The Spectral Decomposition and its Use in Finite Dimensions

Jaclyn R. Sanders
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The formulation of quantum mechanics as generally presented in undergraduate courses in physics relies on a normalized wavefunction. The concept of a wavefunction was developed based upon empirical observation, and is an example of creating mathematics to fit observed phenomena. In the mathematical formulation of quantum mechanics pioneered by John von Neumann, measurable quantities are represented as self-adjoint operators in a Hilbert space. What is remarkable about this formulation is that all information about the observable is contained in the operator, which is a purely mathematical entity. Rather than being fitted to a physical situation, the self-adjoint operator representation is such that the physics follows from the mathematics. The self-adjoint operator contains this information in the form of its *spectral decomposition*. In the finite dimensional case, it is also called its *eigendecomposition*, since in this case the spectrum is nothing else then the set of eigenvalues. As we shall see, in this case this decomposition takes the form

$$A = \sum \lambda P_\lambda,$$

where the summation is over the spectrum of A. In the infinite dimensional case, the general form is

$$A = \int \lambda dP(\lambda)$$

where dP is a *projection valued measure*. Here we shall only consider the finite dimensional case. In this case, considering the spectral decomposition is no more profitable than the usual process of *diagonalization*. In fact, all applications that will be here discussed could also be done by diagonalization. The real advantage of considering spectral decompositions is only evident in infinite dimensions, where one frequently finds self-adjoint operators that do not have any eigenvalues, instead having a "continuous" spectrum. In this case diagonalization loses meaning, while spectral decomposition works just fine. Nevertheless, since the infinite dimensional case requires heavy functional analysis, we will remain in finite dimensions.

As the spectral decomposition applies to the more general normal operator on a Hilbert space, we will use the normal operator for the proofs in this section.

1 The spectral theorem

Let $\mathcal{H}$ be a finite dimensional Hilbert space, and let N be a **normal** operator on $\mathcal{H}$, that is, an operator on $\mathcal{H}$ satisfying

$$NN^* = N^*N.$$

As usual, we shall denote the spectrum of N, or the set of its eigenvalues, by $\mathrm{Sp}(N)$. Recall that $\mathrm{Sp}(N)$ is the set of solutions for $\lambda \in \mathbb{C}$ for the polynomial equation $\det(N - \lambda \mathbb{1}) = 0$ and hence the cardinality of $\mathrm{Sp}(N)$ is less or equal to the dimension of $\mathcal{H}$. Obviously, $Ax = A^*x \;\; \forall x \in \mathcal{H}$ for any self-adjoint operator A. In the more general case of normal operators, this is not true. However, we still have the property

$$\|Nx\| = \|N^*x\|.$$

In order to prove that this is indeed true, we consider the norm-squares,

$$\begin{aligned}\|Nx\|^2 &= \langle Nx, Nx\rangle = \langle x, N^*Nx\rangle \\ &= \langle x, NN^*x\rangle = \langle N^*x, N^*x\rangle = \|N^*x\|^2,\end{aligned}$$

which implies the norms are indeed equal. As a consquence, $\mathrm{Ker}(N) = \mathrm{Ker}(N^*)$. This is because the kernel of N consists of those vectors which are mapped to zero by the operator; this is the set of $x \in \mathcal{H}$ such that $Nx = 0$, which is equivalent to $\|Nx\| = 0$. As we have seen, $\|Nx\| = \|N^*x\|$, so $\|N^*x\| = 0$ if and only if $\|Nx\| = 0$.

Now we shall see that x is an eigenvector for N with eigenvalue $\lambda \in \mathbb{C}$ if and only if it is an eigenvector for N^* with eigenvalue $\overline{\lambda}$. Consider that a nonzero vector $x \in \mathcal{H}$ is an eigenvector for N with eigenvalue λ if and only if $x \in \mathrm{Ker}(N - \lambda \mathtt{1})$. However, a trivial check shows that $N - \lambda \mathtt{1}$ is still a normal operator. Hence $x \in \mathrm{Ker}(N - \lambda \mathtt{1}) \Leftrightarrow x \in \mathrm{Ker}(N - \lambda \mathtt{1})^*$, and the latter is equivalent to saying that x is an eigenvector for N^* with eigenvalue $\overline{\lambda}$, since $(N - \lambda \mathtt{1})^* = N^* - \overline{\lambda}\mathtt{1}$.

