class operator $D_\eta \in \mathcal{B}(\mathcal{H})$ such that

$$\eta(A) = \mathrm{Tr}(D_\eta A) \tag{3.3}$$

for all $A \in \mathcal{B}(\mathcal{H})$. Since $\varphi_n = \varphi \circ \mathrm{Ad}(U_n)$, we have that $D_n := D_{\varphi_n} = U_n D_\varphi U_n^*$. Now let $\tilde{\varphi}$ be the assumed (norm) limit of $n \mapsto \varphi_n$, and consider the operator $D_{\tilde{\varphi}}$. We know that $\varphi_n \to \tilde{\varphi}$ in norm as $n \to \infty$. What does this tell us about D_n $(n \in \mathbb{N})$ and $D_{\tilde{\varphi}}$? Since $\|D_n - D_{\tilde{\varphi}}\| \leq 2$, as $n \to \infty$ we have

$$0 \leq \mathrm{Tr}((D_{\tilde{\varphi}} - D_n)^2) = (\tilde{\varphi} - \varphi_n)(D_\Psi - D_n) \leq 2\|\tilde{\varphi} - \varphi_n\| \to 0. \tag{3.4}$$

In particular, $D_n \to D_{\tilde{\varphi}}$ in norm. Let us see now what can we say about the convergence of spectrums and spectral projections.

Let f be a continuous real function on $[0, 1]$ and $\epsilon > 0$. Then by the Stone-Weierstrass theorem there is a real polynomial p such that $|f(x) - p(x)| < \epsilon/3$ for all $x \in [0, 1]$. As $\mathrm{Sp}(D_n)$, $\mathrm{Sp}(D_{\tilde{\varphi}}) \subset [0, 1]$, we have that both $\|f(D_{\tilde{\varphi}}) - p(D_{\tilde{\varphi}})\| < \epsilon/3$ and $\|f(D_n) - p(D_n)\| < \epsilon/3$ for all $n \in \mathbb{N}$. On the other hand, as p is a polynomial and $D_n \to D_{\tilde{\varphi}}$ in norm, there exists a $N \in \mathbb{N}$ such that $\|p(D_n) - p(D_{\tilde{\varphi}})\| < \epsilon/3$ for all $n > N$. Thus for $n > N$ we have that

$$\|f(D_n) - f(D_{\tilde{\varphi}})\| \leq \|f(D_n) - p(D_n)\| + \|p(D_n) - p(D_{\tilde{\varphi}})\| \\ + \|p(D_{\tilde{\varphi}}) - f(D_{\tilde{\varphi}})\| < \epsilon, \tag{3.5}$$

showing that $\|f(D_n) - f(D_{\tilde{\varphi}})\| \to 0$ as $n \to \infty$.

As was already noted, $\mathrm{Sp}(D_n) = \mathrm{Sp}(D_\varphi)$ for every $n \in \mathbb{N}$ because of unitary equivalence. Now D_φ, $D_{\tilde{\varphi}}$ are *density operators*, so their spectrum is contained in $[0, 1]$ and have at most one point of accumulation; namely, at zero. Moreover, each positive point of their spectrum must be an eigenvalue corresponding to a finite dimensional eigenspace.

Since the spectrum is compact, if $x \notin \mathrm{Sp}(D_\varphi)$, then there exists a continuous function f on $[0, 1]$ such that $f|_{\mathrm{Sp}(D_\varphi)} = 0$ but $f(x) = 1$. Thus $f(D_\varphi) = 0 = f(D_n)$ so by the established convergence property $f(D_{\tilde{\varphi}}) = 0$ showing that x cannot be an eigenvalue for $D_{\tilde{\varphi}}$. On the other hand, let us fix an eigenvalue $\lambda \in \mathrm{Sp}(D_\varphi)\backslash\{0\}$ and choose a continuous function f on $[0, 1]$ such that $f(x) = 0$ for all $x \in \mathrm{Sp}(D_\varphi)\backslash\{\lambda\}$ and $f(x) = 1$ if and only if $x = \lambda$. Then $f(D_n)$ is exactly the spectral projection associated to the eigenvalue λ of D_n; in particular $\|f(D_n)\| = 1$ and $f(D_n)^2 = f(D_n)^* = f(D_n)$. This shows that $f(D_{\tilde{\varphi}})$ — which is the norm-limit of $f(D_n)$ — is also a nonzero projection.

Now $0 \in \mathrm{Sp}(D_\varphi) \cap \mathrm{Sp}(D_{\tilde{\varphi}})$ since we are dealing with density operators given on an infinite-dimensional space. Moreover, 0 is not an eigenvalue for D_φ since φ was assumed to be faithful.

Let us sum up what we have obtained so far. We have shown that $\mathrm{Sp}(D_\varphi) = \mathrm{Sp}(D_n) = \mathrm{Sp}(D_{\tilde{\varphi}})$ and that for each eigenvalue λ of D_φ, the spectral projections $E_{n,\lambda}$ of D_n corresponding to the eigenvalue λ converge in norm to the spectral projection $E_{\tilde{\varphi},\lambda}$ of $D_{\tilde{\varphi}}$ corresponding to the same eigenvalue λ.

