1 The dynamical system

This program simulates the motion of a 1-dimensional infinite system of point particles, where every particle has mass 1. Two neighbouring particles are tagged, and there is a repelling force with potential $\tilde{U}(d) = \frac{c^2}{2d^2}$ acting between these tagged particles (here d is the distance of the tagged particles). The other particles fly freely, they only collide with the two tagged particles, elastically and momentarily. The value of c is constant during the motion, but its value is random – that is, c is fixed together with the initial conditions.

At the beginniing of the process the positions of the particles form a Poisson point process with intensity 1, the velocities are standard normally distributed random variables, independent of everything else, and

$$c = W d_0, \tag{1}$$

where W is standard normally distributed, independent of everything else, and d_0 is the initial distance of the two tagged particles.

This system is in some sense the $M \to 0$ limit of a system without potential, where every particle flies freely until it hits another, but there is an extra particle with a very small mass M between the two tagged particles, which has an initial velocity independent of everything else, normally distributed with mean 0 and variance $\frac{1}{M}$. This extra particle also collides elastically with its neighbours.

2 The problem

We would like to find the variance of the position of the center of mass for the two tagged particles, as a function of time. We hope that the assymptotic behaviour of the variance can be seen from the simulation results. For this purpose we perform N simulations of legth T with independent initial conditions, and calculate the empyrical variance from the sample of size N. This is simply the average of the squares, since the expectation is zero.

3 About the operation of the program

The goal of the simulation is to follow the two tagged particles for some time $T > 10$. For this purpose we don't need to follow infinitely many particles: it's more than enough to follow those that are closer than $10T$ in the beginning. In this way, even if $T = 10^6$ and we perform $N = 10^{10}$ experiments, then with probability 0.9999 it will never happen that a particle we didn't follow would collide with a tagged one. To further increase the speed of the simulation, we could throw away the particles which don't have a sufficient velocity in the direction of the tagged ones. However, the present version of the program does not do that.

The motion oqis observed from the center of mass of the system formed by the two tagged particles. That is, we always use $x = \frac{d}{2}$ instead of d, which is the position of the tagged particles in the chosen reference frame (up to sign). Between two collisions, the change of this x is governed by the equation of motion for a *single* point particle moving in the potential $U(x) = \frac{c^2}{16x^2}$, that is, $\ddot{x} = \frac{c^2}{8x^3}$. The solution of this equation is a hyperbola in the (t, x) plane:

$$\left(\frac{x}{\frac{c}{4\sqrt{E}}}\right)^2 - \left(\frac{t - t^*}{\frac{c}{4\sqrt{2E}}}\right)^2 = 1, \tag{2}$$

where t^* is the moment in time when the two particles are (would be) the closest to each other, and E is the total energy:

$$t^* = \begin{cases} t_1 + \sqrt{\frac{x_1^2}{2E} - \frac{c^2}{32E^2}}, & \text{ha } v_1 \le 0 \\ t_1 - \sqrt{\frac{x_1^2}{2E} - \frac{c^2}{32E^2}}, & \text{ha } v_1 > 0 \end{cases},$$

$$E = \frac{1}{2}v_1^2 + \frac{c^2}{4x_1^2}.$$

Here x_1 and v_1 are the position and the velocity at the time t_1. The value of E is constant between two collisions, but it changes at collision times (as opposed to c, which doesn't change at collisions either).

For the sake of convenience we store the position of every particle as a positive number, and register the particles on the laft and right separately. So we only need to consider positive values of x at all tomes.

The time of the next collision could be obtained by solving for t the system of equation formed by

$$x = x_i + v_i(t - t_1)$$

and (2), where x_i and v_i are the position and velocity of the i-th particle at t_1 (in the central frame), and finding the smallest positive solution (which is the first future collision). However, much of the calculation can be saved if we remember which one of the particles investigated so far would collide first, and we only solve the system of equations for a new particle if it would collide *even sooner*. This is true exactly if at the time of collision of the "so far best" particle, it would be left of the collision place.

Since the i-th particle always starts from the right of the tagged one (that is, $x_i > x_1$), the above system of equations can have at most one positive solution, namely

- if $v_i > \sqrt{2E}$, then there is no positive solution
- if not, then

$$t = \frac{x_1 v_i + 2Et^*}{2E - v_i^2} + \frac{1}{2E - v_i^2}\sqrt{2E(v_i t^* + x_i)^2 + \frac{v_i^2 c^2}{16E} + \frac{c^2}{8}}.$$

So during a simulation step we first find which particle will collide first, when and where, and what the velocity of the tagged particles will be at that time. Than we exchange the velocities of the colliding particles, and we switch to the new central frame. Thus we only keep track of relative positions and velocities: even the position and velocity of the center of mass is only calculated because this is the purpose of the simulation.

4 Modifications

When the investigated system is born as the zero-mass limit of a system with momentary elastic collisions only, the natural choice of the interaction is the one described in the beginning. Still it is an interesting question, how the results are modified if the interaction between the tagged particles is different, specifically, if the constant c in the potential is not chosen according to (1). For this reason we investigate three possible modifications:

- when c is constant times the one given by (1)
- when c is a deterministic constant independent of the initial conditions
- when $c = Wd_0$, as in (1), but W is not normally distributed and independent of everything else, but a deterministic constant independent of the initial conditions. This corresponds to fixing the initial velocity of the small particle in the approximating system.

5 Visual check

The correctness of the simulation of the trajectory can be partially checked by drawing the positions of all particles (modified by the position of the center of mass, that is, in the resting frame). So one can see whether the trajectories are indeed lines, whether we miss a collision, and whether the post-collision trajectories are differentiable continuations of the pre-collision ones. For this purpose the program has a "test" mode, when the trajectories are printed as functions of time.