Another important property of a normal operator N is that its eigenspaces corresponding to different eigenvalues are orthogonal. Let us denote for an eigenvalue $\lambda \in \mathbb{C}$ the corresponding eigenspace by V_λ; that is, let

$$V_\lambda := \mathrm{Ker}(N - \lambda \mathtt{1}).$$

If $x \in V_\lambda$ and $y \in V_\mu$, then $\langle x, Ny \rangle = \mu \langle x, y \rangle$. On the other hand, using that y is an eigenvector for N^* with eigenvalue $\overline{\lambda}$, we have that

$$\langle x, Ny \rangle = \langle N^* x, y \rangle = \langle \overline{\lambda} x, y \rangle = \lambda \langle x, y \rangle.$$

Hence

$$(\lambda - \mu)\langle x, y \rangle = 0,$$

and so in case $\lambda \neq \mu$ it follows that $\langle x, y \rangle = 0$ showing that V_λ and V_μ are orthogonal.

Now we shall show that the only vector which is orthogonal to all eigenvectors of N is the zero vector. Let $W \subset \mathcal{H}$ be the set of vectors that are orthogonal to all eigenvectors N. It is clearly a linear subspace, but we shall actually show that W is an *invariant* linear subspace for N, or $NW \subset W$. Indeed, if $w \in W$ and x is an eigenvector of N with eigenvalue λ, then $N^* x = \overline{\lambda} x$ and so $\langle Nw, x \rangle = \langle w, N^* x \rangle = \overline{\lambda} \langle w, x \rangle = 0$. As x was an arbitrary eigenvector, we may conclude that $Nw \in W$.

Let us suppose now that $W \neq \{0\}$. Then we may consider the restriction of N onto W, which is a linear operator from W to W. As we are above $\mathbb{C}$ and $W \neq \{0\}$, this restriciton must have an eigenvector. However, it would also be an eigenvector of N which is clearly impossible: all eigenvectors of N are orthogonal to W (so in particular W cannot contain any eigenvector of N).

As a consequence of all these properties, we can now announce and prove the spectral theorem.

Theorem 1.1 (The spectral decomposition). *With P_λ denoting the orthogonal projection onto the eigenspace of N corresponding to the eigenvalue*

$\lambda \in \mathrm{Sp}(N)$, we have that

$$N = \sum_{\lambda \in \mathrm{Sp}(N)} \lambda P_\lambda.$$

Moreover, the **spectral projections** *satisfy the relations*

$$P_\lambda P_\mu = \delta_{\lambda,\mu} P_\lambda, \quad P_\lambda^* = P_\lambda, \quad \sum_{\lambda \in \mathrm{Sp}(N)} P_\lambda = \mathbf{1}.$$

Proof. The first relation about the spectral projections is a trivial consequence of the fact that the eigenspaces of N are pairwise orthogonal, and the second relation is just another way of saying that P_λ is an orthogonal projection. What remains is to prove that

$$N - \sum_{\lambda \in \mathrm{Sp}(N)} \lambda P_\lambda = \mathbf{1} - \sum_{\lambda \in \mathrm{Sp}(N)} P_\lambda = 0,$$

which we shall do by showing that for all $z \in \mathcal{H}$, both $(N - \sum_{\lambda \in \mathrm{Sp}(N)} \lambda P_\lambda)z$ and $(\mathbf{1} - \sum_{\lambda \in \mathrm{Sp}(N)} P_\lambda)z$ are orthogonal to all eigenvectors of N (which, by our previous lemma, shows that they are equal to zero).