Let Φ be an eigenvector of D_φ with eigenvalue λ. Then $\Phi_n := U_n\Phi$ is an eigenvector of $D_n = U_n D_\varphi U_n^*$ with the same eigenvalue. To put it in another way, $(E_{n,\lambda} - \mathbb{1})\Phi_n = 0$ implying that $\|(E_{\tilde{\varphi},\lambda} - \mathbb{1})\Phi_n\| \to 0$ as $n \to \infty$ and so $n \mapsto \Phi_n$ (or a subsequence of it) converges if and only if $n \mapsto E_{\tilde{\varphi},\lambda}\Phi_n$ (or its subsequence in question) does so. But $n \mapsto E_{\tilde{\varphi},\lambda}\Phi_n$ "runs" in the unit ball of the finite dimensional space $\mathrm{Im}(E_{\tilde{\varphi},\lambda})$, so it admits a convergent subsequence.

Since D_φ is a density operator, there exists a complete orthonormal system consisting of eigenvectors of D_φ, only. However, since $\mathcal{H}$ is separable, this system is countable. Thus by what was established, we may conclude the existence of a subsequence s such that $n \mapsto U_{s(n)}$ strongly converges on each vector of this system, and hence — as we are dealing with a sequence of unitary operators — on every vector of $\mathcal{H}$.

We are almost finished: we have proved the existence of a convergent subsequence. However, the limit of a strongly convergent sequence of unitary operators may not be again a unitary operator (in general, it is only an isometry). To show the existence of a unitary limit, we have to check the strong convergence of the adjoints. Then to conclude our proof, all we have to note is that if $\tilde{\varphi}$ is faithful, then we may repeat our argument with the unitary sequence $n \mapsto U^*_{s(n)}$ and with the role of φ and $\tilde{\varphi}$ exchanged. □

Recall that in this section we are dealing with a standard split inclusion $(\Omega, \mathcal{N} \subset \mathcal{M})$ in which at least one of the algebras is a factor, and a certain sequence of unitary operators $n \mapsto W_n$. In our case the adjoint action of W_n does not necessarily preserve the canonical type I factor $\mathcal{R}_\Omega$. Rather, we have that $W_n \mathcal{R}_\Omega W_n^* = \mathcal{R}_{\Omega_n}$, where $\mathcal{R}_{\Omega_n}$ is the canonical type I factor given by the vector Ω_n. (Note that $\Omega_n = W_n\Omega$ is automatically a standard vector for the inclusion $\mathcal{N} \subset \mathcal{M}$.) All we can hope now is that since $\Omega_n \to \Psi$, the type I factors $\mathcal{R}_{\Omega_n}$ will get "closer and closer" to the type I factor $\mathcal{R}_\Psi$. At this point, our previous lemma resolves only the rather particular case when the adjoint action of W_n actually *does* preserve $\mathcal{R}_\Omega$. However, this in turn will serve to prove the general case.

Proposition 3.2. *Suppose that for all $n \in \mathbb{N}$, the adjoint action of W_n also preserves the canonical type I factor $\mathcal{R}_\Omega$. Then there exists a subsequence s such that $n \mapsto W_{s(n)}$ strongly converges to a unitary operator.*

Proof. First let us note that if $n \mapsto A_n$ is a sequence of uniformly bounded operators converging strongly to a bounded operator A then also $A_n \otimes \mathbb{1} \to A \otimes \mathbb{1}$ strongly, as $n \to \infty$. Indeed, convergence is clear on vectors of tensorial form, and hence on every vector as our sequence was assumed to be uniformly bounded. In particular, if we identify $\mathcal{R}_\Omega$ with $\mathcal{B}(\mathcal{K})$ (via an isomorphism) where $\mathcal{K}$ is some Hilbert space, then a sequence of unitary operators $n \mapsto U_n \in \mathcal{R}_\Omega$ is strongly converging to the unitary operator of $\mathcal{R}_\Omega$ if and only if we have convergence in the topology given by the strong operator topology of $\mathcal{B}(\mathcal{K})$.

So let now ω and ψ be the normal states on $\mathcal{R}_\Omega$ given by the vectors Ω and Ψ, respectively. These states are faithful since the vectors in question are separating for $\mathcal{M}$ which contains $\mathcal{R}_\Omega$.

Since $\mathcal{R}_\Omega$ is a type I factor, the adjoint action of W_n in $\mathcal{R}_\Omega$ can be implemented by a unitary $U_n \in \mathcal{R}_\Omega$. We have that $\omega \circ \mathrm{Ad}(U_n) \to \psi$ in norm, since $\|W_n\Omega - \Psi\| \to 0$. Thus our previous lemma can be applied, and by what was noted in the beginning of our proof, it shows that there exists a unitary $U \in \mathcal{R}_\Omega$ and a subsequence s such that $U_{s(n)} \to U$ strongly (on our original Hilbert space, not only on $\mathcal{K}$) as $n \to \infty$. Then for an $A \in \mathcal{R}_\Omega$ we have that as $n \to \infty$,

$$W_{s(n)}A\Omega = (U_{s(n)}AU^*_{s(n)})W_{s(n)}\Omega \to UAU^*\Psi, \tag{3.6}$$

since $\|W_{s(n)}\Omega - \Psi\| \to 0$ and since the strong limit of a product of strongly convergent, uniformly bounded sequences is simply the product of the limits. Thus $n \mapsto W_{s(n)}$ is strongly convergent on $\overline{\mathcal{R}_\Omega\Omega}$, and $n \mapsto W^*_{s(n)}$ is strongly convergent on $\overline{\mathcal{R}_\Omega\Psi}$. Now both Ω and Ψ are cyclic for $\mathcal{N}$ and hence for $\mathcal{R}_\Omega$, too; so actually we have shown that $n \mapsto W_{s(n)}$ converges strongly to a unitary operator. □

In our previous proposition we assumed $\mathcal{R}_n := \mathcal{R}_{\Omega_n}$ to coincide with $\mathcal{R}_\Omega$. It is rather clear that this assumption is too strong; it will not hold in general. So now we shall see how we can "correct" W_n by another unitary in order to have this property.

Lemma 3.3. *Let Ψ, Ψ_n $(n \in \mathbb{N})$ be standard vectors for the split inclusion $\mathcal{N} \subset \mathcal{M}$ in which at least one of the algebras is a factor. If $\|\Psi_n - \Psi\| \to 0$ as $n \to \infty$, then there exists a sequence of unitaries $n \mapsto U_n \in \mathcal{N}' \cap \mathcal{M}$ strongly converging to the operator $\mathbb{1}$ such that $U_n\mathcal{R}_{\Psi_n}U_n^* = \mathcal{R}_\Psi$ for all $n \in \mathbb{N}$.*

Proof. We may assume that the smaller algebra $\mathcal{N}$ is a factor. (If only $\mathcal{M}$ is a factor, then instead of the original inclusion we may consider $(\Omega, \mathcal{M}' \subset \mathcal{N}')$ in which it is again the smaller algebra which is a factor.) Let us denote by ψ, $\psi_n (n \in \mathbb{N})$ the faithful normal states on $\mathcal{N}' \cap \mathcal{M}$ given by the vectors Ψ, $\Psi_n (n \in \mathbb{N})$, respectively. The state ψ_n has a unique vector-representation $\tilde{\Psi}_n$ in the natural cone of $(\Psi, \mathcal{N}' \cap \mathcal{M})$. Note that by construction, the modular conjugation $J_{\tilde{\Psi}_n}$ associated to $(\tilde{\Psi}_n, \mathcal{N}' \cap \mathcal{M})$ coincides with J_Ψ. As both cyclic and separating vectors Ψ_n and $\tilde{\Psi}_n$ implement the same state on $\mathcal{N}' \cap \mathcal{M}$, there exists a unitary $U_n' \in (\mathcal{N}' \cap \mathcal{M})'$ such that $U_n'\Psi_n = \tilde{\Psi}_n$, or equivalently, that $U_n'^*\tilde{\Psi}_n = \Psi_n$. As the adjoint action of U_n' preserves both $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N}' \cap \mathcal{M}$, we have that

$$U_n'^* J_\Psi U_n' = U_n'^* J_{\tilde{\Psi}_n} U_n' = J_{U_n'^*\tilde{\Psi}_n} = J_{\Psi_n}. \tag{3.7}$$

Moreover, as is rather evident, $J_\Psi U_n'^* \mathcal{N} U_n' J_\Psi \subset \mathcal{N}' \cap \mathcal{M}$, we also have that U_n' is in the commutant of $J_\Psi U_n'^* \mathcal{N} U_n' J_\Psi$ and

$$\begin{aligned} J_{\Psi_n}\mathcal{N} J_{\Psi_n} &= U_n'^* J_\Psi U_n' \mathcal{N} U_n'^* J_\Psi U_n' = J_\Psi U_n' \mathcal{N} U_n'^* J_\Psi \\ &= (J_\Psi U_n' J_\Psi) J_\Psi \mathcal{N} J_\Psi (J_\Psi U_n' J_\Psi)^* = U_n J_{\Psi_n} \mathcal{N} J_{\Psi_n} U_n^*, \end{aligned} \tag{3.8}$$

where $U_n = J_\Psi U_n' J_\Psi$ is a unitary in the relative commutant $\mathcal{N}' \cap \mathcal{M}$. Now the sequence of states $n \mapsto \psi_n$ clearly converges to ψ in norm (since $\Psi_n \to \Psi$ as $n \to \infty$). It follows that the distance between the vectors $\tilde{\Psi}_n$ and Ψ, both elements of the natural cone of $(\Psi, \mathcal{N}' \cap \mathcal{M})$, also goes to zero as $n \to \infty$. Now $U_n \Psi = J_\Psi U_n' J_\Psi \Psi = J_\Psi U_n' \Psi$ and $\| J_\Psi U_n' \Psi - J_\Psi U_n' \Psi_n \| = \| \Psi - \Psi_n \| \to 0$, so $n \mapsto U_n \Psi$ is convergent as in fact

$$\lim_n (U_n \Psi) = \lim_n (J_\Psi U_n' \Psi_n) = \lim_n (J_\Psi \tilde{\Psi}_n) = J_\Psi \Psi = \Psi. \tag{3.9}$$

Since $U_n \in \mathcal{N}' \cap \mathcal{M} \subset \mathcal{M}$, the above shows that $n \mapsto U_n$ strongly converges to the identity on the closure of $\mathcal{M}'\Psi$ and hence everywhere (as Ψ is cyclic and separating for $\mathcal{M}$ and so for $\mathcal{M}'$, too). □

Corollary 3.4. *Under the assumptions explained in the beginning of this section, it follows that there exists a subsequence s such that* $n \mapsto W_{s(n)}$ *strongly converges to a unitary operator.*

4. Equivalence of Models

Fix a space-like hyperplane H, and let further τ be a nonzero translation such that $\tau(H) = H$. Fix a plane N in H which is orthogonal to the direction of the translation τ. Then $H \backslash N$ is the disjoint union of two open "half-spaces" H^+ and H^-. Here the "+" and "−" signs are given in such a way that $\tau(H^+) \subset H^+$ while $\tau(H^-) \supset H^-$.