So let x be an eigenvector of N with eigenvalue λ. Then $P_\lambda x = x$ and so

$$\langle z, P_\mu x\rangle = \langle z, P_\mu P_\lambda x\rangle = \delta_{\lambda,\mu}\langle z, P_\lambda x\rangle = \delta_{\lambda,\mu}\langle z, x\rangle.$$

Therefore, using also that $N^* x = \overline{\lambda} x$, we have that

$$\begin{aligned} \langle (N - \sum_{\mu \in \mathrm{Sp}(N)} \mu P_\mu) z, x\rangle &= \langle z, N^* x\rangle - \sum_{\mu \in \mathrm{Sp}(N)} \overline{\mu}\langle z, P_\mu x\rangle \\ &= \langle z, x\rangle(\overline{\lambda} - \sum_{\mu \in \mathrm{Sp}(N)} \delta_{\lambda,\mu}\overline{\mu}) = \langle z, x\rangle(\overline{\mu} - \overline{\mu}) = 0 \end{aligned}$$

and by a similar calculation, that $\langle (\mathbf{1} - \sum_{\mu \in \mathrm{Sp}(N)} P_\mu) z, x\rangle = 0$, too. □

We may "invert" the spectral theorem in the following sense.

Lemma 1.2. *Let $S \subset \mathbb{C}$ be a finite set of values, and let P_λ $(\lambda \in S)$ be a system of nonzero operators satisfying*

$$P_\lambda P_\mu = \delta_{\lambda,\mu} P_\lambda, \quad P_\lambda^* = P_\lambda, \quad \sum_{\lambda \in S} P_\lambda = \mathbf{1}.$$

Then the operator $N := \sum_{\lambda \in S} \lambda P_\lambda$ is normal, and its defining sum is exactly its spectral decomposition.

Proof. The normality of N follows easily from the fact that by assumption the operators P_λ ($\lambda \in S$) are self-adjoint and pairwise commuting. Let us determine now the spectrum of N. If x is a nonzero vector for which $Nx = \alpha x$ for some $\alpha \in \mathbb{C}$, then $(N - \alpha\mathbf{1})x = 0$ and so

$$\begin{aligned} 0 &= P_\lambda(N - \alpha\mathbf{1})x = P_\lambda(\sum_{\mu\in S}(\mu - \alpha)P_\mu x \\ &= (\sum_{\mu\in S}(\mu - \alpha)P_\lambda P_\mu x = (\lambda - \alpha)P_\lambda x. \end{aligned}$$

Hence either $\alpha \in S$ or $P_\lambda x = 0$ for all $\lambda \in S$. This latter however is impossible, since

$$x = \mathbf{1}x = \sum_{\lambda\in S} P_\lambda x.$$

Therefore each eigenvalue of N must be an element of S. On the other hand, if $\lambda \in S$, then there exists a vector x such that $P_\lambda x \neq 0$ (since P_λ was assumed to be nonzero), and using the assumed relations it is easy to see, that $P_\lambda x$ is an eigenvector of N with eigenvalue λ. So actually S is exactly the spectrum of N. The proof is finished by similar simple checks that can be easily carried out by the reader. □

2 Taking Functions of a Normal Operator

As we have seen in the last section, a normal operator N can always be decomposed into a sum, $N = \sum \lambda P_\lambda$, where P_λ is the orthogonal projection onto the eigenspace V_λ. For simplicity we shall omit "Sp(N)" under the summation sign; however, the summation — when it is not indicated otherwise — will always be for the spectrum of N. The orthogonal projections P_λ satisfy the relations

$$P_\mu P_\lambda = \delta_{\lambda,\mu} P_\lambda, \quad {P_\lambda}^* = P_\lambda,$$

or, the product of two orthogonal projections is zero unless they are the same, and orthogonal projections are self-adjoint. Knowing these properties, we shall investigate how to calculate a function of N, say, $f(N)$. To begin, consider the case when f is the square function. By a straightforward calculation, we have

$$N^2 = \left(\sum \lambda P_\lambda\right)\left(\sum \mu P_\mu\right) = \sum \lambda_\mu P_\lambda P_\mu = \sum \lambda^2 P_\lambda.$$