Note that $\mathcal{W}^\pm := (H^\pm)^\diamond$ are wedge-regions such that the causal complement of any of them is exactly (the closure of) the other. Moreover, we have $\tau(\mathcal{W}^+) \subset \mathcal{W}^+$ and $\tau(\mathcal{W}^-) \supset \mathcal{W}^-$, and that $\cup_{n\in\mathbb{N}} \tau^n(\mathcal{W}^-)$ is the full spacetime. Hence if $(\mathcal{A}, U_\mathcal{A})$ is an algebraic QFT given on the Hilbert space $\mathcal{H}$ satisfying axioms (1, 2, 3, 4, 5) discussed in the Introduction, then $n \mapsto \mathcal{A}(\tau^n(\mathcal{W}^-))$ is an increasing sequence of von Neumann algebras such that its union is dense (w.r.t. the strong op. topology) in $\mathcal{B}(\mathcal{H})$. Then for the decreasing sequence $n \mapsto \mathcal{A}(\tau^n(\mathcal{W}^+))$, by *locality* we have that $\cap_{n\in\mathbb{N}} \mathcal{A}(\tau^n(\mathcal{W}^+)) = \mathbb{C}\mathbb{1}$.

In our main theorem — apart from many other things — we shall also use a rather well-known fact concerning a decreasing sequence of von Neumann algebras and distances of restrictions of states. However, for reasons of self-containment we shall outline the proof of this fact (which is anyway short).

Lemma 4.1. *Let* $\mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \ldots$ *be a decreasing sequence of von Neumann algebras on a Hilbert space* $\mathcal{H}$ *with* $\cap_{n\in\mathbb{N}} \mathcal{M}_n = \mathbb{C}\mathbb{1}$ *(or equivalently: with*

$\{\cup_{n\in\mathbb{N}}\mathcal{M}'_n\}'' = \mathcal{B}(\mathcal{H})$). *Let further* $\psi, \tilde{\psi}$ *be two normal states on* $\mathcal{M}_1$. *Then for the restriction of states* $\psi_n := \psi|_{\mathcal{M}_n}$, $\tilde{\psi}_n := \tilde{\psi}|_{\mathcal{M}_n}$ *we have*

$$\|\psi_n - \tilde{\psi}_n\| \to 0$$

as $n \to \infty$

Proof. Clearly, the validity of the statement does not depend on the "underlying" Hilbert space. So we may assume that both states on $\mathcal{M}_1$ can be represented by vectors in $\mathcal{H}$ (e.g. we may work on the direct sum of the two GNS representations given by the two states); say Ψ is a representative vector for ψ and $\tilde{\Psi}$ is a representative vector for $\tilde{\psi}$.

Any two unit-vectors can be connected by a unitary operator, so let V be a unitary operator such that $V\Psi = \tilde{\Psi}$. We may write V in the form $V = e^{iA}$, where A is self-adjoint operator with spectrum $\mathrm{Sp}(A) \subset [-\pi, \pi]$, and hence $\|A\| \leq \pi$. Now $\cup_{n\in\mathbb{N}}\mathcal{M}'_n$ is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology. Thus by an application of Kaplansky's density theorem there exists a sequence of self-adjoints $n \mapsto A'_n \in \mathcal{M}'_n$ strongly converging to A such that $\|A_n\| \leq \pi$ for all $n \in \mathbb{N}$. Then $n \mapsto U'_n := e^{iA'_n} \in \mathcal{M}'_n$ is a sequence of unitary operators strongly converging to V; in particular $U'_n\Psi \to \tilde{\Psi}$ as $n \to \infty$.

For the von Neumann algebra $\mathcal{M}_n$ the vectors Ψ and $U'_n\Psi$ represent the same state. Hence as $n \to \infty$,

$$\|\psi_n - \tilde{\psi}_n\| \leq 2\|U'_n\Psi - \tilde{\Psi}\| \to 0, \tag{4.1}$$

which is exactly what we have claimed. □

For what follows, recall our definition of a *double-cone* $K^\diamond$ with base K. Recall also that in the beginning of this section we have fixed a spacelike hyperplane H and some further objects related to H.

Theorem 4.2. *Let* $(\mathcal{A}, U)$ *and* $(\tilde{\mathcal{A}}, \tilde{U})$ *be two algebraic QFT models on the* $d+1$ *dimensional Minkowskian spacetime satisfying the basic requirements* (1, 2, 3, 4, 5, 6) *as well as the Bisognano-Wichmann* (7) *and split* (8) *properties. If there exists a unitary* V *such that*

$$V\mathcal{A}(K^\diamond)V^* = \tilde{\mathcal{A}}(K^\diamond)$$

for every double-cone $K^\diamond$ *with base* $K \subset H^+$, *then the two models are equivalent. That is, there exists a unitary operator* W *such that* $W\mathcal{A}(\mathcal{O})W^* = \tilde{\mathcal{A}}(\mathcal{O})$ *for all double-cones* $\mathcal{O}$ *and* $WU(g)W^* = \tilde{U}(g)$ *for all elements* g *of the connected part of the Poincaré group.*

Proof. Let Ω and $\tilde{\Omega}$ be the (up to phase unique, normalized) vacuum vectors for U and $\tilde{U}$, respectively. We may assume that the two models are given on the same Hilbert space $\mathcal{H}$ and that V is the identity operator so that actually $\mathcal{A}(K^\diamond) = \tilde{\mathcal{A}}(K^\diamond)$ for every double-cone $K^\diamond$ with base $K \subset H^+$.

Remember we defined $\mathcal{A}(\mathcal{W}^+)$ to be the von Neumann algebra generated by *all* local algebras $\mathcal{A}(\mathcal{O})$ with $\mathcal{O} \subset \mathcal{W}^+$. That is, theoretically we should take account of *all* double-cones included in $\mathcal{W}^+$ and not only those with bases on H^+. However, it is easy to see that one can take an increasing sequence of double-cones $n \mapsto K_n^\diamond$ with bases on H^+ such that not only $\cup_{n\in\mathbb{N}}K_n^\diamond = \mathcal{W}^+$, but actually every bounded region $\mathcal{O} \subset \mathcal{W}^+$

is included in $K_n^\diamond$ for *some* $n \in \mathbb{N}$. Then by isotony $\mathcal{A}(\mathcal{W}^+) = \{\cup_{n\in\mathbb{N}} \mathcal{A}(K_n^\diamond)\}''$. So the assumed equality implies that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ coincide on $\mathcal{W}^+$, too.

We may assume that $\tilde{\Omega}$ is in the natural cone of $(\Omega, \mathcal{A}(\mathcal{W}^+))$. Indeed, suppose originally it was not so, and consider the state on $\mathcal{A}(\mathcal{W}^+)$ given by the vector $\tilde{\Omega}$. This state has a unique representative vector $\tilde{\Omega}^\natural$ in the cone in question. Since $\tilde{\Omega}$ is cyclic and separating for $\mathcal{A}(\mathcal{W}^+)$, the corresponding state is faithful, $\tilde{\Omega}^\natural$ is also cyclic and separating for $\mathcal{A}(\mathcal{W}^+)$ and there exists unitary $V' \in \mathcal{A}(\mathcal{W}^+)'$ such that $V'\Omega = \tilde{\Omega}^\natural$. Then we may replace $(\tilde{\mathcal{A}}, \tilde{U})$ with vacuum vector $\tilde{\Omega}$ by $(V\tilde{\mathcal{A}}V'^*, V\tilde{U}V'^*)$ with vacuum vector $\tilde{\Omega}^\natural$. For this latter choice we have the desired property that its vacuum vector is in the required cone, and since V' commutes with all algebras $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{W}^+)$ $(\mathcal{O} \subset \mathcal{W}^+)$, we still have that $\mathcal{A}(K^\diamond) = \tilde{\mathcal{A}}(K^\diamond) = V'\tilde{\mathcal{A}}(K^\diamond)V'^*$ for every double-cone $K^\diamond$ with base $K \subset H^+$.

Let γ_n be the adjoint action of the product $U(\tau^n)^*\tilde{U}(\tau^n)$. By what was assumed, we have that for every $n \in \mathbb{N}$,

$$\gamma_n(\mathcal{A}(K^\diamond)) = \mathcal{A}(K^\diamond) \text{ for every double cone } K^\diamond \text{ with base } K \subset H^+. \tag{4.2}$$

Now let ω and $\tilde{\omega}$ be the faithful normal states on $\mathcal{A}(\mathcal{W}^+)$ given by the vectors Ω and $\tilde{\Omega}$, respectively. Then $\omega \circ \gamma_n$ is nothing else than the state given by the vector $\tilde{U}(\tau^n)^*U(\tau^n)\Omega = \tilde{U}(\tau^n)^*\Omega$; so $\omega \circ \gamma_n = \omega \circ \mathrm{Ad}(\tilde{U}(\tau^n))$. On the other hand, $\tilde{\omega} \circ \mathrm{Ad}(\tilde{U}(\tau^n)) = \tilde{\omega}$ since $\tilde{\Omega}$ is an invariant vector for $\tilde{U}(\tau)$. Putting it together, and applying our previous lemma we have that

$$\|\omega \circ \gamma_n - \tilde{\omega}\| = \|(\omega - \tilde{\omega}) \circ \mathrm{Ad}(\tilde{U}(\tau^n))\| = \|(\omega|_{\mathcal{A}(\tau^n(\mathcal{W}^+))} - \tilde{\omega}|_{\mathcal{A}(\tau^n(\mathcal{W}^+))}\| \to 0 \tag{4.3}$$

as $n \to \infty$, since on $\tau^n(\mathcal{W}^+) \subset \mathcal{W}^+$ (by a similar argument to that used for $\mathcal{W}^+$) the nets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ coincide and hence $\tilde{U}(\tau^n)\mathcal{A}(\mathcal{W}^+)\tilde{U}(\tau^n)^* = \mathcal{A}(\tau^n(\mathcal{W}^+))$ for every $n \in \mathbb{N}$.