Similarly, one can easily find that $N^3 = \sum \lambda^3 P_\lambda$. Continuing in this fashion, and taking also account of the fact that $\mathbf{1} = \sum P_\lambda$, we find that in general, if p is the polynomial given by the formula,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

then

$$p(N) := a_0 \mathbf{1} + a_1 N + a_2 N^2 \ldots + a_n N^n = \sum p(\lambda) P_\lambda.$$

So, for example,

$$3N^2 - N^3 + \mathbf{1} = \sum (3\lambda^2 - \lambda^3 + 1) P_\lambda.$$

Next, we shall consider expressions containing both N and N^*; say, for example, $2N^2 N^* + 5N$. In some sense these are still polynomials. In fact, setting $p(x, y) := 2x^2 y + 5x$, one may say that the mentioned expression is nothing else than the polynomial p of two variables evaluated at (N, N^*). Let us note though, that in general it would be ambiguous to subsitute two arbitrary operators — say X and Y — in p. Should $p(X, Y)$ mean $2X^2 Y + 5X$? Or maybe $2XYX + 5X$? In general two operators need not commute. However, since N is normal, in our case it indeed commutes with its adjoint. Hence, we are completely justified in talking about polynomials of N and N^*.

Since orthogonal projections are self-adjoint, we have that

$$N^* = \left(\sum \lambda P_\lambda\right)^* = \sum \overline{\lambda} {P_\lambda}^* = \sum \overline{\lambda} P_\lambda.$$

Consequently, we have

$$2N^2 N^* + 5N = 2 \sum \lambda^2 \overline{\mu} P_\lambda P_\mu + 5 \sum \lambda P_\lambda = (\sum 2\lambda^2 \overline{\mu} + 5\lambda) P_\lambda,$$

and in general, if p is a polynomial of two variables, then

$$p(N, N^*) = \sum p(\lambda, \overline{\lambda}) P_\lambda.$$

We would also like to consider functions of N that are in no sense polynomials, for example, trigonometric and exponential functions. First we need to discuss if it has any meaning.

Suppose we have a power series

$$f(x) = \sum a_k x^k$$

which is convergent at least on the set of eigenvalues of N (i.e. it is meaningful to write $f(\lambda)$ for each $\lambda \in \mathrm{Sp}(N)$). We would like to define $f(N)$ by the formula

$$f(N) := \lim_{n\to\infty} \sum_{k=0}^{n} a_k N^k.$$

Of course, in order to take limits, we need to fix the topology. Since we are in finite dimensions, this is no problem. We choose a norm, and we consider the topology given by it: in finite dimensions, all norms give the same topology. Here we shall use the operator-norm. So "$f(N)$ exists and equals to the operator A" means that $||A - \sum_{k=0}^{n} a_k N^k|| \to 0$ when $n \to \infty$.

Let now $A = \sum f(\lambda)P_\lambda$. Then by what was so far established about polynomials of N, and by the triangle inequality, as $n \to \infty$, we have

$$\begin{aligned} ||A - \sum_{k=0}^{n} a_k N^k|| &= ||\sum \left(f(\lambda) - \sum_{k=0}^{n} a_k \lambda^k \right) P_\lambda|| \\ &\leq \sum |f(\lambda) - \sum_{k=0}^{n} a_k \lambda^k| ||P_\lambda|| \to 0, \end{aligned}$$

since $\sum_{k=0}^{n} a_k \lambda^k \to f(\lambda)$, as we have assumed that our power series is convergent on the spectrum of N. To sum it up: $f(N)$ exists, and it equals to $\sum f(\lambda)P_\lambda$.