Since Ω is cyclic and separating for $\mathcal{A}(\mathcal{W}^+)$, we can find a unitary W_n implementing γ_n on $\mathcal{A}(\mathcal{W}^+)$ such that $W_n\Omega$ is in the natural cone of $(\Omega, \mathcal{A}(\mathcal{W}^+))$. Then $W_n\Omega$ and $\tilde{\Omega}$ are exactly the vector representatives in the specified natural cone of the states $\omega \circ \gamma_n$ and $\tilde{\omega}$, respectively. Thus by the established norm convergence of states we have that $\|W_n\Omega - \tilde{\Omega}\| \to 0$ as $n \to \infty$.

Let $K^\diamond$ be a nonempty double-cone with base $K \subset \tau(H^+) \subset H^+$. Then the *split property* together with the *isotony* and *Reeh-Schlieder property* imply that $\mathcal{A}(K^\diamond) \subset \mathcal{A}(\mathcal{W}^+)$ is a split inclusion for which the vacuum vectors $\Omega, \tilde{\Omega}$ are standard vectors. Moreover, as was discussed in Sect. 2, $\mathcal{A}(\mathcal{W}^+)$ is a factor. Hence by Corollary 3.4 there exists a unitary operator W and a subsequence s such that $n \mapsto W_{s(n)}$ converges strongly to W.

It is evident that for the limit W we still have that $W\mathcal{A}(K^\diamond)W^* = \mathcal{A}(K^\diamond) = \tilde{\mathcal{A}}(K^\diamond)$ for every double-cone $K^\diamond$ with base $K \subset H^+$ (and so also for regions like $\mathcal{W}^+$ and $\tau^n(\mathcal{W}^+)$), but now we also have that $W\Omega = \tilde{\Omega}$. By the *Bisognano-Wichmann property* it immediately follows that WUW^* and $\tilde{U}$ coincide on both the boosts associated to $\mathcal{W}^+$ and to $\tau(\mathcal{W}^+)$.

Now a quick check shows that the subgroup generated by such boosts contains τ so actually we also have that $WU(\tau)W^* = \tilde{U}(\tau)$. Since every double-cone with base on H can be shifted into H^+ by a repeated use of τ, this further implies that $W\mathcal{A}(K^\diamond)W^* = \tilde{\mathcal{A}}(K^\diamond)$ for every double-cone $K^\diamond$ with base $K \subset H$, and hence also for infinite regions like wedges whose "edges" are included in H. Then in turn — again by the *Bisognano-Wichmann property* — we have that WUW^* and $\tilde{U}$ coincides on every boost that is associated to *some* wedge with edge in H. But elementary geometric arguments show that such boosts generate the entire connected Poincaré group so at this point we have that $WUW^*(g) = \tilde{U}(g)$ for every element g. Moreover, since every double-cone can be moved by a suitable Poincaré transformation so that its base will be on H, we now have that $W\mathcal{A}(K^\diamond)W^* = \tilde{\mathcal{A}}(K^\diamond)$ for *every* double-cone. Thus W establishes an equivalence between the two models, which is exactly what we wanted to prove. □

5. The Conformal Case

The conformal chiral QFT, though originally defined on a lightline, can be naturally extended to the compactified lightline which is customarily identified with the circle $S^1 \equiv \{z \in \mathbb{C} | \, |z| = 1\}$. On the circle the theory becomes *Möbius covariant*; that is, it will carry a symmetry action of the group of diffeomorphisms of S^1 of the form $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$, which is called the **Möbius group** and is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. The connection between the "circle picture" and the "line picture" (here "line"$\equiv \mathbb{R}$) is made by puncturing the circle at $-1 \in S^1$ and using a Cayley-transformation:

$$x = i\frac{1-z}{1+z} \in \mathbb{R} \iff z = \frac{i-x}{i+x} \in S^1 \backslash \{-1\}. \tag{5.1}$$

Via the line picture one can view *translations* and *dilations* as diffeomorphisms of S^1 and in this sense they are elements of the Möbius group.

A Möbius covariant net of von Neumann algebras on S^1 is a map $\mathcal{A}$ which assigns to every nonempty, nondense open "arc" (or simply *interval*) $I \subset S^1$ a von Neumann algebra $\mathcal{A}(I)$ acting on a fixed Hilbert space $\mathcal{H}$, together with a given strongly continuous representation U of the Möbius group satisfying certain properties. Here we shall not dwell much either on the defining properties of a Möbius covariant net of von Neumann algebras on S^1, or on their known consequences. We only assert that the defining properties are adopted versions of (1, 2, 3, 4, 5) whereas (the adopted versions of) property (6, 7) — that is, the *Reeh-Schlieder* and *Bisognano-Wichmann properties* — are consequences. One also has *irreducibility, factoriality of local algebras* and moreover *additivity* even for an infinite set of intervals: $\vee_{I_\alpha}\mathcal{A}(I_\alpha) \supset \mathcal{A}(I)$ whenever $\cup_\alpha I_\alpha \supset I$ for *any* collection $\{I_\alpha\}$. For details we refer to [4,8,9,11,12]. Note however that one cannot derive *split property* (i.e. that $\mathcal{A}(K) \subset \mathcal{A}(I)$ is a split inclusion whenever $\overline{K} \subset I$), since by taking infinite tensorial products it is easy to construct non-split Möbius covariant nets. Nevertheless, it is known to hold in the majority of "interesting" models.