As an example, consider the exponential function. Since it is everywhere convergent, it can be evaluated at our normal operator N regardless of its spectrum, and by what was explained, we have that

$$e^N = \mathbf{1} + N + \frac{N^2}{2!} + \frac{N^3}{3!} + ... = \sum e^\lambda P_\lambda.$$

3 A General "Recipe" for Finding the Spectral Decomposition

In the last section we have done our calculations assuming that we already have the spectral decomposition of the normal operator N at hand. However, for actual examples, we will need to find it. As the spectral projection is composed of eigenvalues and projections, it is natural that to determine the projections one must begin with finding the eigenvalues. To do so, one must solve the characteristic polynomial,

$$p(\lambda) = \det(N - \lambda \mathbb{1}) = 0.$$

Once the eigenvalues are known, we consider the system of equations,

$$\mathbb{1} = \sum P_\lambda$$

$$N = \sum \lambda P_\lambda$$

$$N^2 = \sum \lambda^2 P_\lambda$$

$$...$$

$$N^{m-1} = \sum \lambda^{m-1} P_\lambda.$$

Since the operator N is given, we know the left-hand side of each equation. Hence, we are dealing with a system of inhomogenous linear equations for P_λ. In order to find all P_λ, we need at least as many equations as we have projections, or equivalently, as we have eigenvalues. In fact, we only need exactly as many equations as there are eigenvalues. Indeed, with m being the number of eigenvalues, we see that there exists a unique solution if and only if

$$\det \begin{bmatrix} 1 & 1 & 1 & ... \\ \lambda_1 & \lambda_2 & \lambda_3 & ... \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & ... \\ ... & ... & ... & ... \end{bmatrix} \neq 0.$$

This famous determinant is called the *Vandermonde determinant* and it equals zero if and only if $\lambda_a = \lambda_b$ for some a and b. As we have already defined our lambdas so that none coincide, the Vandermonde determinant is guaranteed to be non-zero, and the equations are linearly independent.

As it was already mentioned, for our purposes (e.g. for calculating functions of operators in a simple way), instead of using the spectral decomposition we could just use diagonalization. It was also explained that the real use of spectral decomposition is in infinite dimensions where diagonalization may lose meaning. Nevertheless, in some cases spectral decomposition can be "better" than diagonalization in finite dimensions; i.e. easier to do. If we had, for instance, a 100-by-100 matrix with only two eigenvalues, after finding the spectral decomposition, we would still only need to calculate two spectral projections. On the other hand, diagonalization would require finding 100 linearly independent eigenvectors (regardless of the fact that there are only two distinct eigenvalues!) and working out the matrices of the change of bases.

4 Examples

In this section we shall consider examples. For the purpose of examples and applications we will work in $\mathbb{C}^n$, which we shall consider with its "standard" scalar product

$$\langle x, y\rangle = \sum_{k=1}^{n} \overline{x}_k y_k.$$

As is well-known, linear operators of $\mathbb{C}^n$ are nothing else than multiplcation by $n \times n$ matrices. The actual form of the adjoint however, depends on the chosen scalar product. In general, the adjoint M^* of M is the linear operator that for all x and y vectors satisfies the relation

$$\langle Mx, y\rangle = \langle x, M^* y\rangle.$$

In our case, $x, y \in \mathbb{C}^n$ and

$$\langle Mx, y\rangle = \sum_k \overline{(\sum_l M_{k,l} x_l)} y_k = \sum_{k,l} \overline{M_{k,l}}\, \overline{x_l} y_k$$

which should be equal to

$$\langle x, M^* y\rangle = \sum_l \overline{x_l} (\sum_k (M^*)_{l,k} y_k) = \sum_{k,l} (M^*)_{l,k} \overline{x_l} y_k.$$

Since x and y was arbitrary, it follows that

$$(M^*)_{l,k} = \overline{M_{k,l}},$$

or that with our choice of the scalar product, the matrix of the adjoint is simply obtained by transposing and complex conjugating. So for example, if

$$M = \begin{bmatrix} i & -1+i \\ 1-i & i \end{bmatrix},$$

then

$$M^* = \begin{bmatrix} -i & 1+i \\ -1-i & -i \end{bmatrix},$$

and by a straightforward calculation

$$MM^* = M^*M = \begin{bmatrix} 3 & 2i \\ -2i & 3 \end{bmatrix}.$$

Hence M is normal. So let us find its spectral decomposition!