Theorem 5.1. *Let $(\mathcal{A}, U)$ and $(\tilde{\mathcal{A}}, \tilde{U})$ be Möbius covariant nets of von Neumann algebras on S^1 with at least one of them being split. Then any of the following* 4 *conditions:*

- *∃ a unitary W s.t. $W\mathcal{A}(I)W^* = \tilde{\mathcal{A}}(I)$ and $WU(g)W^* = \tilde{U}(g)$ for all I, g,*
- *∃ a unitary V s.t. $V\mathcal{A}(I)V^* = \tilde{\mathcal{A}}(I)$ for all I,*
- *∃ an (open, nonempty) I and a unitary V s.t. $V\mathcal{A}(K)V^* = \tilde{\mathcal{A}}(K)$ for all $K \subset I$,*
- *∃ a unitary V s.t. with $\mathbb{R}$-picture notations $V\mathcal{A}(j,k)V^* = \tilde{\mathcal{A}}(j,k)$ for all $j, k \in \mathbb{N}$,*

implies the remaining three.

Proof. It is clear that any of the conditions implies that if one of the nets is split then so is the other and that each condition implies the next one. All we have to show is that the last one implies the first one, which can be done by simply copying the argument of the proof of the main theorem of the previous section.

Note that by (the infinite version of) *additivity* the last condition implies that for the unitary V appearing in the condition we also have $V\mathcal{A}(k,\infty)V^* = \tilde{\mathcal{A}}(k,\infty)$ for every $k \in \mathbb{N}$. So we may replace the wedge $\mathcal{W}^+$ in our former proof by the half-line $(0,\infty)$. We have to be careful to use a translation τ by an integer length; say we let τ be the unit translation $x \mapsto x+1$. For a split inclusion we can choose $\mathcal{A}(1,2) \subset \mathcal{A}(0,\infty)$. Then the argument of the mentioned proof shows that there exists a unitary W such that $W\mathcal{A}(j,k)W^* = \tilde{\mathcal{A}}(j,k)$ for all $j,k \in \mathbb{N}$ and moreover $W\Omega = \tilde{\Omega}$, where Ω and $\tilde{\Omega}$ are vacuum vectors for U and $\tilde{U}$, respectively.

From here on the proof is actually even simpler than in the "normal" case. Indeed, whereas there the respective modular unitaries did *not* generate the Poincaré group and so we needed to consider further regions, here we do not need any further argument. It is easy to see that the "dilations" associated to intervals of the form $(j,k)(j,k \in \mathbb{N})$ generate the entire Möbius group. Moreover, the Möbius group acts transitively on the set of (open, nondense, nonempty) intervals. So we immediately have that $WU(g)W^* = \tilde{U}(g)$ for all g and $W\mathcal{A}(I)W^* = \tilde{\mathcal{A}}(I)$ for all I. □

In [5] it was shown[3] that the Möbius symmetry of a Möbius covariant net $(\mathcal{A}, U)$ admits at most one extension to the group $\mathrm{Diff}^+(S^1)$ which is "compatible" with the net $\mathcal{A}$. For details and precise notations regarding the diffeomorphism covariance we refer to [5]. Here we only note and state that the mentioned uniqueness result implies that our algebraic Haag's theorem can be adjusted for the diffeomorphism covariant case, too.

Proposition 5.2. *One may replace "Möbius covariance" by "diffeomorphism covariance" in Theorem 5.1.*

Let us talk now about implications of our result regarding half-sided modular inclusions. A net $\mathcal{A}$ on the circle is **strongly additive** iff $\mathcal{A}(I_1)\vee\mathcal{A}(I_2) = \mathcal{A}(I)$ whenever I, I_1 and I_2 are intervals with the last two being obtained from I by the removal of a point. As was explained in the Introduction, there is a one-to-one correspondence between *strongly additive* Möbius covariant nets and standard half-sided modular inclusions of factors.

For any inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$ with a common cyclic vector Ω, consider the tunnel introduced by R. Longo:

$$\mathcal{N}_0 \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \mathcal{N}_3 \supset \ldots, \tag{5.2}$$

where $\mathcal{N}_0 = \mathcal{M}$, $\mathcal{N}_1 = \mathcal{N}$ and $\mathcal{N}_{k+1} = J_k\mathcal{N}'_{k-1}J_k (k = 1,2,\ldots)$ and J_k is the modular conjugation associated to $(\mathcal{N}_k, \Omega)$.

It is easy to see that the tunnel is well-defined (i.e. that Ω remains cyclic and separating at each step of the induction and hence the modular conjugation can be indeed considered). But how does it depend on the choice of the common cyclic and separating vector Ω? In some sense not much. The following statement is included for reasons of self-containment; it is well-known to experts of the field.

[3] Actually the statement in [5] is slightly less general. There a certain regularity condition was also used which was however later removed by the author of this article; see [17, Sect. 6.1] for details.

Lemma 5.3. *Let both Ω and $\tilde{\Omega}$ be common cyclic vectors for the inclusion of von Neumann algebras $\mathcal{N} \subset \mathcal{M}$, and denote by $\mathcal{N}_0 \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \ldots$ and $\tilde{\mathcal{N}}_0 \supset \tilde{\mathcal{N}}_1 \supset \tilde{\mathcal{N}}_2 \supset \ldots$ the respective tunnels defined after Eq. (5.2). Then for each $n \in \mathbb{N}$ there exists a unitary operator V_n such that*

$$V_n \mathcal{N}_k V_n^* = \tilde{\mathcal{N}}_k \quad \text{for all } k \in \{0, 1, \ldots, n\}.$$

That is, up to any finite level, the two tunnels are unitarily equivalent.