The spectral decomposition, as discussed above, is a collection of projections corresponding to the eigenvalues of the matrix. Hence, the starting point is to find the eigenvalues of the matrix M.

$$M - \mathbf{1}\lambda = \begin{bmatrix} i-\lambda & -1+i \\ 1-i & i-\lambda \end{bmatrix},$$

so by taking the determinant, the characteristic polynomial of M is

$$\det(M - \mathbf{1}\lambda) = (i-\lambda)^2 - (-1+i)(1-i) = \lambda^2 - 2i\lambda - (1+2i)$$

and so the eigenvalues are $\lambda = -1, 1+2i$. The system of equations for finding the projections is

$$M^n = \lambda_1^n P_1 + \lambda_2^n P_2 ... + \lambda_m^n P_m \quad (m = 1, 2, ..).$$

In this case, we shall need two equations, since there are only two unknown projections. We have

$$\begin{aligned} \mathbf{1} &= P_{-1} + P_{1+2i}, \\ M &= -P_{-1} + (1+2i)P_{1+2i}. \end{aligned}$$

From the first equation $P_{1+2i} = \mathbf{1} - P_{-1}$. Thus by substituting into the second equation

$$M = -P_{-1} + (1+2i)(\mathbf{1} - P_{-1}) = (1+2i)\mathbf{1} - (2+2i)P_{-1}.$$

So finally we get that the spectral projections are

$$\begin{aligned} P_{-1} &= \frac{1}{2+2i}((1+2i)\mathbf{1} - M) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \\ P_{1+2i} &= \mathbf{1} - P_{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \end{aligned}$$

Note that the two orthogonal projections P_{-1} and P_{1+2i} came out to be automatically self-adjoint (i.e. transposing and then conjugating P_{-1} we get back P_{-1}. The same applies for P_{1+2i}). A further check that we could do (if worried about having made mistakes in the calculations) is to check that the obtained P_{-1} and P_{1+2i} are indeed projections (i.e. that $P_{-1}^2 = P_{-1}$ and $P_{1+2i}^2 = P_{1+2i}$).

Let us see now an example for finding the spectral projections of a 3×3 matrix. Let

$$M = \begin{bmatrix} 1 & -i & 1-i \\ i & 4 & 4+i \\ 1+i & 4-i & 5 \end{bmatrix}.$$

Without any further check it is clearly normal, as just by looking at it we see that it is in particular self-adjoint: $M^* = M$. By direct calculation the characteristic polynomial is

$$\det(M - \mathbf{1}\lambda) = -\lambda(9-\lambda)(1-\lambda)$$

and hence the eigenvalues are $\lambda = 0, 1, 9$. (Note that as M is self-adjoint, the eigenvalues are real.) Hence, the system of linear equations to be solved for finding the spectral projections is

$$\begin{aligned} \mathbf{1} &= P_0 + P_1 + P_9 \\ M &= P_1 + 9P_9 \\ M^2 &= P_1 + 81P_9. \end{aligned}$$

Making a few substitutions, we see that

$$P_9 = \frac{1}{72}(M^2 - M), \quad P_1 = M - 9P_9 \quad P_0 = \mathbf{1} - P_9 - P_1$$

and so the spectral projections are

$$\begin{aligned} P_9 &= \begin{bmatrix} \frac{1}{24} & \frac{1}{24} - \frac{1}{8}i & \frac{1}{4} + \frac{1}{8}i \\ \frac{1}{24} + \frac{1}{8}i & \frac{1}{4} & -\frac{1}{8} - \frac{1}{8}i \\ \frac{1}{4} - \frac{1}{8}i & -\frac{1}{8} + \frac{1}{8}i & \frac{1}{8} \end{bmatrix} \\ P_1 &= \begin{bmatrix} \frac{5}{8} & -\frac{3}{8} + \frac{1}{8}i & \frac{1}{4} + \frac{1}{8}i \\ -\frac{3}{8} - \frac{1}{8}i & \frac{1}{4} & -\frac{1}{8} - \frac{1}{8}i \\ \frac{1}{4} - \frac{1}{8}i & -\frac{1}{8} + \frac{1}{8}i & \frac{1}{8} \end{bmatrix} \\ P_0 &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

To see case where not all eigenspaces are 1-dimensional, consider the matrix

$$M = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Again, it is self-adjoint (and so it is normal). Taking the determinant, observe:

$$\det(M - \mathbf{1}\lambda) = -\lambda(3-\lambda)^2 - 36 = -\lambda^3 - 6\lambda^2 - 9\lambda - 36 = (8-\lambda)(1+\lambda)^2.$$

So we only have two eigenvalues; $\lambda = -1, 8$. From the characteristic polynomial we see that the algebraic multiplicity of the eigenvalue $\lambda = -1$ is 2. Since we know that M has a spectral decomposition (as it is normal), it must be diagonalizable and so the algebraic multiplicities of its eigenvalues must be equal to the geometrical ones. Hence $\lambda = -1$ is a "double eigenvalue"; i.e. the corresponding eigenspace is 2-dimensional. However, from the point of view of spectral decomposition, this causes no problems. In fact, it the situation is much simpler than in our previous example, since we only need to find two projections, and therefore only need a system of two linear equations:

$$\begin{aligned} \mathbf{1} &= P_8 + P_{-1} \\ M &= 8P_8 - P_{-1} \end{aligned} \tag{1}$$

and the spectral projections turn out to be

$$\begin{aligned} P_8 &= \frac{1}{9}\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \\ P_{-1} &= \frac{1}{9}\begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}. \end{aligned}$$

Note that the rank (the maximal number of independent columns) of P_8 is 1 (since every column of it is a multiple of its first one), so it is a projection onto a 1-dimensional subspace. In contrast, the rank of P_{-1} is 2, as it is in fact a projection onto a 2-dimensional subspace.

5 Application: Recursive Sequences

Consider the recursion for the Fibonacci sequence:

$$a_{n+2} = a_{n+1} + a_n.$$

$$a_0 = 1, a_1 = 1$$

Although there are numerous ways to determine this sequence non-recursively, one can also use spectral decomposition to solve this recursion. We begin by expressing the recursive term a_n in the form of a recursive vector,

$$v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}.$$

By the recursive relation, we have

$$v_{n+1} = \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n + a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}.$$

Similarly, we find that

$$v_{n+2} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix},$$

and so by substituting the already obtained form of v_{n+1} we get that

$$v_{n+2} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 \begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix}.$$

Continuing in this fashion, one sees that

$$v_{n+m} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^m \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}.$$

So, to calculate the n^{th} term of the Fibonacci sequence, we can multiply the initial condition vector $\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ by the n^{th} power of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. This, in itself, does not make finding the solution to the recursion any easier, as it still requires raising a matrix to its n^{th} power. However, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ is self-adjoint, and hence we can calculate its powers using its spectral decomposition: we have

$$M^n = {\lambda_+}^n P_+ + {\lambda_-}^n P_-$$

where $\lambda_\pm$ are the eigenvalues of M and $P_\pm$ are the corresponding spectral projections. In this case, by direct calculations the eigenvalues are

$$\lambda_+ = \frac{1+\sqrt{5}}{2}, \quad \lambda_- = \frac{1-\sqrt{5}}{2}$$

and the projections come out to be

$$\begin{aligned} P_+ &= \begin{bmatrix} \frac{5-\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{5}+5}{10} \end{bmatrix} \\ P_- &= \begin{bmatrix} \frac{5+\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \end{bmatrix} \end{aligned}$$

Thus

$$v_n = \left(\left(\frac{1+\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{5-\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{5}+5}{10} \end{bmatrix} + \left(\frac{1-\sqrt{5}}{2}\right)^n \begin{bmatrix} \frac{5+\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $v_n = \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}$, we can obtain the general formula for the Fibonacci sequence by reading of the first component of the vector v_n:

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n \right).$$

6 Application: Differential equations with values in a $\mathcal{H}$-space.

Consider the representation of Schrödinger's equation as a partial differential equation,

$$i\hbar \frac{d}{dt}\psi_t(x) = [-\frac{\Delta}{2m} + V(x)]\psi_t(x),$$

where $x \mapsto \psi_t(x)$ is a function in $\mathcal{L}^2(\mathbb{R}^3)$, or the Hilbert space of square-integrable functions on $\mathbb{R}^3$. The bracketed portion of the above equation is the self-adjoint Hamiltonian operator, H. Thus, instead of thinking about it as a partial differential equation, we may think of the Schrödinger equation as the ordinary linear differential equation

$$\frac{d}{dt}\psi_t = -\frac{i}{\hbar}H\psi_t$$

with values in $\mathcal{L}^2(\mathbb{R}^3)$. It can be justified rigorously that the solution of this differential equation is exactly what one expects, namely

$$\psi_t = e^{-\frac{i}{\hbar}Ht}\psi_0,$$

where the exponential $e^{-\frac{i}{\hbar}Ht}$ is obtained via the spectral decomposdition of H. However, as we only considered spectral decompositions in finite dimensions (and of course the space of functions $\mathcal{L}^2(\mathbb{R}^3)$ is very far from being finite dimensional), we are forced to discuss here some other (though much less relevant, from the point of view of quantum physics) examples.

Consider the system of differential equations

$$\dot{x} = 2x + y \qquad \dot{y} = x + 3y.$$

As we see, the derivative of x depends on both x and y, and the same applies for the derivative of y. So it seems difficult to determine the solution, though the system of equations is linear. However, we may re-express this system with vectors and matrices:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Thus by setting $v(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, we have that N is normal (as in fact it is self-adjoint) and $\frac{d}{dt}v(t) = Nv(t)$. So it is easy verify that the solution is $v(t) = e^{Nt}v(0)$. Hence our problem is reduced to finding the exponential of N, which we shall do by considering its spectral decomposition. By some calculations the eigenvalues of N are $\lambda = \frac{5+\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}$ and the corresponding spectral projections are

$$\begin{aligned} P_{\frac{5+\sqrt{5}}{2}} &= \begin{bmatrix} \frac{5-\sqrt{5}}{10} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{5+\sqrt{5}}{10} \end{bmatrix} \\ P_{\frac{5-\sqrt{5}}{2}} &= \begin{bmatrix} \frac{5+\sqrt{5}}{10} & -\frac{\sqrt{5}}{5} \\ -\frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{10} \end{bmatrix}. \end{aligned}$$

Thus

$$e^{Nt} = \sum e^{\lambda t} P_\lambda = \left[\begin{array}{cc} \frac{5-\sqrt{5}}{10} e^{\frac{5+\sqrt{5}}{2}t} + \frac{5+\sqrt{5}}{10} e^{\frac{5-\sqrt{5}}{2}t} & \frac{\sqrt{5}}{5} e^{\frac{5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5} e^{\frac{5-\sqrt{5}}{2}t} \\ \frac{\sqrt{5}}{5} e^{\frac{5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5} e^{\frac{5-\sqrt{5}}{2}t} & \frac{5+\sqrt{5}}{10} e^{\frac{5+\sqrt{5}}{2}t} + \frac{5-\sqrt{5}}{10} e^{\frac{5-\sqrt{5}}{2}t} \end{array} \right]$$

from where we get that

$$\begin{aligned} x(t) &= \left(\frac{5-\sqrt{5}}{10} e^{\frac{5+\sqrt{5}}{2}t} + \frac{5+\sqrt{5}}{10} e^{\frac{5-\sqrt{5}}{2}t} \right) x(0) + \left(\frac{\sqrt{5}}{5} e^{\frac{5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5} e^{\frac{5-\sqrt{5}}{2}t} \right) y(0) \\ y(t) &= \left(\frac{\sqrt{5}}{5} e^{\frac{5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5} e^{\frac{5-\sqrt{5}}{2}t} \right) x(0) + \left(\frac{5+\sqrt{5}}{10} e^{\frac{5+\sqrt{5}}{2}t} + \frac{5-\sqrt{5}}{10} e^{\frac{5-\sqrt{5}}{2}t} \right) y(0). \end{aligned}$$