Proof. We set $V_1 = \mathbb{1}$ and define V_n inductively. Now for $n = 1$ the condition is satisfied since by assumption $\mathcal{N}_0 = \tilde{\mathcal{N}}_0 = \mathcal{M}$ and $\mathcal{N}_1 = \tilde{\mathcal{N}}_1 = \mathcal{N}$. So assume V_k is already defined in a way satisfying the requirement made in the statement. Then $V_k\Omega$ is cyclic and separating for $(V_k \mathcal{N}_k V_k^*) = \tilde{\mathcal{N}}_k$ so there is a unitary $U_k \in \tilde{\mathcal{N}}_k$ such that $U_k V_k \Omega$ is in the natural cone of $(\tilde{\Omega}, \tilde{\mathcal{N}}_k)$. Set $V_{k+1} := U_k V_k$; it is then evident that $V_{k+1}\mathcal{N}_j V_{k+1}^* = \tilde{\mathcal{N}}_j$ for all $j = 0, 1, \ldots, k$. Moreover, as $V_{k+1}\Omega = U_k V_k \Omega$ is in the natural cone of $(\tilde{\Omega}, \tilde{\mathcal{N}}_k)$ and $V_{k+1}\mathcal{N}_k V_{k+1}^* = \tilde{\mathcal{N}}_k$, we have that the adjoint action of V_k takes the modular conjugation J_k associated to $(\Omega, \mathcal{N}_k)$ into the modular conjugation $\tilde{J}_k$ associated to $(\tilde{\Omega}, \tilde{\mathcal{N}}_k)$. Thus

$$V_{k+1}\mathcal{N}_{k+1} V_{k+1}^* = V_{k+1} J_k \mathcal{N}_{k-1}' J_k V_{k+1}^* = \tilde{J}_k (V_{k+1}\mathcal{N}_{k-1} V_{k+1}^*)' \tilde{J}_k = \tilde{J}_k \tilde{\mathcal{N}}_{k-1}' \tilde{J}_k = \tilde{\mathcal{N}}_{k+1} \tag{5.3}$$

and hence the statement is proved by induction. □

Let $(\mathcal{A}, U)$ be a Möbius covariant net with vacuum vector Ω and denote the modular objects associated to $(\Omega, \mathcal{A}(k, \infty))$ by J_k and Δ_k. Using the *Bisognano-Wichmann property* and the main theorem [2, Thm. 2.1] of half-sided modular inclusions, the product $J_k J_{k-1}$ can be expressed with the modular unitaries which in turn can be expressed by U resulting in $J_k J_{k-1} = U(\tau)^2$, where τ is the unit-translation defined in the $\mathbb{R}$-picture by the map $x \mapsto x + 1$. Hence

$$\begin{aligned} J_k \mathcal{A}(k-1, \infty)' J_k &= J_k J_{k-1} \mathcal{A}(k-1, \infty) J_{k-1} J_k \\ &= U(\tau)^2 \mathcal{A}(k-1, \infty) U(\tau)^{-2} = \mathcal{A}(k+1, \infty), \end{aligned} \tag{5.4}$$

and so the tunnel (5.2) associated to $(\Omega, \mathcal{A}(0, \infty) \subset \mathcal{A}(1, \infty))$ is nothing else than the sequence of inclusions

$$\mathcal{A}(0, \infty) \subset \mathcal{A}(1, \infty) \subset \mathcal{A}(2, \infty) \subset \ldots. \tag{5.5}$$

Note that in case we have *strong additivity*, by taking relative commutants, this sequence determines all algebras of the form $\mathcal{A}(j, k)$ with $j, k \in \mathbb{N}$. *Vice versa*, if we know $\mathcal{A}(j, k)$ for all $j, k \in \mathbb{N}$, then by (the infinite version of) *additivity* we can compute all algebras of the form $\mathcal{A}(k, \infty)$ with $k \in \mathbb{N}$. So by what was explained we can draw the following conclusion.

Corollary 5.4. *Suppose $(\Omega, \mathcal{N} \subset \mathcal{M})$ and $(\tilde{\Omega}, \tilde{\mathcal{N}} \subset \tilde{\mathcal{M}})$ are two standard half-sided modular inclusions of factors and denote the two corresponding strongly additive Möbius covariant nets by $(\mathcal{A}, U)$ and $(\tilde{\mathcal{A}}, \tilde{U})$, respecively. Then the conditions:*

- $\exists$ *a unitary V s.t. $V\mathcal{M}V^* = \tilde{\mathcal{M}}$, $V\mathcal{N}V^* = \tilde{\mathcal{N}}$,*
- $\forall n \in \mathbb{N} : \exists$ *unitary V_n s.t. $V_n \mathcal{A}(j, k) V_n^* = \tilde{\mathcal{A}}(j, k)$ for all $j, k \in \{0, 1, \ldots, n\}$,*

are equivalent.

The relevance of this statement in light of the conformal version of our algebraic Haag's theorem has been already discussed in the Introduction.

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